CHAPTER 5

Time Value of Money

Learning Objectives

5.1 Explain the importance of the time value of money and how it is related to an investor's opportunity costs.

5.2 Define simple interest and explain how it works.

5.3 Define compound interest and explain how it works.

5.4 Differentiate between an ordinary annuity and an annuity due, and explain how special constant payment problems can be valued as annuities and, in special cases, as perpetuities.

5.5 Differentiate between quoted rates and effective rates, and explain how quoted rates can be converted to effective rates.

5.6 Apply annuity formulas to value loans and mortgages and set up an amortization schedule.

5.7 Solve a basic retirement problem.

The discount rate used to determine the value of our money today is related to the opportunity cost of that money. So what might be the opportunity cost of pursuing a master's degree in Canada? In a study by Statistics Canada (StatsCan), the opportunity cost was defined as tuition + additional fees + books + lost income − part-time income earned during the school year. For the 1995–96 academic year, StatsCan estimated the opportunity cost on average was $29,956. The survey indicated that the returns from earning a post-graduate degree appear to outweigh the costs. Five years after 1990 graduation, master's graduates earned, on average, one-third more than those with bachelor's degrees. The unemployment rate for master's graduates was also lower.

Source: Statistic Canada. "Pursuing a Master's Degree: Opportunity Cost and Benefits" *Education Quarterly Review* 8, no. 4 (2002). Statistic Canada Catalogue no. 81-003-XIE. Ottawa

CHAPTER 5 PREVIEW

Part 1 was an introduction to the study of finance. In Part 2 we examined the importance of company financial statements. Now, in Part 3, we discuss the basic valuation process as it applies to financial securities. This valuation process relies heavily on discounting future expected cash flows, one of the tools discussed in this chapter. Mastery of the tools presented in this chapter is necessary for understanding finance.

This chapter will introduce you to everyday problems, such as taking out a loan, setting up a series of payments, and valuing them. The ideas in this chapter are important for all types of financial problems: determining the payments on a weekly versus a monthly mortgage, buying versus leasing a new car, appropriately valuing a bond or stock, determining whether a company should expand production or abandon a product line, and deciding how much a company should be willing to pay for another company. Although each situation involves unique circumstances that will be covered in subsequent chapters, the basic framework used to evaluate them is the same and relies on material covered in this chapter.

5.1 OPPORTUNITY COST

Learning Objective 5.1
Explain the importance of the time value of money and how it is related to an investor's opportunity costs.

time value of money the idea that money invested today has more value than the same amount invested later

medium of exchange something that can be used to facilitate transactions

required rate of return or **discount rate** the market interest rate (k) or the investor's opportunity cost

In this chapter, we are concerned with the **time value of money**. As we saw in Chapters 1 and 2, the financial system is designed to transfer savings from lenders to borrowers, so that savers have money to spend in the future. To illustrate this concept, we used the example of saving while working in order to have money when retired. Money, in this sense, represents our ability to buy goods and services—that is, it operates as a **medium of exchange** and has no value in and of itself. Of course, an investor could simply store the dollar bills (tuck them under the mattress!) and spend them later; a dollar is always worth at least a dollar in the future.[1] However, this option ignores the fact that the saver has other uses for that dollar, which in economics are called the "opportunity costs" or "alternative uses," which may include investing this dollar to earn a return. These are what produce the time value of money.

The opportunity cost of money is the interest rate that would be earned by investing it. For this reason, we also call the interest rate the price of money. Knowing this rate helps us determine the value of money received at different times. Suppose, for example, a person has three choices: he or she could receive $20,000 today, $31,000 in five years, or $3,000 per year indefinitely. This choice could, for example, be the payoff from a lottery (though we are certainly not advocating gambling!). Making a choice from these different options requires that we know how to value the dollars received at different times—that is, the winner needs to adjust for the time value of money.

To make a decision, the person needs to know what the interest rate is. We will use k as a standard notation throughout the textbook for the market interest rate. We will refer to this market interest rate by several other names later in the textbook, such as the **required rate of return** or **discount rate**. The reason for these different names will become clear later, but in all cases we are looking at the investor's opportunity cost—that is, what he or she can do with the money being invested. However, first we have to make some basic distinctions in terms of how this interest rate is earned and distinguish between simple interest and compound interest.

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[1] This ignores the fact that what we are really concerned about is what that dollar will buy in terms of goods and services—that is, its purchasing power. We discuss this later in the chapter.

CONCEPT REVIEW QUESTIONS

1. Why does money have a "time value"?
2. What is an "opportunity cost"?

5.2 SIMPLE INTEREST

Simple interest is interest paid or received on only the initial investment (the principal). Although, in practice, simple interest is used for a limited number of applications, we introduce it first to contrast it with compound interest, which is the norm.

Learning Objective 5.2
Define simple interest and explain how it works.

EXAMPLE 5-1

simple interest interest paid or received on only the initial investment (the principal)

Simple Interest I

Suppose someone invests $1,000 today for a five-year term and receives 10 percent annual simple interest on the investment. How much would the investor have after five years?

Solution

Annual interest = $1,000 × 0.1 = $100 per year

Year	Beginning Amount	Ending Amount
1	$1,000	$1,100
2	1,100	1,200
3	1,200	1,300
4	1,300	1,400
5	1,400	1,500

The interest earned is $100 every year, regardless of the beginning amount each year, because interest is earned on only the original investment (principal). Interest is *not* earned on the accrued (or earned) interest.

Because the same amount of interest is earned each year—$100 in the example—we can use Equation 5-1 to find the value of the investment at any point in time.

$$\text{Value (time } n) = \text{P} + (n \times \text{P} \times k) \quad [5\text{-}1]$$

Where P = principal and n = number of periods.

Notice that P × k = interest. If we apply this equation to Example 5-1, P = 1,000, n = 5, and k = 0.1. The value in year 5 = 1,000 + (5 × 100) = $1,500. This is the amount shown in the table for Example 5-1 at year 5.

The basic point of simple interest is that in order to determine the future value of an investment, we calculate the annual interest—in our case $100—multiply this by the number of years of the investment, and add it to the starting principal.

EXAMPLE 5-2

Simple Interest II

We'll repeat the example but assume that the investment is for 50 years.

Solution

Annual interest is still \$100 per year (\$1,000 × 0.1), so using Equation 5-1 we get

$$\text{Value in year 50} = 1{,}000 + (50 \times 100) = 1{,}000 + 5{,}000 = \$6{,}000$$

Understanding simple interest helps solve our earlier problem. For example, a person offered the choice between \$20,000 today and \$31,000 in five years can calculate the two annual interest payments. With the same 10 percent interest rate, the annual interest is \$20,000 × 0.1 = \$2,000 per year. In five years, it would generate \$10,000 in interest, meaning that \$20,000 today is worth \$30,000 in five years. In this case, given the choice between \$20,000 today and \$31,000 in five years, with 10 percent simple interest, the correct choice is \$31,000 in five years, because it is worth more. However, how do we solve the choice between these two options and \$3,000 per year forever (indefinitely)?

One way to solve this problem is to assume a very long period, say 100 years. Receiving \$3,000 each year for 100 years produces a future value of \$3,000 × 100 = \$300,000. We then compare this with investing \$20,000 for 100 years, which has a future value of \$20,000 + (\$2,000 × 100) = \$220,000. In this case, by assuming that "indefinitely" is 100 years, the solution would be to choose the \$3,000 per year. However, apart from the fact that simple interest problems are relatively rare, it turns out that we are missing something very important, particularly when we invest for long periods.

CONCEPT REVIEW QUESTIONS

1. Explain how simple interest payments are determined.
2. Why does simple interest take into account the time value of money?

5.3 COMPOUND INTEREST

Compounding (Computing Future Values)

Learning Objective 5.3
Define compound interest and explain how it works.

compound interest interest that is earned on the principal amount invested *and* on any accrued interest

Compound interest is interest that is earned on the principal amount invested *and* on any accrued interest. Compound interest can result in dramatic growth in the value of an investment over time. This growth is directly related to the length of the period, as well as to the level of return earned, which we will demonstrate shortly. Before we get to that, an example will show how compound interest arrangements work.

Unlike the annual interest for an investment earning simple interest, the amount of compound interest earned increases every year; the interest rate is applied to the *principal plus interest earned*, so the value of the investment increases. As a result, the interest received increases from \$100 in year 1, to \$146.41 in year 5; the ending amount of \$1,610.51 is much higher than the \$1,500 earned with simple interest.

EXAMPLE 5-3

Compound Interest I

Suppose someone invests $1,000 today for a five-year term and receives 10 percent annual *compound* interest. How much would the investor have after five years?

Solution

Annual interest is earned on the original $1,000 (principal) *and* on accrued interest.

Year	Beginning Amount	Interest	Ending Amount
1	$1,000	1,000 × 0.1 = $100	$1,100
2	1,100	1,100 × 0.1 = $110	1,210
3	1,210	1,210 × 0.1 = $121	1,331
4	1,331	1,331 × 0.1 = $133.10	1,464.10
5	1,464.10	1,464.10 × 0.1 = $146.41	1,610.51

To make the process clear, let's look at the first two years of interest using a little algebra. For the first year, everything is the same as the example with simple interest. That is, the ending amount is the starting principal plus the interest or

$$\$1{,}000 + (\$1{,}000 \times 0.1) = \$1{,}100 = \$1{,}000 \times (1 + 0.1) \text{ or } PV_0(1 + k)$$

where PV_0 = the present value today (i.e., at time 0). We have factored the $1,000 principal value, so to get the future value, we multiply the principal by one plus the market interest rate.

For year 2, the full $1,100 is **reinvested**—that is, we explicitly do not take the $100 of interest and spend it. As a result, we have the following equation:

reinvest to keep interest earned on an investment fully invested

$$\$1{,}100 + (\$1{,}100 \times 0.1) = \$1{,}210 = \$1{,}100 \times (1 + 0.1) \text{ or } PV_0(1 + k)^2$$

In this case, $1,100 is invested at the beginning of year 2 and earns the 10 percent interest. The interest earned increases to $110: the $100 interest on the starting principal plus $10 interest earned on the $100 of interest reinvested at the end of the first year. We can again factor the starting value of $1,100 and then factor the $1,000 principal value to get the formula for the future value at the end of year 2. This is the starting principal times one plus the interest rate squared. As we increase the period, we get the general formula

$$FV_n = PV_0(1 + k)^n \quad [5\text{-}2]$$

where FV_n = the future value at time n.

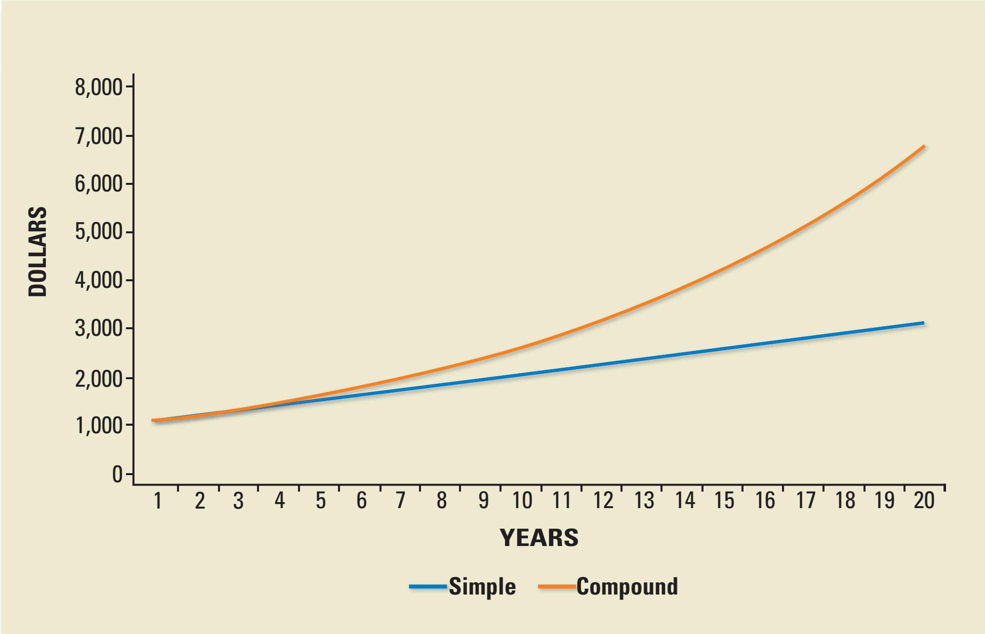
Equation 5-2 is the basic compounding equation, and the last term, $(1 + k)^n$, is the **compound value interest factor (CVIF)**.

compound value interest factor (CVIF) a term that represents the future value of an investment at a given rate of interest and for a stated number of periods: $(1 + k)^n$

Applying this equation to Example 5-3, we get $FV_5 = 1{,}000(1 + 0.1)^5 = 1{,}000(1.61051) = \$1{,}610.51$. This is $110.51 more than for the investment earning simple interest. Figure 5-1 illustrates what happens with the two types of interest over time. Note that for the first few years the difference is minimal, but over time it gets bigger and bigger.

FIGURE 5-1

Simple vs. Compound Interest at 10%



Example 5-3 can also be solved using a financial calculator. Although the keystrokes (and variable names) used will vary from one calculator to the next, the basic procedures do not. We will illustrate with one commonly used calculator: the Texas Instrument (TI) BA II Plus.[2]

solution using a financial calculator

(TI BA II Plus)

Input the following variables:

0 → PMT ; −1,000 → PV ; 10 → I/Y ; and 5 → N

Press CPT (Compute) and then FV

PMT here refers to regular payments and will be discussed in a later section; FV is the future value; I/Y is the period interest rate; and N is the number of periods. The PV is entered with a negative sign on this calculator (this is not the case with all calculators) to reflect the fact that investors must pay money now to get money in the future. Alternatively, we could have left it positive. This would produce a negative sign in front of the FV, which we could simply ignore. We will do this in some of the ensuing applications.

The answer will be 1,610.51.

[2] This is one of only two types of calculators permitted for the Chartered Financial Analyst (CFA) examinations, which are administered by the CFA Institute.

Time value of money problems can also be solved using Excel spreadsheets, which have time value of money functions. We illustrate this below by solving Example 5-3 using Excel.

solution using Excel

= FV	(rate, nper, pmt, pv, type)
where rate =	interest rate (expressed as a decimal)
nper =	number of periods
pmt =	the payment amount
pv =	present value
type =	0 if it is an ordinary annuity, 1 if it is annuity due. (We will explain what the difference is shortly; for now, our examples will involve ordinary annuities.)
For Example 5-3, we would enter the following in the appropriate cell:	
= FV	(0.10, 5, 0, −1000, 0) which yields → 1,610.51.

