1

SCATTERING PARAMETERS AND *ABCD* MATRICES

1.1 INTRODUCTION

In this chapter we recall the most important characterization techniques used in the design of microwave filters [1.1]. These consist of the scattering parameters, which are often based on electromagnetic analysis of the microwave structures, and the *ABCD* parameters, which are useful to make the link with two-port systems and have been studied exhaustively over the years. Several examples are presented to better understand the relations between the two formalisms. The bisection (or Bartlett) theorem is also reviewed and proves to be very useful in the case of symmetrical networks.

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By Pierre Jarry and Jacques Beneat

1.2 SCATTERING MATRIX OF A TWO-PORT SYSTEM

1.2.1 Definitions

The scattering matrix [1.2] of a two-port system provides relations between the input and output reflected waves b_1 and b_2 and the input and output incident waves a_1 and a_2 when the structure is to be connected to a source resistance R_G and a load resistance R_L, as depicted in Figure 1.1. The notion of waves rather than voltages and currents is better suited for microwave structures.

For a two-port system, the equations relating the incident and reflected waves and the S parameters are given by

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

These equations can be summarized in the matrix form $(b) = (S)(a)$, where

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

The parameter S_{11}, called the *input reflection coefficient*, can be computed by setting the output incident wave a_2 to zero and taking the ratio of the input reflected wave over the input incident wave:

$$S_{11} = \left. \frac{b_1}{a_1} \right|_{a_2=0}$$

The output incident wave a_2 is set to zero by connecting the output of the system to the reference resistor R_L. The parameter S_{11} provides a measure of how much of the input incident wave does not reach the output of the system and is reflected back at the input. For microwave filters, ideally, S_{11} should be equal to zero in the passband of the filter.

The parameter S_{21}, called the *forward transmission coefficient*, can be computed by setting the output incident wave a_2 to zero and taking the ratio of the output reflected wave over the input incident wave:

$$S_{21} = \left. \frac{b_2}{a_1} \right|_{a_2=0}$$

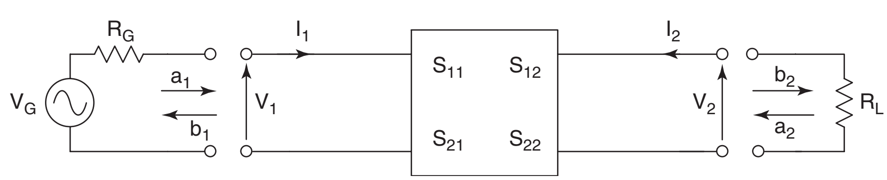

Figure 1.1 Notation used in defining the scattering matrix of a two-port system.

The output incident wave a_2 is set to zero by connecting the output to the reference resistor R_L. The parameter S_{21} provides a measure of how much of the input incident wave reaches the output of the system. For microwave filters, ideally, S_{21} should be equal to 1 in the passband of the filter.

The parameter S_{22}, called the *reflection coefficient at the output of the system*, can be computed by setting the input incident wave a_1 to zero and taking the ratio of the output reflected wave over the output incident wave:

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$

The input incident wave a_1 is set to zero by connecting the input of the system to the reference resistor R_G. As in the case of S_{11}, it is desirable that S_{22} be kept close to zero in the passband of the filter. S_{11} and S_{22} provide a measure of how well the system impedances are matched to the reference terminations.

The parameter S_{12}, called the *reverse transmission coefficient*, can be computed by setting the input incident wave a_1 to zero and taking the ratio of the input reflected wave over the output incident wave:

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0}$$

The input incident wave a_1 is set to zero by connecting the input of the system to the reference resistor R_G. The parameter S_{12} provides a measure of how much of an incident wave set at the output of the system would reach the input. Due to symmetries in the system, S_{21} and S_{12} can have similar values. Since there are no generators at the output of the system, an output incident wave could appear due to a poor S_{22}.

The scattering parameters can be illustrated using a graph, as shown in Figure 1.2. The graph shows that part of the incident wave a_1 results in a reflected wave b_1 through the parameter S_{11}, and in a transmitted wave b_1 through the parameter S_{21}. Similar descriptions can be given for a_2 and the parameters S_{22} and S_{12}. It is always important to remember that the S-parameter values are associated with a given set of termination values. Changing the termination values will change the S-parameter values.

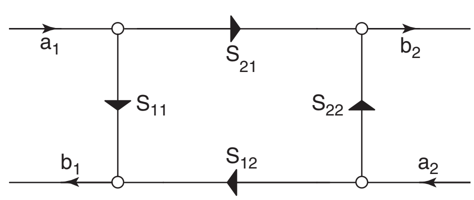

Figure 1.2 Graph of a two-port scattering matrix.