Let us now extend the time horizon for Example 5-3, as we did for the example relating to simple interest.

Compound Interest II

EXAMPLE 5-4

Repeat Example 5-3, but assume the investment is for 50 years.

Solution

Applying Equation 5-2 to this example, we get

$$FV_{50} = 1{,}000(1 + 0.1)^{50} = 1{,}000(117.39085) = \$117{,}390.85$$

Investing $1,000 for 50 years at an annual interest rate of 10 percent produces $117,390.85. Notice the huge difference between this amount and the future value of $6,000 ($1,000 + [50 × $100]) for the same $1,000 invested for 50 years but earning simple interest!

You might be tempted to ask whether a 50-year term is realistic. It is for many investments. Consider someone who begins investing for retirement in his or her early 20s. Those early investments could earn compound returns for 40 years or so before the individual retires. Further, assuming the individual does not withdraw all the savings on the first day of retirement (which would have severe tax consequences), and assuming that this person lives another 20 to 25 years after retirement, some investment dollars may not be touched for more than 50 years. Finance in the News 5-1 shows just how important it is to begin investing early.

Table 5-1 provides some evidence regarding the power of compound returns. It shows the future values that would have resulted from investing $1,000 at the beginning of 1938 and leaving that money invested for 71 years (until the end of 2008) in various investment assets.

TABLE 5-1 Ending Wealth of $1,000 Invested from 1938 to 2008 in Various Asset Classes

	Annual Geometric Mean (%)	Year-End Value, 2008 ($)
Government of Canada treasury bills	5.05	33,135
Government of Canada bonds	6.16	69,561
Canadian stocks	9.90	816,305
U.S. stocks	10.91	1,559,986

Source: Data are from the Canadian Institute of Actuaries.

finance INTHENEWS 5-1

GET A HEAD START ON RRSP: IT PAYS TO BEGIN SAVING FOR RETIREMENT IN YOUR 20s

Ask any financial adviser when young people should start saving for their retirement. The answer is always the same: they should start socking money away inside a registered retirement savings plan in the same year they begin earning income.

For those leaving high school and not going on to university or college, this can mean starting an RRSP as young as age 18. For others whose education years stretch longer, those first paycheques may not come until the late 20s.

No matter when they start working, however, many people sail clear through their 20s without giving a thought to retirement planning. A survey this year for Royal Trust found the average first-time RRSP contributor is 31 years old.

Yet to earn a healthy nest egg for retirement inside a tax-sheltered plan, it really pays to start early. Popular Canadian investment author Gordon Pape points this out in his 1999 Buyer's Guide to RRSPs.

"The greatest growth takes place in the later years," Pape writes. "So the longer you wait to begin, the less your RRSP will be worth when you retire.

"In fact, if you begin contributions in your early 20s and stop when you reach age 35, leaving the balance in your RRSP, you'll end up with more money at 65 than if you waited until you were 35 to begin and contributed every year thereafter."

Judy Willmer, a certified financial planner with Investors Group Inc. in St. Albert, Alta., has an 18-year-old client who just set up an RRSP. But younger people make up only a fraction of her client base; most are between ages 35 and 50.

"Let's say somebody comes in at age 25," Ms. Willmer says. "I'll show them a projection saying if you can save $200 a month up until age 65 you'll have over $1 million. But if you wait 10 years before you save and still want to have that $1-million at age 65, you now have to save $500 a month."

The Royal Trust survey found that first-time RRSP contributors are too cautious in their approach, with only 30 percent saying they planned to put the majority of their investment money into mutual funds. Others were opting for low-risk, low-growth instruments such as savings accounts. And the survey found that only 47 percent of first-time RRSP contributors had consulted with a financial adviser, compared to 73 percent for seasoned investors.

Geoff Anselmo, a 29-year old mutual fund investment specialist with Royal Trust, says it can be difficult to even reach young people to tell them they should start saving for retirement. "You can advertise, but if I'm 21, I'm probably reading the sports section, not the financials," he says.

The young clients who do come his way are often referred to him by their parents. As a way of getting them to see the advantages of investing, he tells them that time is on their side. Ideally the money gets deducted automatically from their savings accounts, so it is relatively "painless."

Mr. Anselmo illustrates this point with an anecdote about Bill and Linda, who are twins.

At age 22, Bill starts investing $2,000 each year, earning a healthy 12 percent per year. He continues for six years, then stops and never invests another dime. Linda waits six years, starts investing $2,000 a year at age 28 and continues to do this until she reaches 65. Like her brother, she gets a 12 percent annual compound rate of return.

At 65, they each have about $1.4-million for their retirement. Yet Bill invested only $12,000 while his sister invested $76,000.

"I find that story in itself is very effective," Mr. Anselmo says. "It talks to people as to why to start early."

Mr. Anselmo generally doesn't develop detailed financial plans for clients in their 20s, mainly because they have a lot of living to do before they can decide what they want their retirement years to look like. Marriage, children, first homes, careers, and career changes all have more importance from where they sit.

"I don't dwell on the things I would dwell on with somebody who is in their prime earning years," he says. "If I talk to somebody young and say, 'At 65, do you want to golf?' they look at me and say, 'I'm worried about tomorrow.'"

Mr. Anselmo encourages the new saver to start with something easy, such as putting $25 or $50 a month into a balanced mutual fund.

As their paycheques get larger, their deductions can increase. Mr. Anselmo finds that after saving up a base of $10,000 or $20,000 and doing some reading on investing, young people can comfortably start to tinker with their investment mixes.

Investors Group's Ms. Willmer takes a slightly different approach to her younger clients. She always works out cash flow statements, showing them where their money is going. Nights out for beer and pizza can eat up $200 a month if you're not careful, she warns them.

"You can show them that if they are earning $25,000 and they put $1,000 into an RRSP they will pay $250 less income tax that year," she says.

Ms. Willmer always recommends automatic bank withdrawals as a painless way to develop saving discipline. And she does a lot of follow-up to ensure clients keep up with their commitments to themselves even while their lives are undergoing big changes in the years between 20 and 30.

Happily, Ms. Willmer has discovered some of her younger clients do think ahead to retirement. "A lot of the kids are very knowledgeable," she says.

Source: Howell, David. "Get a Head Start on RRSP." *Financial Post*, May 20, 1999, p. D4. Material reprinted with the express permission of Edmonton Journal Group Inc., a CanWest Partnership.

The geometric mean represents the average annual growth rate in the value of $1 invested at the start of the period. The geometric mean is thus the compound rate of return or the interest rate that compounds the starting value to the future value. The compound or geometric return differs from the simple arithmetic average, and we will discuss the difference between the two in detail in Chapter 8. Notice the impressive power of compounding over such a long period: $1,000 invested in Canadian equities would have made you almost a millionaire by 2008, and invested in U.S. equities, a millionaire and then some.

The dramatic difference in ending values results from differences in the rate of return. At a rate of return of 5.05 percent (i.e., the T-bill return), $1,000 would have grown to $33,135, while at 10.91 percent (i.e., the U.S. stock return) it would have grown to $1,559,986—more than 47 times as much! The difference in ending values is significant, even when the differences in returns are small. For example, consider the difference in ending values of over $700,000 when $1,000 is invested in Canadian stocks at 9.90 percent versus U.S. stocks at 10.91 percent. These data show why finance professionals struggle to increase the returns on their investments even by very small amounts. In fact, it is normal to look at returns down to 1/100 of 1 percent, which is called a **basis point**. Earning just a few basis points more on one investment causes the future value of the portfolio to compound that much faster. Unfortunately, this search for additional returns often leads investors to underestimate the associated risks, which can backfire, as discussed in Lessons to Be Learned below.

basis point 1/100 of 1 percent

LESSONS TO BE LEARNED

During 2007 and 2008, the yields on government bonds were very low by historical standards—in the 4 to 4.5 percent range. This led many investors—both professionals and individuals—to look for similar investments that would provide them with higher returns (i.e., a few extra basis points). As a result, many

(*continued*)

LESSONS TO BE LEARNED (*continued*)

investors bought corporate debt securities (i.e., corporate bonds) or highly rated asset-backed securities (ABS) and collateralized debt obligations (CDOs), many of which were AAA-rated.[3] Unfortunately, many investors underestimated the risks associated with these investments (as did the debt rating agencies), and instead of earning higher returns, they ended up generating huge losses. The lesson to take from this is that while extra returns do indeed make a difference in the long run, investors should understand the risks underlying their investments.

Let's return to our two choices of $20,000 today or $31,000 in five years' time, assuming the investor now earns *compound* interest of 10 percent. Using the CVIF formula, we get CVIF = $(1 + k)^5 = (1.1)^5 = 1.61051$. The $20,000 compounds to $32,210. So the choice is now $20,000 today, because it will be grow to more than $31,000 in five years if invested at 10 percent with compound interest. However, we still have a problem comparing either of these single sums with the option of getting $3,000 a year forever. We could compound each of the $3,000 annual payments forward to, say, 100 years in the future, but there are easier ways of doing this, as we will discuss in section 5.4.

Discounting (Computing Present Values)

So far we have been concerned with finding future values, but there is a problem with comparing future values: there are many of them! We could choose an arbitrary common period to make the comparisons, which solves this problem. The obvious choice is to compare the values at the *current* time, so instead of calculating future values, we determine *present values*. This process is also called **discounting**. We will explain it with a simple example.

discounting finding the present value of a future value by accounting for the time value of money

EXAMPLE 5-5

Discounting

An investor estimates that she needs $1 million to live comfortably when she retires in 40 years. How much does she have to invest today, assuming a 10 percent interest rate on the investment?

Solution

To solve this example, first start with what is already known: the future value formula of Equation 5-2.

$$FV_n = PV_0(1 + k)^n$$

where CVIF = $(1 + k)^n$. With a starting present value, we multiply by the CVIF to get the future value. This means we can divide the future value by the CVIF to get the present value. Rearranging Equation 5-2 to solve for PV we get

[5-3]
$$PV_0 = \frac{FV_n}{(1 + k)^n} = FV_n \times \frac{1}{(1 + k)^n}$$

(*continued*)

[3] We will discuss debt ratings in greater detail in Chapter 6, but AAA is the highest rating, which means these instruments were considered extremely high quality—similar to government bonds.

EXAMPLE 5-5 *continued*

Equation 5-3 is the basic discounting equation, and the last term, $1/(1+k)^n$, is called the discount factor or **present value interest factor (PVIF)**. Some older textbooks have tables of PVIF and CVIF for various periods and interest rates, although they are simply reciprocals of each other, but the use of computers and calculators makes these tables obsolete.

Let's return to our example. If FV = 1,000,000; $k = 0.1$; and $n = 40$, we get

$$PV = 1{,}000{,}000 \times 1/(1.1)^{40} = 1{,}000{,}000 \times (1/45.259256)$$
$$= 1{,}000{,}000 \times 0.02209493 = \$22{,}094.93$$

An investment of $22,094.93 today, earning a 10 percent return per year, has a future value of $1 million in 40 years. With a 10 percent market interest rate, $22,094.93 today and $1 million in 40 years' time are worth the same amount, so the two figures are economically equivalent.

present value interest factor (PVIF) a formula that determines the present value of $1 to be received at some time in the future, *n*, based on a given interest rate, *k*

Now you know why we call this process "discounting." If people don't want to pay the full price for something, they ask for a discount or, in other words, they ask for something off the price. In the same way, $1 million in 40 years is not worth $1 million today, so you discount, or take something off, to get it to its true value. Discounting future values to find their present value is the same process, except that when we know the market interest rate, we can use Equation 5-3 to calculate the exact true value.

Notice the following important points from Example 5-5 and Equation 5-3:

- Discount factors (the PVIF) are always less than one (as long as $k > 0$). This means that future dollars are worth less than the same dollars today.
- Discount factors are the reciprocals of their corresponding compound factors and vice versa (PVIF = 1/CVIF).

We can also solve Example 5-5 using a financial calculator or Excel, as shown below.

solution using a financial calculator

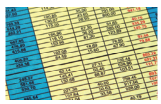
(TI BA II Plus)

Input the following variables:

0 → PMT; −1,000,000 → FV; 10 → I/Y; and 40 → N

Press CPT (Compute) and then PV. This will give an answer of 22,094.93.

solution using Excel



The following function may be used in Excel:
= PV (rate, nper, pmt, fv, type)
For this example, we would enter the following in the appropriate cell:
= PV (0.1, 40, 0, −1000000, 0)
This would yield a PV of 22,094.93.

Determining Rates of Return or Holding Periods

When we looked at the discounting problem, we noted that we simply divided through by the CVIF but essentially used the same future value equation. Let's look at this again.

$$FV_n = PV_0(1 + k)^n$$

We have used this equation to solve for future values (FV) and present values (PV), but notice that we can solve for two other values: the interest rate (k) and the period (n). If both the present and future values are known, and we know either the interest rate or the period, we can solve for the last unknown. We illustrate this in Examples 5-6 and 5-7.

EXAMPLE 5-6

Finding the Rate of Return

Suppose we modify the lottery example used earlier. The "prize" is now a $20,000 investment that has a payoff of $31,000 in five years. We have the present and future values and the period, so we can solve for the interest rate. This is an important interest rate, called the internal rate of return (IRR), because it is the rate of return that is *internal* to the values in the problem. Many problems in finance are IRR problems for which we need to compare the rates of return earned on different investments.

Solution

$$FV = 31{,}000;\ PV = 20{,}000;\ n = 5$$

Using Equation 5-2, we get $20{,}000 = 31{,}000/(1 + k)^5$.
We could solve for k in the following manner:

$$31{,}000/20{,}000 = 1.55 = (1 + k)^5$$

This is a simple problem, but it is still awkward to solve. One way to solve it is through trial and error, using a calculator. Put, say, 1.08 into memory and then enter 1 and press "multiply, memory recall, equals" five times. The result is 1.469. Doing the same thing with 10 percent (1.1) produces 1.6105, so we know that the internal rate of return is in between these two numbers and closer to 10 percent than to 8 percent. All we can then do is iterate to get closer and closer to 1.55. Eventually we would end up with 9.161 percent.[4] However, this is a laborious and inefficient process. Using a financial calculator or Excel makes these types of calculations much simpler, as we illustrate below.

solution using a financial calculator

(TI BA II Plus)

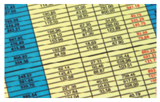
Input the following variables:

0 → PMT; 31,000 → FV; −20,000 → PV; and 5 → N

Press CPT (Compute) and then I/Y. This will give an answer of 9.161 percent. Notice that either FV or PV needs to be input as a negative number, because to "receive" one cash flow (either today or in the future) you need to "pay" (or invest) either today or in the future.

[4] If you know how to use logarithms, we can take the log of both sides and solve for $Ln(1 + k) = Ln(FV/PV)/n$.

solution using Excel



The following function may be used:
= RATE (nper, pmt, pv, fv, type)
For this example, we would enter the following in the appropriate cell:
= RATE (5, 0, 20000, −31000, 0)
This also yields the correct answer of 9.161 percent.

EXAMPLE 5-7

Solving for Time or "Holding" Periods

In this example, we use the same data as before but change the problem to ask how long we have to invest $20,000 at 10 percent to get $31,000.

Solution

With our data, we now have to solve the following for n:

$$31{,}000/20{,}000 = 1.55 = (1.1)^n$$

Unfortunately, solving this equation with a simple calculator is even more complicated than the IRR problem was. We could put 1.1 into memory, enter 1, and again press "multiply, memory recall, equals" five times to find that the period is between four and five years but closer to five. However, we can't be more accurate than this. If you are familiar with logarithms, you can take logs of both sides and solve for n as follows:

$$n = \frac{Ln(\text{FV}/\text{PV})}{Ln(1 + k)} = \frac{Ln(1.55)}{Ln(1.1)}$$

The natural log of 1.55 is 0.438255 and the natural log of 1.1 is 0.09531, so the answer is 4.6 years. However, logarithm tables are rarely used; we used the logarithm function in Excel and typed in = $ln(1.55)$. Using a financial calculator or Excel makes the calculations much easier.

solution using a financial calculator

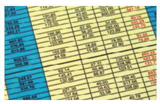
(TI BA II Plus)

Input the following variables:

0 → PMT; 31,000 → FV; −20,000 → PV; and 10 → I/Y

Press CPT (Compute) and then N. This will give an answer of 4.5982 or 4.6 years.

solution using Excel



The following function may be used:
= NPER (rate, pmt, pv, fv, type)
For this example, we would enter the following in the appropriate cell:
= NPER (0.1, 0, −20000, 31000, 0)
This would yield an answer of 4.5982 or 4.6 years.

We'll summarize what we have learned so far. We have Equation 5-2:

$$FV_n = PV_0(1 + k)^n$$

This equation has four values. If we know any three of them, we can solve for the last one. Therefore, four different types of finance problems can be solved:

- *Future value problems*: How much will I have in *w* years at *x* percent if I invest $*y* today?
- *Present value problems*: What is the value today of receiving $*z* in *w* years if the interest rate is *x* percent?
- *IRR problems*: What rate of return will I earn if I invest $*y* today for *w* years and get $*z*?
- *Period problems*: How long do I have to wait to get $*z* if I invest $*y* today at *x* percent?

All the problems that we have looked at are single-sum problems, looking at a single investment today and a single payoff in the future. In principle, we can solve almost any problem using the techniques we have discussed because, for example, valuing a series of receipts in the future can be done by valuing each one individually. However, special formulas exist for standard problems in finance for which the receipts or payments are the same each period. Think back to our first example about choosing between $20,000 today, $31,000 in five years, or $3,000 each year forever. The last choice involves valuing a constant $3,000 each year forever and is an example of an "annuity." In this case, it is a special class of annuity: a perpetual annuity, commonly referred to as a "perpetuity."

CONCEPT REVIEW QUESTIONS

1. Explain how to compute future values and present values when using compound interest.
2. What is the relationship between CVIFs and PVIFs? Why does this make sense?
3. Why does compound interest result in higher future values than simple interest?

5.4 ANNUITIES AND PERPETUITIES

The Importance of Investing Early

Learning Objective 5.4
Differentiate between an ordinary annuity and an annuity due, and explain how special constant payment problems can be valued as annuities and, in special cases, as perpetuities.

The example below illustrates the power of compound interest as time passes. Consider twins who follow two different investing approaches. Assume each earns a 12 percent annual return.

- Twin 1: At age 21, she begins investing $2,000 per year (at year end) for six years, and then she makes no further contributions (total amount invested is $12,000). Note that she invests the same amount each year, so this is an example of an annuity. At age 65, she will have accumulated $1.2 million for retirement.
- Twin 2: She postpones investing for six years, until she reaches age 28, and then invests $2,000 per year for 38 years (total amount invested is $76,000). At age 65, she will also have accumulated $1.2 million for retirement.

Notice that they have the same ending amounts, but Twin 1 invested only $12,000 in total, while Twin 2 invested $76,000. This shows how the compounding effect is magnified as the time horizon increases.

We could solve these problems to find the ending value using "brute force," finding the future value of each payment the twins make individually. However, we will solve them after we develop more formally the concept of an annuity.

Ordinary Annuities

So far, we have dealt with PV and FV concepts as they apply to only two cash flows—one today (i.e., the PV), and one in the future (i.e., the FV). In practice, we will often need to compare different series of receipts or payments that occur through time. An **annuity** is a series of payments or receipts, which we will simply call **cash flows**, that are for the same amount and paid at the same interval—that is, for example, they are paid annually, monthly, or weekly—over a given period. Annuities are common in finance; the one you may be familiar with is a loan or mortgage payment. This involves an identical payment made at regular intervals for a loan based on a single interest rate.

annuity regular payments on an investment that are for the same amount and are paid at the same interval

cash flows the actual cash generated from an investment

Ordinary annuities involve end-of-period payments. We have the same values as in our earlier discussion: FV, PV, *n*, and *k*. However, now we have another term, PMT, for the regular annuity payment or receipt. Example 5-8 demonstrates how to determine the FV and PV of an ordinary annuity.

ordinary annuities equal payments that are made at the end of each period

FV and PV of an Ordinary Annuity

EXAMPLE 5-8

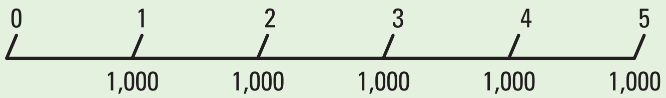
Suppose someone plans to invest $1,000 at the *end* of each year for the next five years and expects to earn 13 percent per year.

a. How much will the investor have after five years?

b. How much would the investor need to deposit today to have the same results?

Solution

a. We can first depict the series of payments on a timeline diagram, which shows when the cash flows occur.



Timelines are very useful in finance. You should get into the habit of displaying the data in a problem in a timeline. For example, from this diagram we can see that by the end of year five, the first deposit of $1,000 will have earned a return for four years, because there are four years from the end of year 1 to the end of the problem in year 5. In contrast, the second payment will earn a return for only three years, the third for only two, the fourth for one year, and the final payment will not earn a return at all. Using this information, we could view this as a five-part problem in which we have to find the future value of each of the five payments.

$FV_5 = 1{,}000(1.13)^4 + 1{,}000(1.13)^3 + 1{,}000(1.13)^2 + 1{,}000(1.13)^1 + 1{,}000$

$= 1{,}000(1.63047) + 1{,}000(1.44290) + 1{,}000(1.27690) + 1{,}000(1.13) + 1{,}000(1)$

$= 1{,}000(6.48027) = \$6{,}480.27$

(*continued*)

EXAMPLE 5-8 *continued*

The investor would have $6,480.27 after five years.

This would be our brute-force calculation, in which we solved the problem with five separate calculations. This approach is fine for a five-period annuity problem, but it would be tedious for a 25-year monthly mortgage problem that has 300 payments. Fortunately, there is a much quicker way (even without the use of a financial calculator or a spreadsheet). If we look closely at our solution, we can see that we are multiplying $1,000 by the sum of five compound value interest factors (CVIF), based on a 13 percent return (i.e., the CVIF for $k = 13$ percent, with $n = 4, 3, 2, 1$, and 0, respectively).

The ordinary annuity equation, Equation 5-4, adds these CVIFs.

[5-4]
$$FV_n = PMT\left[\frac{(1+k)^n - 1}{k}\right]$$

PMT is the end-of-period annuity payment. This formula is usually called the compound value annuity formula or CVAF to distinguish it from the single-sum CVIF. The advantage of Equation 5-4 is that it involves only one formula and can be solved easily, even on a simple calculator.

Using this equation, we can solve Example 5-8a as follows:

$$FV_5 = PMT\left[\frac{(1+0.13)^5 - 1}{0.13}\right] = 1{,}000(6.48027) = \$6{,}480.27$$

We can get the CVIF for five years at 13 percent by using a simple calculator, putting 1.13 into memory, entering 1, and then pressing "times, memory recall, equals" five times, or by using Excel and entering =1.13^5.

(*continued*)

solution using a financial calculator

(TI BA II Plus)

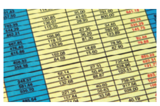
Input the following variables:

1,000 → PMT ; 5 → N ; 0 → PV (i.e., no deposit today); and

13 → I/Y

Press CPT (Compute) and then FV. This will give you an answer of −6,480.27. Remember, you get a negative value because the calculator is programmed to consider cash outflows and cash inflows.

solution using Excel



The following Excel function may be used:
= FV (rate, nper, pmt, pv, type)
For this example, we would enter the following in the appropriate cell:
= FV (0.13, 5, −1000, 0, 0)
This would yield an answer of $6,480.27.

EXAMPLE 5-8 *continued*

b. To find the present value, we can again view this as a five-part problem for which we have to find the present value of each of the five annual payments:

$$PV_0 = 1{,}000(1/1.13)^5 + 1{,}000(1/1.13)^4 + 1{,}000(1/1.13)^3 + 1{,}000(1/1.13)^2 + 1{,}000(1/1.13)$$

$$= 1{,}000(0.54276) + 1{,}000(0.61332) + 1{,}000(0.69305) + 1{,}000(0.78315) + 1{,}000(0.88496)$$

$$= 1{,}000(3.51724) = \$3{,}517.24$$

As before, notice that we are using brute force by multiplying \$1,000 by the sum of the relevant five PVIF or discount factors, which add to 3.51724. Fortunately, the formula for determining the PV of ordinary annuities, Equation 5-5, will do this for us.

$$PV_0 = PMT\left[\frac{1 - \frac{1}{(1+k)^n}}{k}\right] \quad [5\text{-}5]$$

This formula is usually called the present value annuity formula or PVAF to distinguish it from the PVIF used for valuing single-sum problems.

By using this equation to solve Example 5-8b, we get

$$PV_0 = \$1{,}000\left[\frac{1 - \frac{1}{(1.13)^5}}{0.13}\right] = 1{,}000(3.51723) = \$3{,}517.23$$

The difference of \$0.01 is due to rounding errors in the long approach. Again, we can get the PVAF for 5 years at 13 percent by using a simple calculator with memory. For longer-period problems, we can use a financial calculator or Excel.

solution using a financial calculator

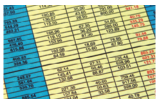
(TI BA II Plus)

Input the following variables:

1,000 → PMT ; 5 → N ; 0 → FV ; and 13 → I/Y

Press CPT (Compute) and then PV. This will give an answer of −3,517.23.

solution using Excel



The following Excel function may be used:
= PV (rate, nper, pmt, fv, type)
For this example, we would enter the following in the appropriate cell:
= PV (0.13, 5, −1000, 0, 0)
This would yield an answer of \$3,517.23.
We can ensure that our answer is correct by checking that \$3,517.23 is the present value of the future value at year 5, calculated earlier, of \$6,480.27 with a 13 percent interest rate. We leave this as an exercise for you.

lessee a person who leases an item

annuity due an annuity (such as a lease) for which the payments are made at the beginning of each period

Annuities Due

Sometimes annuities are structured so that the cash flows are paid at the *beginning* of a period, rather than at the end. For example, leasing arrangements are usually set up like this, with the **lessee** making an immediate payment on taking possession of the equipment, such as a car. Such an annuity is called an **annuity due**. Example 5-9 demonstrates how to evaluate these cash flows.

EXAMPLE 5-9

FV and PV of an Annuity Due

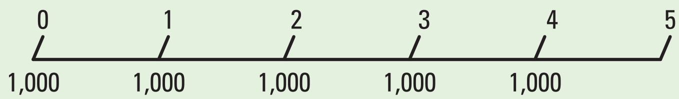
We will repeat Example 5-8, except we assume that the payments are made at the *beginning* rather than the end of each year.

a. How much will the investor have after five years?

b. How much would the investor have to deposit today to have the same results?

Solution

a. We begin as before by depicting the data on a timeline.



Notice that, as in Example 5-8, we have five cash flows of $1,000 each. However, each cash flow appears one period earlier, and thus each receives an extra period of interest at the rate of 13 percent. Using the brute-force approach applied in Example 5-8a, we can find the future value of each of the five payments.

$$FV_5 = 1{,}000(1.13)^5 + 1{,}000(1.13)^4 + 1{,}000(1.13)^3 + 1{,}000(1.13)^2 + 1{,}000(1.13)^1$$

$$= 1{,}000(1.84244) + 1{,}000(1.63047) + 1{,}000(1.44290) + 1{,}000(1.27690) + 1{,}000(1.13)$$

$$= [1{,}000(6.48027)](1.13) = \$7{,}322.71$$

Notice that because each flow gets one extra period of compounding at 13 percent, the net result is that we multiply our answer to Example 5-8a by 1.13. In other words, the FV (annuity due) = [FV (ordinary annuity)]$(1 + k)$.

Therefore, we can alter Equation 5-4 to find the FV of an annuity due as follows:

[5-6]

$$FV_n = PMT\left[\frac{(1+k)^n - 1}{k}\right](1+k)$$

This is CVAF$(1 + k)$, so in practice we don't use a separate formula. However, we can now solve Example 5-9a as follows:

$$FV_5 = PMT\left[\frac{(1+.13)^5 - 1}{0.13}\right](1.13) = [1{,}000(6.48027)](1.13) = \$7{,}322.71$$

Note that the value of the annuity due, $7,322.71, is 1.13 times higher than the value of the ordinary annuity that we calculated earlier, which was $6,480.27.