1.2.2 Computing the *S* Parameters

A common example of a scattering matrix in microwave is that of a waveguide of length l_0 and characteristic impedance Z_0, as shown in Figure 1.3. When the structure is to be connected to a source and load resistance equal to the characteristic impedance of the waveguide, the scattering matrix is given by

$$(S) = \begin{pmatrix} 0 & e^{-j\beta l_0} \\ e^{-j\beta l_0} & 0 \end{pmatrix}$$

where $j\beta$ is the propagation function of a given mode above the cutoff frequency of the waveguide. This matrix tells us that the structure will be perfectly matched to the terminations since S_{11} and S_{22} are equal to zero. It also tells us that b_1 the wave transmitted, will simply be a delayed version of a_1, the incident wave, since the forward transmission coefficient, S_{21}, has a magnitude of 1 and a linear phase of $-\beta l_0$, and the longer the length, the longer the delay. Since we cannot differentiate one end of a waveguide from the other, we would have similar results if connecting the source to the output and the load to the input (e.g., $S_{12} = S_{21}$).

As will be seen, microwave structures will at times have discontinuities that result in the apparition of "scattered" and unwanted electromagnetic fields. For these cases, matching the electromagnetic fields on each side of the discontinuity will provide relations that can be used for defining the scattering parameters of the discontinuity. In that case, the scattering parameters will be defined directly from electromagnetic wave equations. It should be noted, however, that scattering parameters are not limited to microwave structures and electromagnetic field equations.

The incident and reflected waves can be expressed in terms of voltages and currents, as shown in Figure 1.1.

$$a_1 = \frac{V_1 + R_G I_1}{2\sqrt{R_G}} \qquad a_2 = \frac{V_2 + R_L I_2}{2\sqrt{R_L}}$$

$$b_1 = \frac{V_1 - R_G I_1}{2\sqrt{R_G}} \qquad b_2 = \frac{V_2 - R_L I_2}{2\sqrt{R_L}}$$

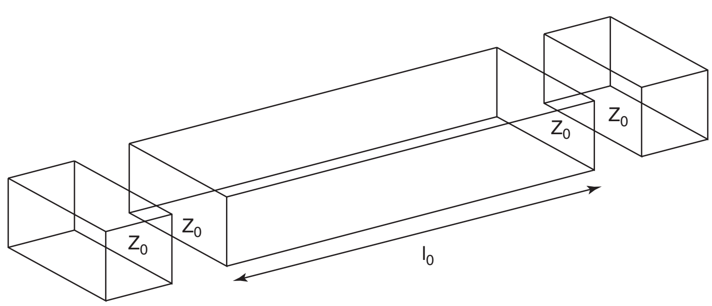

Figure 1.3 Waveguide of length l_0 and characteristic impedance Z_0.

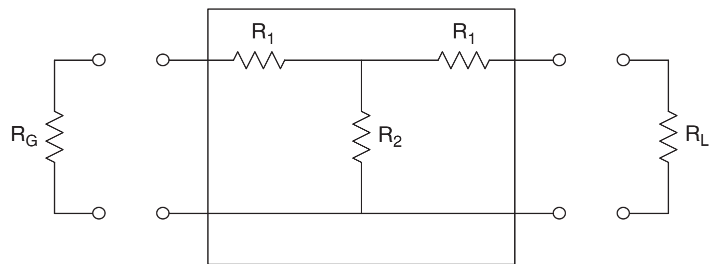

Figure 1.4 Defining the scattering parameters of a resistive two-port system.

For example, the scattering parameters of the resistive two-port system in Figure 1.4 can be defined from these voltages and currents.

The input reflection coefficient S_{11} is defined from the input incident and reflected waves when the system is connected to the reference resistor R_L, as shown in Figure 1.5. Also shown in the figure, the system connected to resistor R_L can be modeled as Z_{in}, an input impedance of the system. In this case, $V_1 = Z_{\text{in}} I_1$ and S_{11} is given by

$$S_{11} = \frac{V_1 - R_G I_1}{V_1 + R_G I_1} = \frac{Z_{\text{in}} I_1 - R_G I_1}{Z_{\text{in}} I_1 + R_G I_1} = \frac{Z_{\text{in}} - R_G}{Z_{\text{in}} + R_G}$$

This gives $Z_{\text{in}} = R_1 + R_2 || (R_1 + R_L)$ for the input impedance of the system. For S_{11} to be equal to zero, the input impedance Z_{in} should be equal to the source resistor R_G.

The forward transmission coefficient S_{21} is defined from the input incident wave and output reflected wave when the system is connected to the reference resistor R_L, as shown in Figure 1.6. Replacing the incident and reflected waves by their voltage and current expressions, S_{21} is given by

$$S_{21} = \sqrt{\frac{R_G}{R_L}} \frac{V_2 - R_L I_2}{V_1 + R_G I_1}$$

Also from the computations of S_{11}, $V_1 = Z_{\text{in}} I_1$ when the system is connected to R_L. From Figure 1.6 it is also seen that $V_2 = -R_L I_2$. Therefore, S_{21} will be

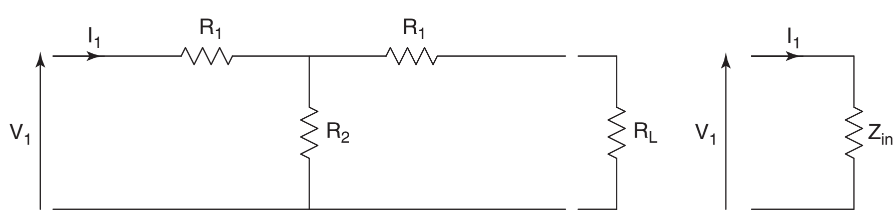

Figure 1.5 Defining the input reflection coefficient S_{11}.

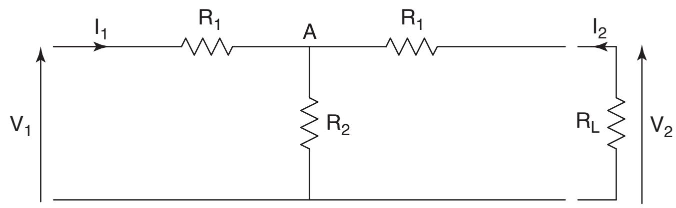

Figure 1.6 Defining the forward transmission coefficient S_{21}.

given by

$$S_{21} = \sqrt{\frac{R_G}{R_L}}\frac{V_2 - (-V_2)}{V_1 + (R_G/Z_{\text{in}})V_1} = 2\sqrt{\frac{R_G}{R_L}}\frac{1}{1 + R_G/Z_{\text{in}}}\frac{V_2}{V_1}$$

Note that in the case where $Z_{\text{in}} = R_G$ (input impedance matching) and $R_G = R_L$ (similar source and load terminations), the forward coefficient reduces to

$$S_{21} = \frac{V_2}{V_1}$$

In Figure 1.6,

$$V_2 = \frac{R_L}{R_L + R_1}V_A \quad \text{and} \quad V_A = \frac{R}{R + R_1}V_1$$

where $R = R_2||(R_1 + R_L)$, so that

$$V_2 = \frac{R_L}{R_L + R_1}\frac{R}{R + R_1}V_1$$

and a general expression for S_{21} is given by

$$S_{21} = 2\sqrt{\frac{R_G}{R_L}}\frac{1}{1 + R_G/Z_{\text{in}}}\frac{R_L}{R_L + R_1}\frac{R}{R + R_1}$$

The S_{22} and S_{12} parameters can be defined using a similar process, where the input is now connected to the reference resistor R_G. In the resistive example above, the S parameters are independent of frequency since the impedances of the resistors are independent of frequency. However, the results can be used to define the S parameters of a more general case, as shown in Figure 1.7. The input reflection coefficient S_{11} will now be a function of the impedances of the system and therefore depend on the frequency of application through the Laplace variable s:

$$S_{11}(s) = \frac{Z_{\text{in}}(s) - R_G}{Z_{\text{in}}(s) + R_G}$$

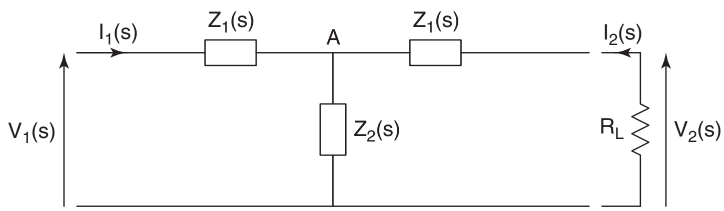

Figure 1.7 Defining the scattering parameters of a general two-port system.

where the input impedance $Z_{\text{in}}(s)$ is given by $Z_{\text{in}}(s) = Z_1(s) + Z_2(s)||[Z_1(s) + R_L]$.

The forward transmission coefficient S_{21} will also depend on the frequency of operation and is given by

$$S_{21}(s) = 2\sqrt{\frac{R_G}{R_L}}\frac{1}{1 + R_G/Z_{\text{in}}(s)}\frac{R_L}{R_L + Z_1(s)}\frac{Z(s)}{Z(s) + Z_1(s)}$$

where the impedance $Z(s) = Z_2(s)||[Z_1(s) + R_L]$. This means that the S parameters will generally have different values depending on the frequency at which they are being evaluated. In the case of a microwave filter, ideally, $S_{21}(f)$ should be equal to 1 at the frequencies of the passband of the filter and be equal to zero at the frequencies of the stopband of the filter. Similarly, $S_{11}(f)$ should be equal to zero at the frequencies of the passband of the filter and be equal to 1 at the frequencies of the stopband of the filter. Figure 1.8 shows the forward transmission coefficient S_{21} and the input reflection coefficient S_{11} versus frequency for a fifth-order Chebyshev filter with passband edges at 10 and 11 GHz and specified a −20-dB return loss.

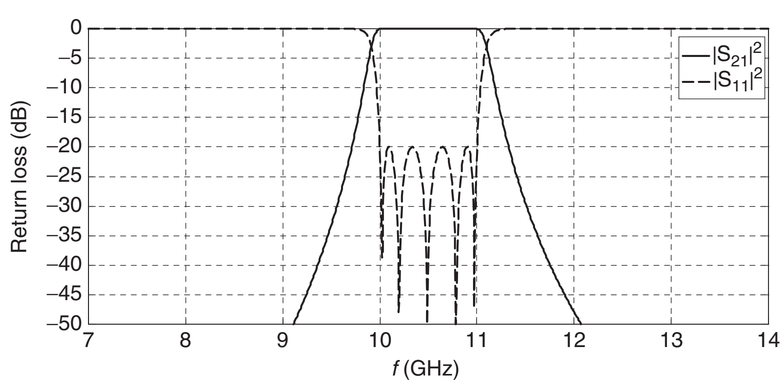

Figure 1.8 Frequency dependence of the scattering parameters.