(continued)

There is a "Begin" mode on the TI BA II Plus, as there is on most financial calculators. We need to activate Begin mode before solving this problem, which can be done as follows for the TI BA II Plus:

Press [2nd] [BGN] [2nd] [Set]. Then input the variables:

[1,000] → [PMT] ; 5 → [N] ; 0 → [PV] ; and 13 → [I/Y]

Press [CPT] (Compute) and then [FV]. This gives an answer of −7,322.71. As before, recognize that the negative sign is due to the need for a series of cash inflows matched by a cash outflow.

solution using a financial calculator

(TI BA II Plus)

The following Excel function may be used:
= FV (rate, nper, pmt, pv, type)
Now, we can see what the "type" stands for in the Excel formula. When type is set = 0, as in our previous examples, it refers to an ordinary annuity; when it is set = 1, it refers to an annuity due. So for this example, we would enter the following in the appropriate cell:
= FV (0.13, 5, −1000, 0, 1)
This yields an answer of $7,322.71.

solution using Excel

EXAMPLE 5-9 *continued*

b. As before, to solve for the present value, we could view this as a five-part problem for which we have to find the present value of each of the five payments.

$$PV_0 = 1{,}000(1/1.13)^4 + 1{,}000(1/1.13)^3 + 1{,}000(1/1.13)^2 + 1{,}000(1/1.13)^1 + 1{,}000$$

$$= [1{,}000(3.51724)](1.13) = \$3{,}974.48$$

Note that, as in Example 5-9a, we are multiplying our answer to Example 5-8b by 1.13—that is $(1 + k)$. Accordingly, we can modify Equation 5-5 to arrive at the formula for determining the PV of an annuity due, which is given below:

$$PV_0 = PMT\left[\frac{1 - \frac{1}{(1+k)^n}}{k}\right](1 + k) \qquad [5\text{-}7]$$

Equation 5-7 is PVAF$(1 + k)$. Using this to solve Example 5-9b, we get

$$PV_0 = \$1{,}000\left[\frac{1 - \frac{1}{(1.13)^5}}{0.13}\right](1.13) = [1{,}000(3.51723)](1.13) = \$3{,}974.47$$

Again, the difference of $0.01 is due to rounding errors in the long approach.

solution using a financial calculator

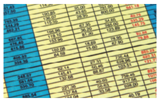
(TI BA II Plus)

First, activate the Begin BGN function. Then input the following variables:

1,000 → PMT ; 5 → N ; 0 → FV ; and 13 → I/Y

Press CPT (Compute) and then PV. This gives an answer of −3,974.47.

solution using Excel



The following Excel function may be used:
= PV (rate, nper, pmt, fv, type)
For this example, we would enter the following in the appropriate cell:
= PV (0.13, 5, −1000, 0, 1)
This gives an answer of $3,974.47.

Perpetuities

perpetuities special annuities that provide payments forever

Perpetuities are special annuities in that they go on forever, so n goes to infinity in the annuity equation. In this case, Equation 5-5 reduces to

[5-8]

$$PV_0 = \frac{PMT}{k}$$

Perpetuities are easy to value because all we do is divide the cash payment or receipt by the interest rate.

EXAMPLE 5-10

Annuities and Perpetuities

a. An annuity pays $3,000 per year at year end and earns an annual return of 12 percent per year for 30 years. What is the present value?

b. What is the PV of a $3,000 per year annuity that goes on forever—that is, in perpetuity—if $k = 12$ percent?

Solution

a. $$PV_0 = \$3{,}000\left[\frac{1 - \frac{1}{(1.12)^{30}}}{0.12}\right] = 3{,}000(8.05518) = \$24{,}165.55$$

(continued)

solution using a financial calculator

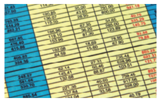
(TI BA II Plus)

Input the following variables:

3,000 → PMT ; 30 → N ; 0 → FV ; and 12 → I/Y

Press CPT (Compute) and then PV. This will give an answer of −24,165.55.

solution using Excel



The following Excel function may be used:
= PV (rate, nper, pmt, fv, type)
For this example, we would enter the following in the appropriate cell:
= PV (0.12, 30, −3000, 0, 0)
This would yield an answer of $24,165.55.

EXAMPLE 5-10 *continued*

b. $PV_0 = \frac{\$3{,}000}{0.12} = \$25{,}000$

Notice the small difference in the present value of these cash streams. This tells you that the PV of the cash flows of $3,000 per year from years 31 to infinity (∞) is only $834.45—that is, $25,000 − $24,165.55. This is a very important result and is behind many financial innovations. It means that cash flows far in the future are of little value because of the discounting involved in the time value of money.[5]

Annuities and Perpetuities Summarized

So far, we have discussed constant annuities and perpetuities—i.e., where the cash flows remain the same for every period. However, sometimes we want to evaluate growing (or shrinking) perpetuities or annuities—i.e., where the cash flows grow (or shrink) at a certain rate every period. We will consider situations where these assumptions may be appropriate in Chapter 7 (for valuing stocks) and in chapters 13 to 15 (for evaluating long-term investment and merger and acquisition decisions). Appendix 5A describes how to evaluate growing annuities and perpetuities.

We conclude by deriving the solution to the investing-early scenario described at the beginning of this section.

Investing Early

EXAMPLE 5-11

Solve for Twin 1 from the scenario on page 000.

Solution

This must be solved as a two-part problem, which can then be solved in several ways. We first estimate the future value of the six $2,000 payments at the end of six years.

$$FV_6 = PMT\left[\frac{(1+k)^n - 1}{k}\right] = (2{,}000)\left[\frac{(1.12)^6 - 1}{0.12}\right] = (2{,}000)(8.11519) = \$16{,}230.38$$

Then we estimate the future value of the accumulated savings after 38 years (i.e., from age 27 to age 65).

$$FV_{38} = PV_0(1+k)^n = (16{,}230.38)(1.12)^{38} = (16{,}230.38)(74.17966) = \$1{,}203{,}964.13$$

(*continued*)

[5] This has also been behind many misleading advertisements for which something is 100 percent backed by a government bond. The small print indicates that the bond pays off in 25 years so is not worth much today!

EXAMPLE 5-11 *continued*

Solve for Twin 2 from the scenario above.

$$FV_{38} = PMT\left[\frac{(1+k)^n - 1}{k}\right] = (2{,}000)\left[\frac{(1.12)^{38} - 1}{0.12}\right]$$

$$= (2{,}000)(609.83053) = \$1{,}219{,}661.07$$

So they both end up with approximately $1.2 million.

CONCEPT REVIEW QUESTIONS

1. Explain how to calculate the present value and future value of an ordinary annuity and an annuity due.
2. Define "perpetuity."
3. Why is the present value of $1 million in 50 years' time worth very little today?

5.5 NOMINAL VERSUS EFFECTIVE RATES

Learning Objective 5.5
Differentiate between quoted rates and effective rates, and explain how quoted rates can be converted to effective rates.

Determining Effective Annual Rates

So far, we have assumed that payments are made annually and that interest is compounded annually, so we have been able to use quoted rates to solve each problem. In practice, in many situations, payments are made (or received) at intervals other than annually (e.g., quarterly, monthly), and compounding often occurs more frequently than annually, so we need to be sure that we use the appropriate effective interest rate.

effective rate the rate at which a dollar invested grows over a given period; usually stated in percentage terms based on an annual period

The **effective rate** for a period is the rate at which a dollar invested grows over that period. It is usually stated in percentage terms based on an annual period. To determine effective rates, we first recognize that the annual rates quoted by financial institutions will equal the annual effective rate only when compounding is done on an annual basis. We will use some examples to illustrate the process for determining effective rates.

EXAMPLE 5-12

Effective versus Quoted Rates

a. Suppose someone invests $1,000 today for one year at a quoted annual rate of 16 percent compounded annually. What is the FV at the end of the year?

b. What if someone invests $1,000 at a quoted rate of 16 percent compounded quarterly?

Solution

a. $FV = 1{,}000(1.16)^1 = \$1{,}160$.

This means that each $1 grows to $1.16 by the end of the period, so we can say that the "effective" annual interest rate is 16 percent.

(continued)

EXAMPLE 5-12 *continued*

b. When the rate is "quoted" at 16 percent, and compounding is done quarterly, the appropriate adjustment (by convention) is to charge 16 percent/4 = 4 percent per quarter, so we have

$$FV = 1{,}000(1.04)^4 = \$1{,}170 \text{ (rounded)}$$

Notice that even though the quoted rate is 16 percent, each dollar invested grows to $1.17—that is, by 17 percent—by the end of the period. In this case, we say that the "effective" annual interest rate is 17 percent.

We can use the following equation to determine the effective annual rate for any given compounding interval:

$$k = \left(1 + \frac{QR}{m}\right)^m - 1 \qquad [5\text{-}9]$$

where k = effective annual rate, QR = quoted rate, and m = the number of compounding intervals per year.

Applying this equation to Example 5-12, we see the following:

For 5-12a, $m = 1$, QR = 0.16, and we get

$$k = \left(1 + \frac{0.16}{1}\right)^1 - 1 = 0.16 = 16\%$$

so the quoted and effective rates are the same.

For 5-12b, $m = 4$, QR = 0.16, so we get

$$k = \left(1 + \frac{0.16}{4}\right)^4 - 1 = 0.1700 = 17\%$$

The effective rate is higher than the quoted rate. This is why it is important to examine the compounding frequency of investments and loans; looking at the rate alone is often not enough.

When compounding is conducted on a continuous basis, we use Equation 5-10 to determine the effective annual rate for a given quoted rate.

$$k = e^{QR} - 1 \qquad [5\text{-}10]$$

where e is the unique Euler number (approximately 2.718) and is found on your calculator and in Excel. It is used frequently in finance. If we use the Excel function "enter = exp(.16)," and subtract 1, we get 17.351 percent.

Example 5-13 shows that as the frequency of compounding increases, the effective annual rate also increases.

Effective Annual Rates for Various Compounding Intervals

EXAMPLE 5-13

What are the effective annual rates for the following quoted rates?

a. 12 percent, compounded annually

b. 12 percent, compounded semi-annually

(*continued*)

EXAMPLE 5-13 *continued*

c. 12 percent, compounded quarterly

d. 12 percent, compounded monthly

e. 12 percent, compounded daily

f. 12 percent, compounded continuously

Solution

a. Annual compounding, $m = 1$: $k = \left(1 + \frac{0.12}{1}\right)^{1} - 1 = 12\%$

b. Semi-annual compounding, $m = 2$: $k = \left(1 + \frac{0.12}{2}\right)^{2} - 1 = 12.36\%$

c. Quarterly compounding, $m = 4$: $k = \left(1 + \frac{0.12}{4}\right)^{4} - 1 = 12.55\%$

d. Monthly compounding, $m = 12$: $k = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 12.68\%$

e. Daily compounding, $m = 365$: $k = \left(1 + \frac{0.12}{365}\right)^{365} - 1 = 12.747\%$

f. Continuous compounding: $k = e^{0.12} - 1 = 12.75\%$

Example 5-13 shows that as the compounding frequency increases, the quoted rate of 12 percent increases to a maximum effective rate of 12.75 percent, achieved with instantaneous or continuous compounding. However, the daily rate is almost the same, at 12.747 percent.

You can, of course, solve effective interest rate problems by using a calculator or Excel.

solution using a financial calculator

(TI BA II Plus)

Perform the following keystrokes:

2ND ICONV

2ND CLR WORK

The screen will show NOM = some value
Make NOM = 12 (this is the nominal rate)
Then

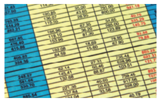
ENTER ↓ ↓

This should show C/Y = some value
For daily compounding, for example, input
C/Y = 365 (this is the number of compounding periods per year)

Then ENTER ↓ ↓ , which should show EFF = some value.

Then press CPT , which gives an answer of 12.747 percent.

solution using Excel



Excel has a special function for solving for effective rates: the Effect(nominal, npery) function, for which nominal is the nominal interest rate and npery is the compounding frequency.[6] For any of the previous periods we would enter

= effect(0.12, n)

with n = 1, 2, 4, 12, 365, and a very large number to approximate $n = \infty$, and we would get the answers above.

Effective Rates for "Any" Period

In Example 5-12b, the effective quarterly rate is 4 percent, because each dollar grows to $1.04 by the end of one quarter. Similarly, in Example 5-13 (which uses a quoted rate of 12 percent), the effective semi-annual rate for 5-13b is 6 percent (i.e., 12 percent/2), the effective quarterly rate for 5-13c is 3 percent (i.e., 12 percent/4), and the effective monthly rate for 5-13d is 1 percent (i.e., 12 percent/12).

However, suppose we need to know the effective monthly rate associated with the annual effective rate of 12.36 percent from Example 5-13b, perhaps to make monthly payments on a loan. It is not appropriate to divide 12.36 percent by 12, because it is an effective rate, not a quoted rate. Remember, we are looking for the effective monthly rate (i.e., how much $1 would grow over a given month), based on an annual effective rate of 12.36 percent. In this case, we know that after 12 compounding intervals at a monthly effective rate of $k_{monthly}$, each $1 would have grown to $1.1236. In other words, we have

$$(1 + k_{monthly})^{12} = 1.1236$$

We could solve this equation by taking the 12th root of each side:

$$(1 + k_{monthly}) = (1.1236)^{1/12}$$

So $k_{monthly} = (1.1236)^{1/12} - 1 = 0.0097588$ or 0.97588 percent.

We can verify this by compounding $1 at the rate of 0.97588 percent per month for 12 months as follows: $(1.0097588)^{12} = 1.1236$. In other words, investing $1 at 0.97588 percent per month produces the same amount at the end of one year (1.1236) as does investing $1 for one year with semi-annual compounding at 6 percent per six-month period.

The following equation, which is a variation of Equation 5-9, can be used to determine the effective rate for any period, given any quoted rates.

$$k = \left(1 + \frac{QR}{m}\right)^{\frac{m}{f}} - 1 \qquad [5\text{-}11]$$

where f = frequency of payments per year (i.e., $f = 1$ when we are looking for an annual effective rate, $f = 12$ when looking for a monthly effective rate, etc.). Notice that when $f = 1$, we have Equation 5-9.

There is no specific function to solve for "other than annual" effective rates by using the TI BA II Plus calculator. We could find the effective annual rate for a 12 percent

[6] You may need to add the Analysis ToolPak in Excel, which can be done by clicking on Tools, then Add-Ins, then Analysis ToolPak, and then OK.

nominal rate with quarterly compounding (as demonstrated in the previous example), and then do the following:

$$(1 + k_{monthly})^{12} = 1.1236$$

$$\text{So } (1 + k_{monthly}) = (1.1236)^{1/12}, \text{ and}$$

$$k_{monthly} = (1.1236)^{1/12} - 1 = 0.0097588 \text{ or } 0.97588\%$$

In Excel, we can go back to the rate function that we used previously. For example, if the annual rate is 12.36 percent and we want to know the effective monthly rate, we can use the following function,

=Rate (nper, pmt, pv, fv, type),

entering the following in the appropriate cell,

=Rate (12, 0, −1, 1.1236, 0),

where there are 12 periods and no intervening payments, and we are interested in a $1 outflow growing to 1.1236. Because we are not interested in annuities, we put in 0 for type. This produces the same answer: 0.975879 percent.