1.2.3 *S*-Parameter Properties

Depending on the properties of the S parameters, the structures can be classified into the following categories:

- *Reciprocity*. A two-port system is said to be *reciprocal* if the S matrix is equal to its transpose:

$$(S) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = (S)^{\mathrm{T}} = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}$$

  In other words, the forward transmission coefficient and the reverse transmission coefficient are equal (e.g., $S_{21} = S_{12}$).
- *Symmetry*. A two-port system is said to be *symmetrical* if in addition to the reciprocity, the input and output reflection coefficients are identical (e.g., $S_{11} = S_{22}$), and *antisymmetrical* if they are opposite in sign (e.g., $S_{11} = -S_{22}$).
- *Lossless*. A two-port system is said to be *lossless* if power is conserved. In this case the complex conjugate of the S matrix is equal to its transpose:

$$(S)^* = \begin{pmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{pmatrix} = (S)^{\mathrm{T}} = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix}$$

In the case of lossless structures, additional relations exist between the transmission and reflection coefficients:

$$|S_{11}|^2 + |S_{12}|^2 = 1$$
$$|S_{21}|^2 + |S_{22}|^2 = 1$$
$$S_{11}S_{21}^* + S_{22}^*S_{12} = 0$$

In the case of a lossless and reciprocal two-port system, the input reflection coefficient and the forward transmission coefficient are such that $|S_{11}|^2 = 1 - |S_{21}|^2$. An example of this is shown in Figure 1.8. Additional results concerning lossless systems are given in Appendix 1.

1.3 *ABCD* MATRIX OF A TWO-PORT SYSTEM

There are several benefits of using the *ABCD* matrix representation when designing microwave filters [1.3,1.4]. They allow simulating entire structures made of a cascade of lumped elements, such as capacitors, inductors, and transformers. Lumped ladder structures are available in the literature for providing specified filtering responses. In addition, S parameters can be converted to *ABCD* parameters, and vice versa. When analyzing a microwave structure element, it is easier to

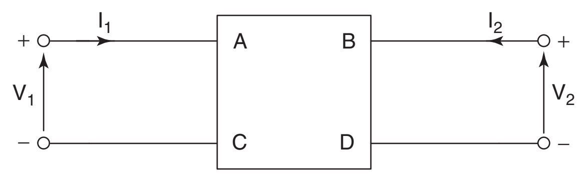

Figure 1.9 Notation used in defining the *ABCD* matrix of a two-port system.

model the element as a combination of lumped elements rather than interpreting the *S* parameters directly.

The *ABCD* matrix of a two-port is defined using voltages and currents as shown in Figure 1.9. The *ACBD* matrix is defined by

$$\begin{aligned} V_1 &= AV_2 + B(-I_2) \\ I_1 &= CV_2 + D(-I_2) \end{aligned}$$

In matrix form this provides the relation

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

The *ABCD* matrices of some of the most basic elements found in microwave structures are described next.

1.3.1 *ABCD* Matrix of Basic Elements

A common element found in many microwave filter design problems consists of a single impedance *Z* placed in series, as shown in Figure 1.10. The equations and the *ABCD* matrix of a series impedance are

$$\begin{aligned} I_1 &= (-I_2) \\ V_1 &= V_2 + Z(-I_2) \end{aligned} \quad \text{or} \quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

Another common element consists of an admittance *Y* placed in parallel, as shown in Figure 1.11. The equations and *ABCD* matrix of a parallel admittance are

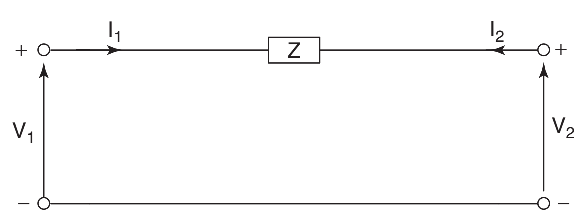

Figure 1.10 *ABCD* matrix of a series impedance.

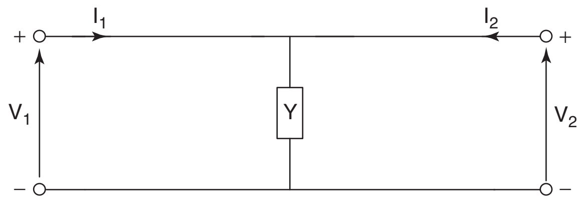

Figure 1.11 *ABCD* matrix of a parallel admittance.