CONCEPT REVIEW QUESTIONS

1. Why can effective rates often be very different from quoted rates?
2. Explain how to calculate the effective rate for any period.

5.6 LOAN OR MORTGAGE ARRANGEMENTS

Learning Objective 5.6
Apply annuity formulas to value loans and mortgages and set up an amortization schedule.

mortgage a loan, usually secured by real property, that involves "blended" equal payments (both interest and a principal repayment) over a specified payment period

amortize to retire a loan over a given period by making regular payments

One common and important application of annuity concepts is in the form of loan or **mortgage** arrangements. Typically, these arrangements involve "blended" payments for equal amounts that include both an interest and a principal repayment component. The loan payments are designed to **amortize** the loan, which means that, at the end of the loan term, the balance due (or principal outstanding) will equal zero—in other words, the loan and all associated interest obligations will have been paid off in their entirety. Note that both "amortize" and "mortgage" contain the French word *mort*, which means "death." So a mortgage is "killed off" over the mortgage period, and no money is owed at the end. Similarly, amortize means to "kill off" financially.

Because these loans involve equal payments at regular intervals, based on one fixed interest rate specified when the loan is taken out, the payments can be viewed as annuities. Therefore, we can determine the amount of the payment, the effective period interest rate, and so on, by using Equation 5-5 and recognizing that the PV equals the amount of the loan.

An amortization schedule divides the blended payments into the interest portion and the principal repayment portion. This is of importance to businesses, where the interest portion is a deductible expense for tax purposes. The interest portion is determined by applying the effective period interest rate to the principal outstanding at the beginning of each period. The remaining portion of the payment is then used to reduce the amount of principal outstanding. The example below shows the development of an amortization schedule.

Loan Payments and Amortization Schedule

EXAMPLE 5-14

Determine the required year-end payments for a three-year $5,000 loan with a 10 percent annual interest rate. Complete an amortization schedule.

Solution

First, determine the required payments by solving Equation 5-5 for PMT.

$$PMT = \frac{PV_0}{\left[\dfrac{1 - \dfrac{1}{(1+k)^n}}{k}\right]} = \frac{5{,}000}{\left[\dfrac{1 - \dfrac{1}{(1.10)^3}}{0.10}\right]} = \frac{5{,}000}{2.48685} = \$2{,}010.57$$

solution using a financial calculator

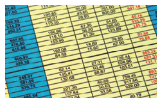
(TI BA II Plus)

Input the following variables:

FV = 0; PV = 5,000; N = 3; I/Y = 10,

Compute PMT = −2,010.57, or $2,010.57.

solution using Excel



Use the Excel payment function (PMT):
=PMT(rate, nper, pv, fv, type)
which for our example gives =PMT(0.1, 3, 5000, 0, 0), and the same answer of $2,010.57.

EXAMPLE 5-14 *continued*

Second, determine the loan amortization schedule:

Period	(1) Beginning Principal Outstanding	(2) PMT	(3) Interest [(1)*k]	(4) Principal Repayment [(2) − (3)]	End Principal Outstanding [(1) − (4)]
1	5,000.00	2,010.57	500.00	1,510.57	3,489.43
2	3,489.43	2,010.57	348.94	1,661.63	1,827.80
3	1,827.80	2,010.57	182.77	1,827.80	0.00

You need to understand how this amortization table is created. The loan is a simple annual payment loan, so the cost of the loan is the annual interest rate times the outstanding balance, in this case 10 percent times $5,000, or $500. This is the first charge on the loan payments; the residual, which is $1,510.57 in the first year, goes to reduce the amount of the loan.

FIGURE 5-2

Mortgage Loan Payments

Enter values	
Loan amount	$ 5,000.00
Annual interest rate	10.00 %
Loan period in years	3
Number of payments per year	1
Start date of loan	06/11/2009
Optional extra payments	$ -

Loan summary	
Scheduled payment	$ 2,010.57
Scheduled number of payments	3
Actual number of payments	3
Total early payments	$ -
Total interest	$ 1,031.72

Lender Name:

Pmt No.	Payment date	Beginning balance	Scheduled payment	Extra payment	Total payment	Principal	Interest	Ending balance	Cumulative interest
1	06/11/2010	$ 5,000.00	$ 2,010.57	$ -	$ 2,010.57	$ 1,510.57	$ 500.00	$ 3,489.43	$ 500.00
2	06/11/2011	3,489.43	2,010.57	-	2,010.57	1,661.63	348.94	1,827.79	848.94
3	06/11/2012	1,827.79	2,010.57	-	1,827.79	1,645.02	182.78	0.00	1,031.72

For the next year, the outstanding balance on the loan is now $3,489.43 and the cost of the loan goes down to $348.94, even though the payment is the same at $2,010.57. As a result, the amount going toward the repayment of the loan increases to $1,661.63.

Note that Excel users can automate the whole process of generating an amortization table by typing amortization into the online help and downloading the amortization template. For our example, Excel produced the information presented in Figure 5-2.

However, even though you can use Excel, you should practise generating your own schedule by using the PMT function and writing out each line separately, since Excel sometimes presents the data in a way that your instructor might not accept. Note the heading of the loan amortization schedule and the last line of the schedule compared with the one we developed.

EXAMPLE 5-15

Determining the Principal Outstanding

Determine the principal outstanding on the loan in Example 5-14 after one year, without referring to the amortization schedule found in the solution.

Solution

This problem can be solved by recognizing that the principal outstanding at any time on a loan equals the PV of all future payments at that time. For this example, we find the PV for the given payments and interest rate when $n = 2$ (i.e., the number of payments remaining on the loan after one year).

$$PV_0 = PMT\left[\frac{1 - \frac{1}{(1+k)^n}}{k}\right] = \$2{,}010.57\left[\frac{1 - \frac{1}{(1.10)^2}}{0.10}\right]$$

$$= (\$2{,}010.57)(1.735537) = \$3{,}489.42$$

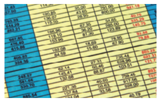
solution using a financial calculator

(TI BA II Plus)

PMT = 2,010.57; I/Y = 10; N = 2; FV = 0,

Compute PV −3,489.42.

solution using Excel



Use the Excel PV function

=PV (rate, nper, pmt, fv, type),

which for our example gives =PV (0.1, 2, 2010.57, 0, 0) and the same answer of $3,489.42.

Notice again that the answers are the same except for $0.01 because of rounding. Remember that the answer will come out as negatives because we are investing money and then receiving a payoff.

Mortgages

In practice, loan repayments are often not made on an annual basis. Many loans call for quarterly, monthly, or even weekly repayments. For these arrangements, we need to convert the quoted annual rates into effective period rates that correspond to the frequency of payments, which can be done using Equation 5-11. This conversion is needed to determine the interest that accrued during the period in question based on the principal outstanding at the beginning of the payment period (i.e., at the beginning of the month, week, etc.), because that amount will be reduced after the payment is made.

Mortgages represent an example of a loan that requires that payments be made more frequently than annually. In fact, mortgage payments must be made at least monthly, and many offer the opportunity to make biweekly or weekly payments. In Canada, mortgages are further complicated by the fact that compounding is done on a semi-annual basis, similar to bonds, which we will discuss in Chapter 6. When we deal with mortgages in Canada, $m = 2$ in Equation 5-11, while $f > 2$ (in fact, f must be greater than or equal to 12, since payments must be made at least monthly).

Finally, there is one other thing to be familiar with: the distinction between the "term" and the "amortization period" associated with long-term loans, such as mortgages. In particular, the term of a loan refers to the period for which investors can "lock in" at a fixed rate. This is usually shorter than the period over which the loan is to be repaid, or amortized, which is called the amortization period. The payments are based on the amortization period. For example, a loan with a 25-year amortization period may be structured so that the investor locks in a fixed rate of 6 percent for five years (which is the term of the loan). The payments for this loan would be determined based on the 6 percent quoted rate, assuming equal payments for 25 years even though after five years the payments will change if the interest rate on the mortgage changes. The example below demonstrates how to apply these concepts in practice.

Determining Mortgage Payments and Amortization Schedule

EXAMPLE 5-16

Determine the monthly payments and the amortization schedule for the first three months of a $200,000 mortgage loan with an amortization period of 25 years, based on a quoted rate of 6 percent and a 10-year term.

Solution

First, determine the effective period rate.

Because it is a Canadian mortgage, we know that $m = 2$. Since payments are made monthly, we need to find the effective monthly rate, so $f = 12$.

(*continued*)

EXAMPLE 5-16 *continued*

Using Equation 5-11, $k_{monthly} = \left(1 + \frac{0.06}{2}\right)^{\frac{2}{12}} - 1 = 0.4938622\%$.

Second, determine the required monthly payments by using Equation 5-5. There are 300 payments in total, since n = 25 years × 12 months = 300. PV = 200,000 (i.e., the loan amount).

$$PMT = \frac{PV_0}{\left[\frac{1 - \frac{1}{(1+k)^n}}{k}\right]} = \frac{200{,}000}{\left[\frac{1 - \frac{1}{(1.004938622)^{300}}}{0.004938622}\right]} = \frac{200{,}000}{156.2972258} = \$1{,}279.61$$

solution using a financial calculator

(TI BA II Plus)

PV = 200,000; I/Y = 0.4938622; N = 300; FV = 0;

Compute PMT = −1,279.61.

EXAMPLE 5-16 *continued*

Third, construct an amortization schedule similar to the one in Example 5-14 above.

Period	(1) Beginning Principal Outstanding	(2) PMT	(3) Interest [k × (1)]	(4) Principal Repayment [(2) − (3)]	End Principal Outstanding [(1) − (4)]
1	200,000.00	1,279.61	987.72	291.89	199,708.11
2	199,708.11	1,279.61	986.28	293.33	199,414.78
3	199,414.78	1,279.61	984.83	294.78	199,120.00

Note that unlike the simple annual payment loan, for which the cost of the loan was the annual interest cost, now the cost is the monthly interest rate of 0.4938622 percent because we have a monthly amortization schedule. It is this monthly rate applied to the outstanding balance that determines how much of the mortgage's monthly payments represent the cost of the loan. As is clear from the amortization schedule, very little of the early payments go toward reducing the principal—of the early payments are for interest. This is true for all long-term loans, because by definition the repayment of the loan is being done over a longer period. As time passes, the interest cost of the fixed payments continues to decrease, and the payment of principal correspondingly increases. The reason for this is simply that the interest rate is the cost of borrowing money, and this cost has to be subtracted first from the monthly payment.

As before, Excel simplifies the whole process. However, note that Excel reports mortgage costs according to U.S., not Canadian, practice. In Canada, mortgage rates are quoted equivalent to bonds, so a 6 percent quoted mortgage rate is actually 3 percent every six months, and the monthly rate is then determined as a monthly rate that compounds to 3 percent over six months or 0.49386 percent. In the United States, the same

FIGURE 5-3

Mortgage Loan Payments

Enter values	
Loan amount	$ 200,000.00
Annual interest rate	0.06
Loan period in years	25
Number of payments per year	12
Start date of loan	06/11/2010
Optional extra payments	$ -

Loan summary	
Scheduled payment	$ 1,288.60
Scheduled number of payments	300
Actual number of payments	300
Total early payments	$ -
Total interest	$ 186,580.84

Lender Name:

Pmt No.	Payment date	Beginning balance	Scheduled payment	Extra payment	Total payment	Principal	Interest	Ending balance	Cumulative interest
1	06/12/2010	$ 200,000.00	$ 1,288.60	$ -	$ 1,288.60	$ 288.60	$ 1,000.00	$ 199,711.40	$ 1,000.00
2	06/01/2011	199,711.40	1,288.60	-	1,288.60	290.05	998.56	199,421.35	1,998.56
3	06/02/2011	199,421.35	1,288.60	-	1,288.60	291.50	997.11	199,129.86	2,995.66

6 percent quote means 0.5 percent per month. In other words, the annual rate is simply divided by 12. As a result, the same quoted mortgage rate in the United States and Canada will produce different results. The first three months of the Excel amortization table for Example 5-16 are shown in Figure 5-3.

Note that for the first month, the interest payment is simply 0.5 percent times the principal of $200,000, or $1,000, versus $987.72 for the true (Canadian) cost. Also note that the monthly payment is $9 higher, at $1,288.60.

Excel is a wonderful software program, but before using the sophisticated functions, always check the answers by using the basic functions first to make sure that the macros written by the programmers are consistent with what you want to estimate. For Canadian mortgages, you can check with a variety of mortgage calculators, available free on the Internet, that also produce amortization schedules.[7]

One item of interest to a mortgagee would be how much of the loan has been retired after a certain time. We can solve this problem in the same manner that we did in Example 5-15.

Determining the Principal Outstanding on a Mortgage

EXAMPLE 5-17

Determine the principal outstanding on the mortgage in Example 5-16 at the end of the 10-year term.

Solution

Trying to solve this by constructing an amortization schedule to the end of 10 years would be cumbersome. However, it is easily solved by recognizing that the principal outstanding is the PV of all future payments after 10 years. We find the PV for the payments when $n = 15$ years $\times$ 12 months = 180 months—that is, for the balance of the payments after 10 years.

$$PV_0 = \$1,279.61\left[\frac{1-\dfrac{1}{(1.004938622)^{180}}}{0.004938622}\right] = (\$1,279.61)(119.0642325) = \$152,355.78$$

[7] A well-presented calculator is available on the website of Bankrate.com: www.bankrate.com/goocan/mortgage-calculator.asp.

solution using a financial calculator

(TI BA II Plus)

PMT = 1,279.61; I/Y = 0.4938622; N = 180; FV = 0;

Compute PV = −152,355.78.

solution using Excel

=PV (0.004938622, 180, 1279.61, 0, 0) gives the same answer of −$152,355.78. Note as before that the interest rate is inserted in decimals.

Notice in Example 5-17 that slightly less than 25 percent of the loan has been repaid after 10 years, even though 40 percent of the amortization period (i.e., 10 out of 25 years) has elapsed. This is consistent with our earlier observation with respect to the high proportion of each of the early payments that goes toward interest versus principal reduction. However, remember that this $152,355.78 is the amount owed after 10 years; the present value of this would be a lot less. We leave it as an exercise to work out the present value of the amount of the mortgage still outstanding at the end of 10 years.

CONCEPT REVIEW QUESTIONS

1. Explain how loan and mortgage payments can be determined using annuity concepts.
2. What complications arise when dealing with mortgage loans in Canada?
3. Why is a 6 percent U.S. mortgage not the same as a 6 percent Canadian mortgage?

5.7 COMPREHENSIVE EXAMPLES

Learning Objective 5.7
Solve a basic retirement problem.

We conclude this chapter by providing a few comprehensive examples that involve applying the concepts you have learned to more challenging situations. The second example is a common problem facing investors with respect to planning for retirement.

EXAMPLE 5-18

Multiple Annuities

a. What is the PV of $1,000 received at year end for the next four years, followed by $2,000 per year end for years 5 to 7, assuming a 10 percent rate of interest, compounded annually?

b. Suppose an investor needed $15,000 at the end of seven years and can only invest $1,000 per year for years 1 to 4 (as above). How much would the investor need to deposit in each of years 5 to 7 to achieve this objective, assuming a 10 percent interest rate as above?

(*continued*)

EXAMPLE 5-18 *continued*

Solution

a. This problem is best solved by first constructing a timeline:

0	1	2	3	4	5	6	7
	1,000	1,000	1,000	1,000	2,000	2,000	2,000

Notice that we can view this problem in several ways. It can be viewed as a four-year annuity of $1,000 followed by a three-year annuity of $2,000, or as a seven-year annuity of $1,000, with a three-year annuity of an "extra" $1,000 beginning at the end of year 5, and so on. Also, we could solve it by finding the future value of both annuities at $t = 7$ and then discounting them, or by finding the PV of the second annuity at $t = 4$ and discounting it back to $t = 0$, which is the approach we have chosen.

Viewing this problem as a four-year ordinary annuity (A1) that pays $1,000 per year, followed by a three-year ordinary annuity (A2) that pays $2,000 per year, we can determine the PV as follows:

$$\text{PV(A1)} = 1{,}000\left[\frac{1 - \frac{1}{(1.10)^4}}{0.10}\right] = 1{,}000(3.16987) = \$3{,}169.87$$

$$\text{PV(A2)} = 2{,}000\left[\frac{1 - \frac{1}{(1.10)^3}}{0.10}\right]\left[\frac{1}{(1.10)^4}\right] = 2{,}000(2.48685)(0.683013) = \$3{,}397.10$$

Notice that for PV(A2), we first determine the PV of the three-year annuity starting at time $t = 4$ (i.e., 4,973.70), then we discount this amount back to today.

solution using a financial calculator

(TI BA II Plus)

Enter the following:

0 → FV; 1,000 → PMT; 10 → I/Y; 4 → N

Then CPT → PV will = −3,169.87

Then enter the following:

0 → FV; −2000 → PMT; 10 → I/Y; 3 → N

Then CPT → PV will = 4,973.70

This is the PV of the three-year annuity at time $t = 4$, which must be discounted back to time $t = 0$.

Next, enter the following:

4,973.70 → FV; 0 → PMT; 10 → I/Y; 4 → N

Then CPT → PV will = −3,397.10

solution using Excel

Using Excel, calculate PV(A1) = PV(0.1, 4, −1000, 0, 0)

We can calculate Excel PV (A2) by first calculating the PV of the three-year annuity as

=PV (0.1, 3, −2000, 0, 0),

which gives $4,937.70. Then use

=PV (0.1, 4, 0, −4937.70, 0),

which gives $3,397.10.

However, we can do this in one step with Excel because we can nest Excel functions within each other. In this case, we can write it in one step as

=PV (0.1, 4, 0, PV (0.1, 3, −2000, 0, 0), 0)

and we would get the same answer.

EXAMPLE 5-18 *continued*

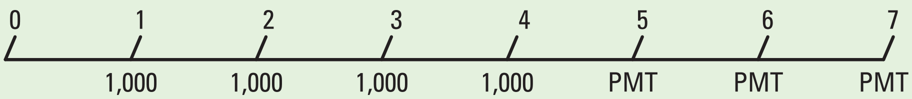
So the PV(Total) = PV(A1) + PV(A2) = 3,169.87 + 3,397.10 = $6,566.97.

In all these problems, the interest rate need not be the same over the two periods. For our example, the interest rate for the second three-year period could be 10 percent, while for the first four periods it could be 5 percent. In this case, the Excel solution would be

=PV (0.05, 4, 0, PV (0.1, 3, −2000, 0, 0), 0)

and the value increases to $6,943.05.

b. This problem can also be best represented by using a timeline:



We can solve this problem in several ways. We have chosen to solve it by finding the future value of the first annuity (A1) at $t = 7$, and then determining the FV of the second annuity (A2) at $t = 7$, required to achieve the $15,000 target. Then, knowing the required FV of the three-year annuity, we can determine the payments for years 5 to 7.

Determine the FV of A1 at time $t = 4$:

$$FV_4(A1) = 1{,}000 \times \left[\frac{(1 + 0.10)^4 - 1}{0.10}\right] = (1{,}000)(4.641) = \$4{,}641$$

Second, determine the FV of A1 at time $t = 7$:

$$FV_7(A1) = 4{,}641 \times (1.10)^3 = (4{,}641)(1.331) = \$6{,}177.17$$

Third, determine the required FV of A2 at time $t = 7$:

Required $FV_7(A2)$ = Required amount $-FV_7(A1)$ = 15,000 − 6,177.17

= $8,822.83

(*continued*)

EXAMPLE 5-18 *continued*

Fourth, determine the required payments for A2:

$$\text{PMT} = \frac{8{,}822.83}{\left[\frac{(1.10)^3 - 1}{0.10}\right]} = \frac{8{,}822.83}{3.31} = \$2{,}665.51$$

solution using a financial calculator

(TI BA II Plus)

First, enter the following:

0 → PV ; −1,000 → PMT ; 10 → I/Y ; 4 → N

then CPT → FV will = 4,641

Second, enter the following:

4,641 → PV ; 0 → PMT ; 10 → I/Y ; 3 → N

Then CPT → FV will = −6,177.17

Third, enter the following:

8,822.83 → FV ; 0 → PV ; 10 → I/Y ; 3 → N

Then CPT → PMT will = −2,665.51

solution using Excel

=FV (0.1, 4, −1000, 0, 0) gives an answer of $4,641.

=FV (0.1, 3, 0, 4641, 0) gives an answer of $6,177.17.

Then we can use the payment function,

=PMT (rate, nper, pv, fv, type), so that inserting our values

=PMT (0.1, 3, 0, 8822.83, 0) gives the same answer of $2,665.51.

If we are really ambitious, we can collapse all three Excel functions into a single one and input

=PMT (0.1, 3, 0, 15000 + FV(0.1, 3, 0, FV(0.1, 4, −1000, 0, 0), 0), 0)

and get the right answer. However, although this shows the power of Excel, it is not recommended. Much can be learned by breaking the problem into its three constituent parts because you need to develop an understanding of what is approximately the right answer. This opportunity is lost when you collapse it all into one function. Furthermore, you have to be careful; Excel provides a negative present value if the payoff is positive because it treats the values as those from an investment. In our case, we have to add the future value from the annuity since Excel provides a negative value for this.

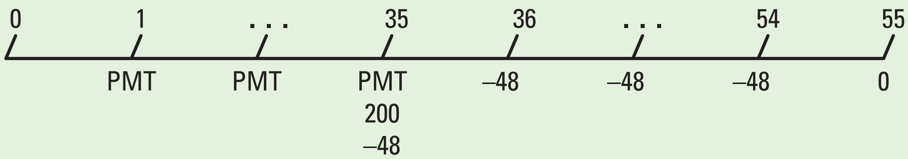
EXAMPLE 5-19

Retirement Problem

An investor plans to retire 35 years from today and have sufficient savings to guarantee $48,000 each year for 20 years. Assume retirement withdrawals will be made at the beginning of each of the 20 years. The investor estimates that at the time of retirement, he can sell his business for $200,000. The expectation is that interest rates will be relatively stable at 8 percent a year for the next 35 years. Thereafter, the interest rate is expected to decline to 6 percent forever. The investor wants to make equal annual deposits at the end of each of the next 35 years. How much should be deposited each year in order to meet the stated objective?

Solution

A timeline helps visualize the problem:



k = 8 percent for years 1 to 35 k = 6 percent for years 35 to 55

First, determine how much is needed after 35 years. At this time, the investor wants a 20-year annuity due of $48,000, because he is drawing down funds immediately on retirement, at a 6 percent interest rate. In other words, find the PV of a 20-year annuity due, when $k = 6$ percent:

$$PV_{35} = 48{,}000\left[\frac{1 - \frac{1}{(1.06)^{20}}}{0.06}\right](1.06) = (48{,}000)(11.46992122)(1.06) = \$583{,}589.59$$

solution using a financial calculator

(TI BA II Plus)

First, set your calculator to Begin mode and then enter

FV = 0; PMT = 48,000; I/Y = 6; N = 20;

and compute PV = −583,589.59 or $583,589.59.

EXAMPLE 5-19 *continued*

Second, subtract the $200,000 expected from the sale of the business, leaving the amount needed to raise through investments:

Need $FV_{35} = 583{,}589.59 - 200{,}000 = \$383{,}589.59$

Third, determine the required year-end payments over the next 35 years:

$$PMT = \left[\frac{383{,}589.59}{\frac{(1.08)^{35} - 1}{0.08}}\right] = \frac{383{,}589.59}{172.3168037} = \$2{,}226.07$$

Finally, take your calculator out of Begin mode. Enter

PV = 0; FV = 383,589.59; I/Y = 8; N = 35

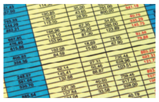
then compute PMT = −2,226.07 or $2,226.07

solution using a financial calculator

(TI BA II Plus)

Or, using Excel,
=PV (0.06, 20, 48000, 0, 1)
gives the same answer of $583,589.59. However, now that we need an annuity due, we have to put a 0 in for the future value to let Excel know that the 1 in the last column refers to an annuity due and is not a future value. If we then use the payment function to calculate the annuity, we get
=PMT (0.08, 35, 0, 383589.59, 0)
The answer is $2,226.07. As before, we can collapse this into one function as =PMT (0.08, 35, 0, PV(0.06, 20, −48000, 0, 1) −200000, 0)
But this is not advisable until you have more experience with Excel.

solution using Excel



Notice that the two problems in this section appear complicated at first but are quite manageable if you break them down into their components. Timelines are very useful for this purpose because they help you visualize what information you have and what is needed to solve the problem. If you are able to solve these problems, then you have a good understanding of the basic concepts involving the time value of money.

CONCEPT REVIEW QUESTIONS

1. Explain how timelines can be used to break a complicated time value of money problem into manageable components.
2. Demonstrate how to solve a typical retirement problem.

SUMMARY

In this chapter, we demonstrate how to compare cash flows that occur at different points in time, after accounting for the time value of money. This process is applied to virtually every topic that is studied in finance, so it is extremely important. We show how to determine economically equivalent future values from values that occur in previous periods by applying the process of compounding at an appropriate rate of return. Similarly, we show how to determine economically equivalent present values (in today's dollars) for cash flows that occur in the future by discounting them, which is the reciprocal of compounding. These processes can be applied to several cash flows simultaneously.

Annuities represent a special type of cash flow stream involving equal payments at the same interval, with the same interest rate being applied throughout the period. We see that these kinds of cash flow streams are commonplace in finance applications (e.g., loan payments) and that there are relatively simple formulas that enable us to determine the present or future value of these cash flows. We also illustrate how to adjust quoted interest rates to find effective rates, which is important because compounding often takes place at other than annual intervals. Finally, we conclude with some more involved applications of the concepts discussed in the chapter.

APPENDIX 5A
GROWING ANNUITIES AND PERPETUITIES

Growing Perpetuities

Sometimes we may want to evaluate a stream of cash flows that will grow (or shrink) at a constant rate per period (g) indefinitely.[8] In this case, our payments (PMT) will change from one period to the next, as shown below:

$$PMT_1 = PMT_0(1 + g)$$

$$PMT_2 = PMT_1(1 + g) = PMT_0(1 + g)^2$$

$$PMT_3 = PMT_2(1 + g) = PMT_0(1 + g)^3$$

and so on.[9]

The present value of this stream of payments can be calculated as follows:

[5A-1]
$$PV_0 = \frac{PMT_0(1 + g)}{(1 + k)^1} + \frac{PMT_0(1 + g)^2}{(1 + k)^2} + \frac{PMT_0(1 + g)^3}{(1 + k)^3} + \ldots + \frac{PMT_0(1 + g)^\infty}{(1 + k)^\infty}$$

Notice that what we are doing here is multiplying PMT_0 by a factor of $(1 + g)/(1 + k)$ every period. This is easily solved because it represents the sum of a geometric series, so we can reduce this to the following expression:

[5A-2]
$$PV_0 = \frac{PMT_0(1 + g)}{k - g} = \frac{PMT_1}{k - g}$$

Equation 5A-2 has several important points:

1. This relationship holds only when $k > g$. Otherwise, the answer is negative, which is uninformative.
2. Only *future* estimated cash flows and estimated growth in these cash flows are relevant.
3. The relationship holds only when growth in payments is expected to occur at the same rate indefinitely.

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[8] For example, in Chapter 7, we will make this assumption about dividends paid on common shares.

[9] Notice that if the cash flows are shrinking, it merely means that g is a negative rate.

Valuing a Growing Perpetuity

EXAMPLE 5A-1

You are attempting to determine the present value of cash flows to be generated from a rental property you are considering purchasing. You have estimated that the after-tax cash flows from this property will grow at 4 percent per year indefinitely due to rental increases. The cash flow this past year was $100,000, and the appropriate discount rate is 15 percent. Find the present value of these cash flows.

Solution

$PMT_0 = 100{,}000$; $g = 4\%$; $k = 15\%$.

Using the equation above, we get

$$PV_0 = \frac{PMT_0(1+g)}{k-g} = \frac{(100{,}000)(1.04)}{.15-.04} = \frac{104{,}000}{.11} = \$945{,}454.55$$

Growing Annuities

Sometimes we may want to evaluate a stream of cash flows that will grow (or shrink) at a constant rate per period (g) over a given period of time, ending at some terminal point (n). Such a stream represents a growing annuity.