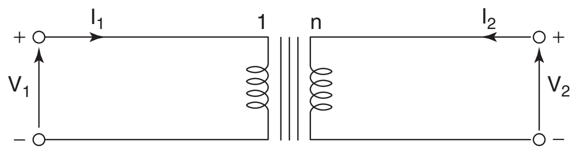

Figure 1.12 *ABCD* matrix of an ideal transformer.

$$\begin{aligned} V_1 &= V_2 \\ YV_2 &= I_1 + I_2 \end{aligned} \quad \text{or} \quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

Another common element encountered in microwave systems is the ideal transformer shown in Figure 1.12. The equations and *ABCD* matrix of the ideal transformer are

$$\begin{aligned} V_2 &= nV_1 \\ (-I_2) &= \frac{I_1}{n} \end{aligned} \quad \text{or} \quad \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & n \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

1.3.2 Cascade and Multiplication Property

One of the main advantages of the *ABCD* representation is that the *ABCD* matrix of a system made of the cascade of two systems, as shown in Figure 1.13, is equal to the multiplication of the individual *ABCD* matrices. The equations of this system are given by

$$\begin{aligned} V_1 &= A_1V_2 + B_1(-I_2) \\ I_1 &= C_1V_2 + D_1(-I_2) \end{aligned} \quad \text{and} \quad \begin{aligned} V_1' &= A_2V_2' + B_2(-I_2') \\ I_1' &= C_2V_2' + D_2(-I_2') \end{aligned}$$

but since $V_2 = V_1'$ and $(-I_2) = I_1'$ we have

$$\begin{cases} V_1 = A_1(A_2V_2' + B_2(-I_2')) + B_1(C_2V_2' + D_2(-I_2')) \\ \quad = (A_1A_2 + B_1C_2)V_2' + (A_1B_2 + B_1D_2)(-I_2') \\ I_1 = C_1(A_2V_2' + B_2(-I_2')) + D_1(C_2V_2' + D_2(-I_2')) \\ \quad = (C_1A_2 + D_1C_2)V_2' + (C_1B_2 + D_1D_2)(-I_2') \end{cases}$$

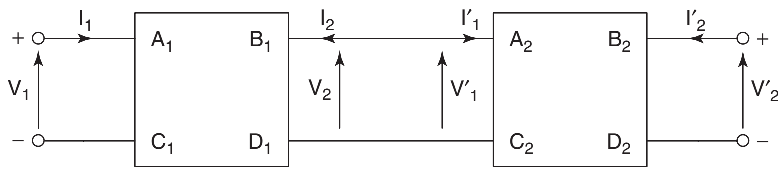

Figure 1.13 Cascade of two *ABCD* systems.

On the other side, if we multiply the *ABCD* matrices, we find that

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}$$

Since both techniques provide the same answers, in the future the *ABCD* matrix of the cascade of two systems will be given by multiplication of their *ABCD* matrices. This is very powerful since it will be easier to define individual *ABCD* matrices of a microwave structure and then multiply these matrices for simulating the entire structure.

Application to a T Network A *T network* is often described as three impedances forming a structure that looks like a T, as shown in Figure 1.14. In this case the *ABCD* matrix is given by the multiplication of three *ABCD* matrices (impedance Z_1, admittance $1/Z_2$, and impedance Z_3):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & Z_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{1}{Z_2} & 1 \end{pmatrix}\begin{pmatrix} 1 & Z_3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{Z_1}{Z_2} & Z_1 + Z_3 + \dfrac{Z_1 Z_3}{Z_2} \\ \dfrac{1}{Z_2} & 1 + \dfrac{Z_3}{Z_2} \end{pmatrix}$$

Application to a* Π *Network A Π *network* is often described as three admittances forming a structure that looks like a Π, as shown in Figure 1.15. In this case, the *ABCD* matrix is given by the multiplication of three *ABCD* matrices

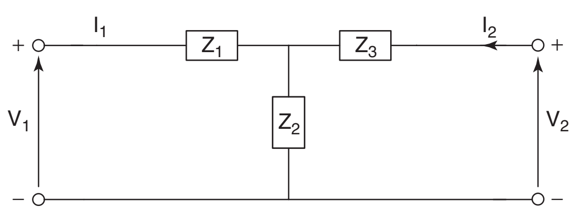

Figure 1.14 T network.

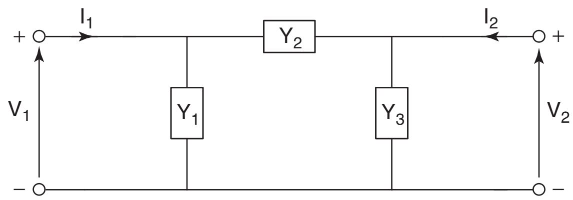

Figure 1.15 Π network.

(admittance Y_1, impedance $1/Y_2$, and admittance Y_3):

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Y_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \dfrac{1}{Y_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{Y_3}{Y_2} & \dfrac{1}{Y_2} \\ Y_1 + Y_3 + \dfrac{Y_1 Y_3}{Y_2} & 1 + \dfrac{Y_1}{Y_2} \end{pmatrix}$$

1.3.3 Input Impedence of a Loaded Two-Port

When a two-port is connected to a load Z_L(s) as shown in Figure 1.16, the output voltage V_2 and current I_2 will be such that $V_2 = Z_L(-I_2)$. The *ABCD* equations then become

$$\begin{aligned} V_1 &= AV_2 + B(-I_2) = AZ_L(-I_2) + B(-I_2) \\ I_1 &= CV_2 + D(-I_2) = CZ_L(-I_2) + D(-I_2) \end{aligned}$$

from which it is straightforward to extract the input impedance of the two-port network:

$$Z_{\text{in}} = \left.\frac{V_1}{I_1}\right|_{Z_L} = \frac{AZ_L + B}{CZ_L + D}$$

1.3.4 Impedance and Admittance Inverters

An *impedance inverter* is a two-port network that can provide an input impedance that is the inverse of the load impedance. This property is illustrated in

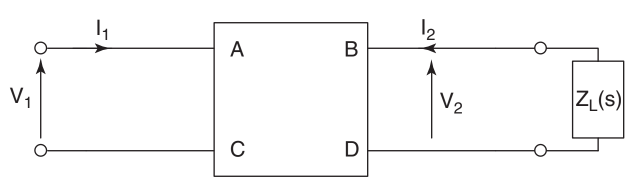

Figure 1.16 Two-port terminated on a load impedance $Z_L(s)$.

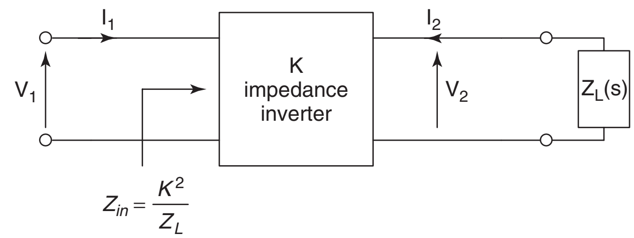

Figure 1.17 Impedance inverter principles.

Figure 1.17. An *ideal impedance inverter* will have an *ABCD* matrix of the form

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 0 & jK \\ \dfrac{j}{K} & 0 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

The input impedance of the two-port network when connected to a load impedance $Z_L(s)$ will be given by

$$Z_{\text{in}} = \left.\frac{V_1}{I_1}\right|_{Z_L} = \frac{AZ_L + B}{CZ_L + D} = \frac{0Z_L + jK}{(j/K)Z_L + 0} = \frac{K^2}{Z_L}$$

An *admittance inverter* is a two-port network that can provide an input admittance that is the inverse of the load admittance. This property is illustrated in Figure 1.18. An *ideal admittance inverter* will have an *ABCD* matrix of the form

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{j}{J} \\ jJ & 0 \end{pmatrix} \begin{pmatrix} V_2 \\ (-I_2) \end{pmatrix}$$

The input admittance of the two-port network when connected to a load admittance $Y_L(s)$ will be given by

$$Y_{\text{in}} = \left.\frac{I_1}{V_1}\right|_{Y_L} = \frac{CV_2 + DY_LV_2}{AV_2 + BY_LV_2} = \frac{C + DY_L}{A + BY_L} = \frac{jJ + 0Y_L}{0 + (j/J)Y_L} = \frac{J^2}{Y_L}$$

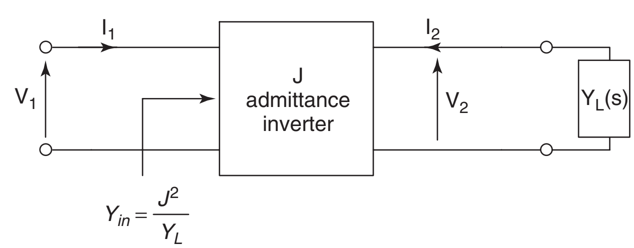

Figure 1.18 Admittance inverter principles.

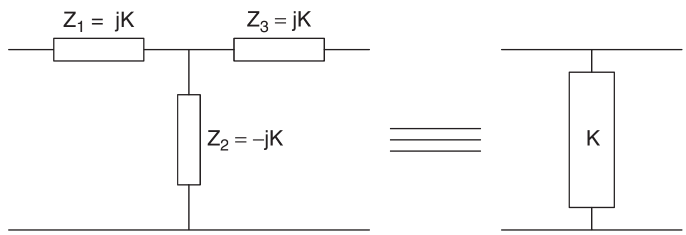

Figure 1.19 Realization of an ideal impedance inverter.

An ideal impedance inverter can be realized as a T network, as shown in Figure 1.19. This is shown by checking that the *ABCD* matrix of the T network, when $Z_1 = Z_3 = jK$ and $Z_2 = -jK$ is that of an impedance inverter:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{Z_1}{Z_2} & Z_1 + Z_3 + \dfrac{Z_1 Z_3}{Z_2} \\ \dfrac{1}{Z_2} & 1 + \dfrac{Z_3}{Z_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \dfrac{jK}{-jK} & jK + jK + \dfrac{(jK)(jK)}{-jK} \\ \dfrac{1}{-jK} & 1 + \dfrac{jK}{-JK} \end{pmatrix} = \begin{pmatrix} 0 & jK \\ \dfrac{j}{K} & 0 \end{pmatrix}$$

An ideal admittance inverter can be constructed using the Π network of Figure 1.20. This is shown by checking that the *ABCD* matrix of a Π network when $Y_1 = Y_3 = jJ$ and $Y_2 = -jJ$ is that of an admittance inverter:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{Y_3}{Y_2} & \dfrac{1}{Y_2} \\ Y_1 + Y_3 + \dfrac{Y_1 Y_3}{Y_2} & 1 + \dfrac{Y_1}{Y_2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \dfrac{jJ}{-jJ} & \dfrac{1}{-jJ} \\ jJ + jJ + \dfrac{(jJ)(jJ)}{-jJ} & 1 + \dfrac{jJ}{-jJ} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{j}{J} \\ jJ & 0 \end{pmatrix}$$

An impedance inverter can be approximated using two identical inductors and a capacitor as shown in Figure 1.21. The *ABCD* matrix of this T network is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 - LC\omega^2 & jL\omega(2 - LC\omega^2) \\ jC\omega & 1 - LC\omega^2 \end{pmatrix}$$

At the frequency ω_0 where $LC\omega_0^2 = 1$, this T network behaves as an ideal impedance inverter $K = L\omega_0$.