The present value of this stream of payments can be calculated as follows:

$$PV_0 = \frac{PMT_0(1+g)}{(1+k)^1} + \frac{PMT_0(1+g)^2}{(1+k)^2} + \frac{PMT_0(1+g)^3}{(1+k)^3} + \ldots + \frac{PMT_0(1+g)^n}{(1+k)^n} \quad [5A\text{-}3]$$

Notice that this equation is identical to Equation 5A-1, except that the last payment occurs at time "n"—i.e., the payments do not go on to infinity. Equation 5A-3 reduces to the following expression:

$$PV_0 = \frac{PMT_1}{k-g} \times \left[1 - \left(\frac{1+g}{1+k}\right)^n\right] \quad [5A\text{-}4]$$

Notice that this equation estimates the present value of a growing perpetuity and then subtracts the present value of the payments from period "$n+1$" to infinity from this value.

Valuing a Growing (Shrinking) Annuity

EXAMPLE 5A-2

A mining company is attempting to determine the present value of cash flows to be generated from a new mining operation. They have estimated that the after-tax cash flows from this mine will shrink at a rate of 10 percent per year as the reserves are depleted, and that after 10 years the mine will be abandoned. Next year's cash flow is estimated to be $200,000, and the appropriate discount rate is 20 percent. Find the present value of these cash flows.

Solution

$PMT_1 = 200{,}000$; $g = -10\%$; $k = 20\%$; $n = 10$

(*continued*)

EXAMPLE 5A-2 *continued*

Using Equation 5A-4, we get

$$PV_0 = \frac{PMT_1}{k - g} \times \left[1 - \left(\frac{1 + g}{1 + k}\right)^n\right] = \frac{200{,}000}{.20 - (-.10)} \times \left[1 - \left(\frac{1 + (-.10)}{1 + .20}\right)^{10}\right]$$

$$= \$629{,}124.32$$

Summary of Learning Objectives

5.1 Explain the importance of the time value of money and how it is related to an investor's opportunity costs.

Time value of money is the idea that money invested today has more value than the same amount invested later. This concept helps us to understand how interest is earned and why investors are indifferent to investment today and future value later.

The opportunity cost of money is the interest rate that would be earned by investing it. For this reason, we also call the interest rate the price of money.

5.2 Define simple interest and explain how it works.

Simple interest is interest earned on the original principal. The growth in the value of an investment is simply the sum of annual interest earned.

5.3 Define compound interest and explain how it works.

Compound interest is interest earned on the principal amount invested and on any accrued interest. Compound interest can result in dramatic growth in the value of an investment over time.

5.4 Differentiate between an ordinary annuity and an annuity due, and explain how special constant payment problems can be valued as annuities and, in special cases, as perpetuities.

Annuities are streams of level payments at regular time intervals. An ordinary annuity has payments at the end of each period. An annuity due has the same number of payments as an ordinary annuity, but the payments occur at the beginning of each period. The present value of an ordinary annuity can be found with a formula that is equal to the sum of the present value factors. The future value of an ordinary annuity can be found with a formula that is equal to the sum of the future value factors. To get the present and future value factors for an annuity due, just multiply the ordinary annuity factors by $(1 + k)$.

5.5 Differentiate between quoted rates and effective rates, and explain how quoted rates can be converted to effective rates.

Quoted rates are also called stated rates or annual percentage rates, which are measured annually and used for quoting purposes. The effective rate for a period is the rate at which a dollar invested grows over that period. It is usually stated in percentage terms based on an annual period. The relation can be found in the formula $k = \left(1 + \frac{QR}{m}\right)^m - 1$, where QR is the quoted rate, m is the compounding frequency, and k is the annual effective rate.

5.6 Apply annuity formulas to value loans and mortgages and set up an amortization schedule.

Loans can be valued as an annuity since they meet the three characteristics of an annuity in that they equal payments, for a fixed period of time, and based on the same discount rate.

To set up an amortization schedule, five variables are calculated for each period: beginning principal outstanding, payment, interest, principal repayment, and ending principal outstanding.

5.7 Solve a basic retirement problem.

A simple retirement problem can be solved by equating the present value of the retirement annuity and the future value of the savings annuity. Comprehensive examples are presented in section 5.7.

Key Terms

amortize, p. 00
annuity, p. 00
annuity due, p. 00
basis point, p. 00
cash flows, p. 00
compound interest, p. 00
compound value interest factor (CVIF), p. 00
discount rate, p. 00
discounting, p. 00
effective rate, p. 00
lessee, p. 00
medium of exchange, p. 00
mortgage, p. 00
ordinary annuities, p. 00
perpetuities, p. 00
present value interest factor (PVIF), p. 00
reinvested, p. 00
required rate of return, p. 00
simple interest, p. 00
time value of money, p. 00

Formulas/Equations

(5-1) (pg. 000) $\text{Value (time } n) = \text{P} + (n \times \text{P} \times k)$

(5-2) (pg. 000) $\text{FV}_n = \text{PV}_0(1 + k)^n$

(5-3) (pg. 000) $\text{PV}_0 = \dfrac{\text{FV}_n}{(1 + k)^n} = \text{FV}_n \times \dfrac{1}{(1 + k)^n}$

(5-4) (pg. 000) $\text{FV}_n = \text{PMT}\left[\dfrac{(1 + k)^n - 1}{k}\right]$

(5-5) (pg. 000) $\text{PV}_0 = \text{PMT}\left[\dfrac{1 - \dfrac{1}{(1 + k)^n}}{k}\right]$

(5-6) (pg. 000) $\text{FV}_n = \text{PMT}\left[\dfrac{(1 + k)^n - 1}{k}\right](1 + k)$

(5-7) (pg. 000) $\text{PV}_0 = \text{PMT}\left[\dfrac{1 - \dfrac{1}{(1 + k)^n}}{k}\right](1 + k)$

(5-8) (pg. 000) $\text{PV}_0 = \dfrac{\text{PMT}}{k}$

(5-9) (pg. 000) $k = \left(1 + \dfrac{\text{QR}}{m}\right)^m - 1$

(5-10) (pg. 000) $k = e^{\text{QR}} - 1$

(5-11) (pg. 000) $k = \left(1 + \dfrac{QR}{m}\right)^{\frac{m}{f}} - 1$

(5A-1) (pg. 000) $\text{PV}_0 = \dfrac{\text{PMT}_0(1 + g)}{(1 + k)^1} + \dfrac{\text{PMT}_0(1 + g)^2}{(1 + k)^2} + \dfrac{\text{PMT}_0(1 + g)^3}{(1 + k)^3} + \ldots + \dfrac{\text{PMT}_0(1 + g)^\infty}{(1 + k)^\infty}$

(5A-2) (pg. 000) $\text{PV}_0 = \dfrac{\text{PMT}_0(1 + g)}{k - g} = \dfrac{\text{PMT}_1}{k - g}$

(5A-3) (pg. 000) $$PV_0 = \frac{PMT_0(1+g)}{(1+k)^1} + \frac{PMT_0(1+g)^2}{(1+k)^2} + \frac{PMT_0(1+g)^3}{(1+k)^3} + \ldots + \frac{PMT_0(1+g)^n}{(1+k)^n}$$

(5A-4) (pg. 000) $$PV_0 = \frac{PMT_1}{k-g} \times \left[1 - \left(\frac{1+g}{1+k}\right)^n\right]$$

Questions and Practice Problems

Multiple Choice Questions

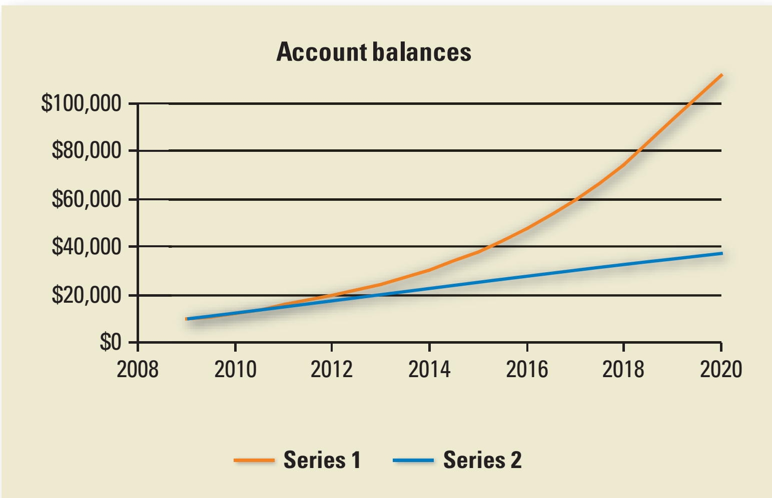
1. What is the total amount accumulated after five years if someone invests $1,000 today with a simple annual interest rate of 8 percent? With a compound annual interest rate of 8 percent?

   a. $600, $680
   b. $1,400, $1,469
   c. $1,469, $1,400
   d. $5,400, $1,016

2. An investment of $5,000 invested for three years has an expected future value of $6,100. The simple annual interest rate is closest to ______ and the compound annual interest rate is closest to ______.

   a. 6.01%; 6.41%
   b. 7.33%; 6.85%
   c. 6.85%; 7.33%
   d. 6.41%; 6.01%

3. At the end of 2009, Malcolm invested $10,000 in two bank accounts. The expected value of each bank account for the years 2009 to 2020 are represented in the following graph. In the graph, "Series 1" represents an account paying ______ interest, while "Series 2" represents an account paying ______ interest.



   a. simple; compound
   b. compound; simple
   c. simple; simple
   d. compound; compound

4. Which of the following has the largest future value if $1,000 is invested today?
   a. Five years with a simple annual interest rate of 10%
   b. 10 years with a simple annual interest rate of 8%
   c. Eight years with a compound annual interest rate of 8%
   d. Eight years with a compound annual interest rate of 7%

   Interest rates in the following questions are compound rates unless otherwise stated.

5. Suppose an investor wants to have $15 million to retire 25 years from now. How much would she have to invest today with an annual rate of return equal to 5 percent?
   a. $6,666,666.67
   b. $4,429,541.58
   c. $600,000
   d. $21,345

6. Which of the following is *false?*
   a. The longer the time period, the smaller the present value, given a $100 future value and holding the interest rate constant.
   b. The greater the interest rate, the greater the present value, given a $100 future value and holding the time period constant.
   c. A future dollar is always less valuable than a dollar today if interest rates are positive.
   d. The discount factor is the reciprocal of the compound factor.

7. Maggie deposits $10,000 today and is promised a return of $17,000 in eight years. What is the implied annual rate of return?
   a. 6.86%
   b. 7.06%
   c. 5.99%
   d. 6.07%

8. How long will it take Mike to triple his investment if he can earn an annual rate of return of 9%?
   a. 15.5 years
   b. 13.9 years
   c. 12.7 years
   d. 10 years

9. Which of the following statements is *incorrect?*
   a. An ordinary annuity has payments at the end of each year.
   b. An annuity due has payments at the beginning of each year.
   c. A perpetuity is considered a perpetual annuity.
   d. An ordinary annuity has a greater PV than an annuity due, if they both have the same periodic payments, discount rate, and time period.

10. Jan plans to invest an equal amount of $2,000 in an equity fund every year end beginning this year. The expected annual return on the fund is 15 percent. How much would she expect to have at the end of 20 years?
    a. $237,620
    b. $176,424
    c. $204,887
    d. $178,424

11. Jan plans to invest an equal amount of $2,000 in an equity fund every year end beginning this year. The expected annual return on the fund is 15 percent. She plans to invest for 20 years. What is the present value of Jan's investments?

   a. $12,625
   b. $12,519
   c. $14,396
   d. $12,396

12. What is the present value of a perpetuity with an annual year-end payment of $1,500 and expected annual rate of return equal to 12 percent?

   a. $14,000
   b. $13,500
   c. $11,400
   d. $12,500

13. What is the present value of a perpetuity with an annual payment of $1,500 and the first payment is due immediately? The expected annual rate of return is equal to 12 percent.

   a. $14,000
   b. $13,500
   c. $11,400
   d. $12,500

14. Ten years ago you borrowed $250,000. The term of the loan was 20 years with monthly payments of $2,752.72. The interest rate on the loan was 12 percent compounded monthly. You have just made the 120th payment. What is the principal outstanding?

   a. $200,000.00
   b. $196,696.74
   c. $191,865.70
   d. $125,000.00

15. Which of the following statements is correct?

   a. The future value of a perpetuity cannot be computed.
   b. The future value of a perpetuity can be computed.
   c. The present value of a perpetuity cannot be computed.
   d. The present value of a perpetuity can be computed.

Practice Problems

16. Franklin is trying to decide whether or not to take a philosophy course next semester. He finds the topic interesting, but, being a business student, he wants to measure the cost of taking the course. After detailed thought and analysis he has identified the following items that he feels may be relevant to his decision:

   Tuition fees for the course: $500

   Textbook for the course: $200

   Rent for his apartment: $5,000

   Food for next semester: $3,200

   Currently Franklin earns $2,000 per semester as a teaching assistant.

Franklin estimates that the philosophy course will require considerable study so he will have to reduce his teaching assistant hours by 25 percent. Determine the cost to Franklin of taking the philosophy course.

17. Dmitri Chekov has invested $25,000 in an investment that promises to pay him 8 percent simple interest per year for 10 years. Determine how much interest he will earn in the:
    a. First year
    b. Ninth year
18. After a summer of travelling (and not working), a student finds himself $1,500 short for this year's tuition fees. His parents have agreed to loan him the money for three years at a simple interest rate of 6 percent, with interest due at the end of each year.
    a. How much interest will he owe his parents after one year?
    b. How much will he have paid, in total, after three years?
19. Your sister has been forced to borrow money to pay her tuition this year. If she makes annual interest payments on the loan at year end for the next three years, and the loan is for $2,500 at a simple interest rate of 6 percent, how much will she pay each year?
20. Khalil's summer job has given him $1,200 more than he needs for tuition this year. The local bank pays simple interest at a rate of 0.5 percent per month. How much interest will he earn in one year?
21. A new Internet bank pays compound interest of 0.5 percent per month on deposits. How much interest will Khalil's summer savings of $1,200 earn in one year with this online bank account?
22. History tells us that a group of Dutch colonists purchased the island of Manhattan from the Native American residents in 1626. Payment was made with *wampum* (likely glass beads and trinkets), which had an estimated value of $24. Suppose the Dutch had invested this money back home in Europe and earned an average return of 5 percent per year. Determine how much this investment would be worth 380 years later, using:
    a. simple interest
    b. compound interest
23. David has been awarded a scholarship that will pay $2,500 one year from now. However, he really needs the money today, and has decided to take out a loan. If the interest rate is 8 percent, how much can he borrow so that the scholarship will just pay off the loan?
24. Grace, a retired librarian, would like to donate some money to her alma mater to endow a $5,000 annual scholarship. The university will manage the funds, and expects to earn 3 percent per year. How much will Grace have to donate so that the endowment fund never runs out?
25. Grace, a retired librarian, would like to donate some money to her alma mater to endow a $5,000 annual scholarship. The first scholarship will be awarded in 5 years. The university will manage the funds, and expects to earn 3 percent per year. How much will Grace have to donate so that the endowment fund never runs out?
26. Muriel would like to support the education of her favourite grand-nephew, Stephen, who plans to begin university in three years. How much will Muriel have to invest today, at 7 percent, to be able to give Stephen $4,000 at the end of each year for four years?
27. You have just won $50 million in a lottery and are offered two options: receive $40 million today or receive $5 million per year for the next 10 years. At what interest rate are you indifferent between the two options? If the interest rate is greater than the indifference rate, which do you prefer and why? If the interest rate is less than the indifference rate, which do you prefer and why?