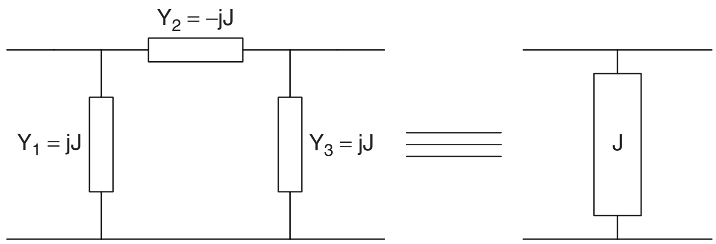

Figure 1.20 Realization of an ideal admittance inverter.

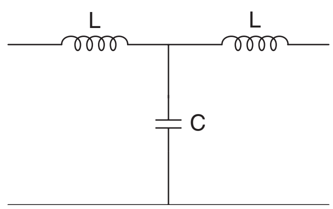

Figure 1.21 Approximate realization of an impedance inverter.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\Bigg|_{\omega=\omega_0} = \begin{pmatrix} 0 & jL\omega_0 \\ \dfrac{j}{L\omega_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & jK \\ \dfrac{j}{K} & 0 \end{pmatrix}$$

As will be seen in the filter design chapters, one will often represent a microwave structure as an equivalent circuit based on impedance or admittance inverters. In some cases it will be possible to greatly reduce or even remove the frequency dependence of the inverter. For other cases, the ideal inverter behavior can only be assumed around the center frequency of the filter (narrowband designs).

1.3.5 *ABCD*-Parameter Properties

Depending on the properties of the *ABCD* matrix, the structures can be classified into the following categories:

- *Reciprocal*. In this case, the determinant of the ABCD matrix is equal to unity:

$$AD - BC = 1$$

- *Symmetrical*. In this case, the parameters A and D are equal:

$$A = D$$

1.4 CONVERSION FROM FORMULATION *S* TO *ABCD* AND *ABCD* TO *S*

The *ABCD* matrix can be defined from the *S* parameters and termination conditions using the following conversion equations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{R_G}{R_L}}\dfrac{(1+S_{11})(1-S_{22})+S_{21}S_{12}}{2S_{21}} & \sqrt{R_G R_L}\dfrac{(1+S_{11})(1+S_{22})-S_{21}S_{12}}{2S_{21}} \\ \dfrac{1}{\sqrt{R_G R_L}}\dfrac{(1-S_{11})(1-S_{22})-S_{21}S_{12}}{2S_{21}} & \sqrt{\dfrac{R_L}{R_G}}\dfrac{(1-S_{11})(1+S_{22})+S_{21}S_{12}}{2S_{21}} \end{pmatrix}$$

The *S* matrix can be defined from the *ABCD* parameters and termination conditions using the following conversion equations:

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{AR_L+B-CR_GR_L-DR_G}{AR_L+B+CR_GR_L+DR_G} & \dfrac{2\sqrt{R_GR_L}(AD-BC)}{AR_L+B+CR_GR_L+DR_G} \\ \dfrac{2\sqrt{R_GR_L}}{AR_L+B+CR_GR_L+DR_G} & \dfrac{-AR_L+B-CR_GR_L+DR_G}{AR_L+B+CR_GR_L+DR_G} \end{pmatrix}$$

From these definitions it is possible to check the effects of adding redundant elements such as impedance inverters at the input and output of a system on the scattering parameters. This is done in Appendix 2.

1.5 BISECTION THEOREM FOR SYMMETRICAL NETWORKS

When a network is symmetrical, it can be modeled as the cascade of a half network and a reverse half network, as shown in Figure 1.22. In this case, the *ABCD* matrix of the symmetrical network is given by the product of the half network matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} D_1 & B_1 \\ C_1 & A_1 \end{pmatrix} = \begin{pmatrix} A_1D_1+B_1C_1 & 2A_1B_1 \\ 2C_1D_1 & A_1D_1+B_1C_1 \end{pmatrix}$$

This matrix corresponds to a symmetrical network since $A = D$. Furthermore, the symmetrical network is reciprocal [e.g., $AD - BC = (A_1D_1 - B_1C_1)^2 = 1$] when the half network is reciprocal [e.g., $A_1D_1 - B_1C_1 = 1$].