28. Two friends, Abe and Betty, are planning for their retirement. Both are 20 years old and plan on retiring in 30 years with $1 million each. Betty plans on making annual deposits beginning in one year (total of 30 deposits) while Abe plans on waiting and then depositing twice Betty's deposits. If both can earn 5 percent per year, how long can Abe wait before he has to start making his deposits?

29. You are planning on buying your first home and need to borrow $250,000 from the bank. The manager offers you two mortgages: Long will take 25 years to be paid off, and your annual payments will be $17,738. Short will only take 10 years to be paid off, and your annual payment will be $35,200. The manager offers you the following advice: if you take the Long option, you will pay $193,450 in interest ($17,738 × 25 – $250,000) while if you take the Short option, you will only pay $102,000 in interest ($35,200 × 10 – $250,000)—a savings of over $91,000. Do you agree or disagree with the manager's advice? Briefly explain your reasoning.

30. Bank A pays 7.25 percent interest compounded semi-annually, Bank B pays 7.20 percent compounded quarterly, and Bank C pays 7.15 percent compounded monthly. Which bank pays the highest effective annual rate?

31. Jimmie is buying a new car. His bank quotes a rate of 9.5 percent per year for a car loan. Calculate the effective annual rate if the compounding occurs:

    a. annually
    b. quarterly
    c. monthly

32. If Alysha puts $50,000 in a savings account paying 6 percent per year, determine how much money she will have in total at the end of the first year if interest is compounded:

    a. annually
    b. monthly
    c. daily

33. A bank is currently offering a savings account paying an interest rate of 8 percent compounded quarterly. It would like to offer another account, with the same effective annual rate, but compounded monthly. What is the equivalent rate compounded monthly?

34. Public corporations have no fixed life span; as such, they are often viewed as entities that will pay dividends to their shareholders in perpetuity. Suppose a firm pays a dividend of $2 per share every year. If the discount rate is 12 percent, what is the present value of all the future dividends?

35. Mary-Beth is planning to live in a university residence for four years while completing her degree. The annual cost for food and lodging is $5,800 and must be paid at the start of each school year. What is the total present value of Mary-Beth's residence fees if the discount rate (interest rate) is 6 percent per year?

36. Calculate the effective annual rates for the following:

    a. 24%, compounded daily
    b. 24%, compounded quarterly
    c. 24%, compounded every four months
    d. 24%, compounded semi-annually
    e. 24%, compounded continuously
    f. Calculate the effective monthly rate for a to d.

37. Amanda would like to borrow $50,000 to pay one year's tuition at a private U.S. university. She would like to make quarterly payments and finish repaying the loan in 5 years. If

the bank is quoting her a rate of 6 percent compounded monthly, determine her quarterly payment.

38. Wilma would like to borrow $150,000 to start her own business. She would like to make monthly payments to repay the loan in 5 years. If the bank is quoting her a rate of 6 percent compounded quarterly, determine her monthly payment.

39. On the advice of a friend, Gilda invests $20,000 in a mutual fund which has earned 10 percent per year, on average, in recent years. If this rate of return continues, determine how much her investment will be worth in:

    a. one year
    b. five years
    c. 10 years

40. When Jon graduates in three years, he wants to throw a big party, which will cost $800. To have this amount available, how much does he have to invest today if he can earn a compound return of 5 percent per year?

41. At the age of 10, Felix decided that he wanted to attend a very prestigious (and expensive) university. How much will his parents have to save each year to accumulate $40,000 by the time Felix needs the funds in eight years? Assume Felix's parents can earn 7 percent (compounded annually) on their savings, and that each year's savings are deposited at the end of the year.

42. Jane's parents save $1,000 per year for 17 years to pay for her university tuition costs. They deposit the money into a Registered Education Savings Plan (RESP) account so that no tax is payable on the interest income. This RESP account provides a return of 6 percent per year.

    a. How much will Jane's account be worth when she begins her university studies?
    b. As an incentive to save for higher education, the government will add 20 percent to any money contributed to an RESP each year. Including these grants, how much will Jane have in her account?

43. Stephen has learned that his great-aunt intends to give him $4,000 each year he is studying at university. Tuition must be paid in advance, so Stephen would like to receive his payments at the beginning of each school year. How much will his great-aunt have to invest today at 7 percent, to make the four annual (start-of-year) payments? Assume that Stephen will be starting school in 4 years.

44. Jimmie wishes to buy a new car that will cost $29,000.

    a. How much will his monthly car payments be if he obtains a loan that is amortized over 60 months, and the nominal interest rate is 8.5 percent per year with monthly compounding?
    b. Create an amortization schedule for Jimmie's car loan. What portion of the first monthly payment goes toward repaying the principal amount of the loan? What portion of the last monthly payment goes toward the principal?
    c. Using the amortization schedule, determine how much Jimmie still owes on the car loan after three years of payments on the five-year loan. What is the present value of this amount?

45. Michelle is offered a loan of $29,000 that requires 60 monthly payments of $588.02. What is the effective annual interest rate on this loan? What would the quoted rate be?

46. To start a new business, Su Mei intends to borrow $25,000 from a local bank. If the bank asks her to repay the loan in five equal annual instalments of $6,935.24, determine the bank's effective annual interest rate on the loan transaction. With annual compounding, what nominal rate would the bank quote for this loan?

47. The Business Development Bank is willing to loan Su Mei the $25,000 she needs to start her new business. The loan will require monthly payments of $556.11 over five years.

    a. What is the effective monthly rate on this loan?

    b. With monthly compounding, what is the nominal (annual) interest rate on this loan?

48. After losing money playing online poker, Scott visits a loan shark for a $750 loan. To avoid a visit from the "collection agency," he will have to repay $800 in just one week.

    a. What is the nominal interest rate per week? Per year?

    b. What is the effective annual interest rate?

49. Josephine needs to borrow $180,000 to purchase her new house in Yarmouth, Nova Scotia. She would like to pay off the mortgage in 20 years, making monthly payments. For the initial three-year term, Providence Bank has offered her a quoted annual rate of 6.40 percent.

    a. What is the effective annual interest rate?

    b. What is the effective monthly interest rate?

    c. How much will Josephine's monthly mortgage payments be?

    d. Yarmouth Credit Union will provide Josephine with a mortgage at a rate of 6.36 percent, but unlike most Canadian mortgages, the compounding will occur monthly. Should Josephine take out the mortgage loan from Yarmouth Credit Union, or from Providence Bank?

50. A lakefront house in Kingston, Ontario is for sale with an asking price of $499,000. The real-estate market has been quite active, so the house will almost certainly attract several offers, and may sell for more than the asking price. Charlie is very eager to purchase this house, but is concerned that he may not be able to afford it. He has $130,000 available for a down payment, and can pay up to $1,950 per month on a mortgage loan. As a long-time customer, Charlie's bank has offered him a great mortgage rate of 3.90 percent on a one-year term. If the loan will be amortized over 25 years, what is the most that Charlie can afford to pay for the house?

51. Five years ago, Franklin borrowed $300,000 to purchase a house in Sandy Lake. At the time, the quoted rate on the mortgage was 6 percent, the amortization period was 25 years, the term was 5 years, and the payments were made monthly. Now that the term of the mortgage is complete, Franklin must renegotiate his mortgage. If the current market rate for mortgages is 8 percent, what is Franklin's new monthly payment?

52. Timmy sets himself a goal of amassing $1 million in his retirement fund by the time he turns 61. He begins saving $3,000 each year, starting on his 21st birthday (40 years of saving).

    a. If his savings earn 10 percent per year, will Timmy achieve his goal?

    b. At what age will the value of Timmy's savings plan be worth $1 million?

53. Tommy has a goal of amassing $1 million by the time he retires. However, there always seemed to be a reason not to save money, so he put it off for many years. Finally, with just 15 years to retirement, he began to save. Fortunately, Tommy's executive-level job allowed him to save $30,000 per year. If these savings earn 10 percent per year, will Tommy achieve his $1-million goal at the desired time?

54. Jack is 28 years old now and plans to retire in 35 years. He works in a local bank and has an annual after-tax income of $45,000. His expected annual expenditure is $36,000 and the rest of his income will be invested at the beginning of each of the next 35 years at an expected annual rate of return of 12.6 percent. Calculate the amount Jack will have accumulated when he retires.

55. a. Determine the month-end payment for a $200,000, 10-year loan with an interest rate of 12 percent, compounded monthly, assuming there is no down payment.

    b. Calculate the outstanding loan amount after 18 months.

    c. Redo part (a), assuming it is a mortgage loan with monthly payments.

56. Investor A just turned 20 years old and currently has no investments. She plans to invest \$5,500 at the end of each year for eight years, beginning in five years. The rate of return on her investment is 15 percent, continuously compounded. Investor B is 40 years old and he just started to invest at the beginning of every year an equal amount of money starting today. He will invest for 10 years. The rate of return on his investment is 16 percent, compounded quarterly. Determine the yearly payment Investor B has to make in order to have the same present value as Investor A.

57. Paul and Maria want to have enough money to travel around the world when they retire. They both just turned 30 and will retire when they turn 60. They earn a total of \$9,000 after taxes each month. Their monthly expenditures include \$3,000 in mortgage payments, \$850 in car payments, and \$1,450 in other expenses. They approached a fund manager and decided to invest the rest of their income at the end of each year. They expect to earn a 10 percent expected annual rate of return for each of the next 30 years. When they retire, they will sell their cottage for an expected price of \$50,000.

    a. Determine how much they will have when they retire.
    b. How much can Paul and Maria withdraw annually at the beginning of the year for travelling after they retire if they expect to live until they are 90?

58. Veda has to choose between two investments that have the same cost today. Both investments will ultimately pay \$1,300 but at different times, as shown in the table below. If Veda does not choose one of these investments, she could leave the funds in a bank account paying 5 percent per year. Which investment should Veda choose?

Year	Investment A	Investment B
1	\$0	\$200
2	\$500	\$400
3	\$800	\$700

59. Felix will need \$10,000 per year for four years to pay for tuition. How much will Felix's parents have to invest at the end of each year for the eight years before he begins his studies if their savings earn compound interest at 7 percent per year? Assume the tuition payments also occur at the end of each year.

60. Roger has his eye on a new car that will cost \$20,000. He has \$15,000 in his savings account, earning interest at a rate of 0.5 percent per month.

    a. How long (to the nearest month) will it be before he can buy the car?
    b. How long will it be before Roger can buy the car if, in addition to his existing savings, he can save \$250 per month?

61. How many years will it take for an investment to double in value if the rate of return is 9 percent, and compounding occurs:

    a. annually?
    b. quarterly?

62. Céline has just won a lottery. She will receive a payment of \$6,000 at the end of each year for nine years. As an alternative, she can choose an immediate payment of \$50,000.

    a. Which alternative should she pick if the interest rate is 5 percent?
    b. What would the interest rate have to be for Céline to be indifferent about the two alternatives?

63. Alysha has decided to use her $50,000 in savings to make a down payment on a house. She will live in the house for the next two years while still at university, and then sell it when she graduates. The bank has offered her a mortgage rate of 5.1 percent compounded semi-annually on a two-year term, with an amortization period of 25 years. The house she is interested in purchasing costs $280,000.

64. A firm has just declared that its dividend next year will be $3 per share. That rate of payment will continue for an additional four years, after which the dividends will fall back to their usual $2 per share. What is the present value of all the future dividends?

65. After one year living in a university residence, Mary-Beth decides to rent an apartment for the remaining three years of her degree. She has found a nice location that will cost $450 per month; rent for the first and last month must be paid up front. How much money would Mary-Beth need to have in her bank account right now to be sure she will always have enough for rent? The bank account pays 3.75 percent interest, compounded monthly.

66. A 65-year-old man intends to use his retirement funds to purchase an annuity from a life insurance company. Given the amount of money the man has available to invest, the insurance company is able to offer two alternatives. The first option is to receive $2,785 each month for as long as he lives; the second option is to receive $3,500 each month, but for only 20 years (payments will be made to his estate if he should die before that time). The relevant interest rate is 6 percent per year. How long must the man live so that the first option is a better deal?

67. An investment promises to pay you $100 per year starting in 1 year. The cash flow from the investment is expected to increase by 3 percent per year forever. If alternative investments of similar risk earn a return of 9 percent per year, determine the maximum you would be willing to pay for the investment.

68. An investment promises to pay you $100 per year starting immediately. The cash flow from the investment is expected to increase by 3 percent per year forever. If alternative investments of similar risk earn a return of 9 percent per year, determine the maximum you would be willing to pay for the investment.

69. An investment promises to pay you $100 per year starting in 5 years. The cash flow from the investment is expected to increase by 3 percent per year forever. If alternative investments of similar risk earn a return of 9 percent per year, determine the maximum you would be willing to pay for the investment.

70. Shirley has been offered two perpetuities: Grow and Shrink. Grow promises her $100 in one year and that the annual cash flow will increase by 4 percent per year forever. Shrink, in contrast, promises her $1,000 in one year but the annual cash flow will decline by 2 percent forever. If her opportunity cost is 5 percent per year and both annuities cost $1,000, which annuity offers her the greater value?

71. Xiang wishes to have $1 million in 30 years. He cannot afford to make large deposits at the moment; however, he believes that he will be able to increase his deposits by 3 percent per year for the next 30 years. He will make his first deposit in one year. If his opportunity cost is 5 percent, how large an initial deposit is needed? If instead of increasing his deposit each year, Xiang invested the same amount each year, how large a deposit would he need to make each year?