If we define the even-mode input impedance of the half network Z_e as

$$Z_e = \left.\frac{V_1}{I_1}\right|_{\text{half-circuit open}} = \frac{A_1 \times \infty + B_1}{C_1 \times \infty + D_1} = \frac{A_1}{C_1}$$

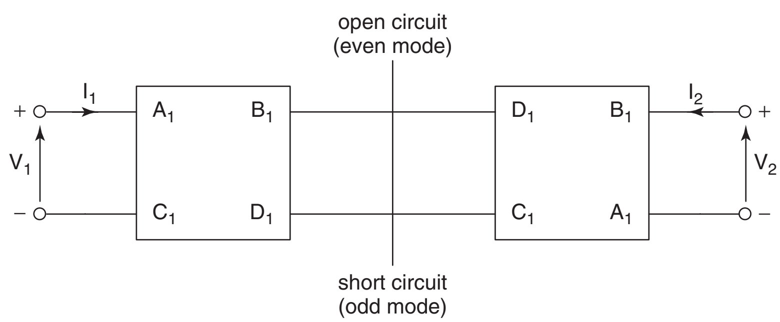

Figure 1.22 Decomposition of a symmetrical network.

and the odd-mode input impedance of the half network Z_o as

$$Z_o = \left.\frac{V_1}{I_1}\right|_{\text{half-circuit shorted}} = \frac{A_1 \times 0 + B_1}{C_1 \times 0 + D_1} = \frac{B_1}{D_1}$$

then for a reciprocal network (e.g., $A_1 D_1 - B_1 C_1 = 1$), the matrix of a symmetrical network can be expressed as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{Z_e - Z_o} \begin{pmatrix} Z_e + Z_o & 2Z_e Z_o \\ 2 & Z_e + Z_o \end{pmatrix}$$

Using the conversion formulas from $ABCD$ to S parameters, the S matrix of a symmetrical and a reciprocal network can be represented as

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{2(Z_e Z_o - R_G R_L) - (Z_e + Z_o)(R_G - R_L)}{2(Z_e + R_G)(Z_o + R_L) + (Z_e - Z_o)(R_G - R_L)} & \dfrac{2\sqrt{R_G R_L}(Z_e - Z_o)}{2(Z_e + R_G)(Z_o + R_L) + (Z_e - Z_o)(R_G - R_L)} \\ \dfrac{2\sqrt{R_G R_L}(Z_e - Z_o)}{2(Z_e + R_G)(Z_o + R_L) + (Z_e - Z_o)(R_G - R_L)} & \dfrac{2(Z_e Z_o - R_G R_L) + (Z_e + Z_o)(R_G - R_L)}{2(Z_e + R_G)(Z_o + R_L) + (Z_e - Z_o)(R_G - R_L)} \end{pmatrix}$$

When the terminations are equal, $R_G = R_L = R_0$, and using the normalized even and odd input impedances $z_e = Z_e/R_0$ and $z_o = Z_o/R_0$, the S matrix reduces to

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{z_e z_o - 1}{(z_e + 1)(z_o + 1)} & \dfrac{z_e - z_o}{(z_e + 1)(z_o + 1)} \\ \dfrac{z_e - z_o}{(z_e + 1)(z_o + 1)} & \dfrac{z_e z_o - 1}{(z_e + 1)(z_o + 1)} \end{pmatrix}$$

In addition, if we note that

$$S_{21} = S_{12} = \frac{1}{2}\frac{z_e - 1}{z_e + 1} - \frac{1}{2}\frac{z_o - 1}{z_o + 1} = \frac{1}{2}(S_{11e} - S_{11o})$$
$$S_{22} = S_{11} = \frac{1}{2}\frac{z_e - 1}{z_e + 1} + \frac{1}{2}\frac{z_o - 1}{z_o + 1} = \frac{1}{2}(S_{11e} + S_{11o})$$

the S parameters of a symmetrical network can be expressed in terms of the input reflection coefficient of the half network under open or shorted conditions:

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} S_{11e} + S_{11o} & S_{11e} - S_{11o} \\ S_{11e} - S_{11o} & S_{11e} + S_{11o} \end{pmatrix}$$

Similar formulations can be found using normalized even and odd input admittances $y_e = 1/z_e$ and $y_o = 1/z_o$.

The bisection theorem will be used in the case of symmetrical networks. Often, these networks will be composed of impedance or admittance inverters. The half network and even- and odd-mode impedances of an impedance inverter are given below.

In the case of an ideal impedance inverter K, the *ABCD* matrix can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & jK \\ \dfrac{j}{K} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & jK \\ \dfrac{j}{K} & 1 \end{pmatrix}^2$$

The half network of an impedance inverter K and its reverse half network can then be given by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & jK \\ \dfrac{j}{K} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} D_1 & B_1 \\ C_1 & A_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & jK \\ \dfrac{j}{K} & 1 \end{pmatrix}$$

We can check that the product of *ABCD* matrices of the half networks provides the *ABCD* matrix of the ideal inverter:

$$\begin{aligned}
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} D_1 & B_1 \\ C_1 & A_1 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 & jK \\ \dfrac{j}{K} & 1 \end{pmatrix}\begin{pmatrix} 1 & jK \\ \dfrac{j}{K} & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (1)(1) + (jK)\left(\dfrac{j}{K}\right) & (1)(jK) + (jK)(1) \\ \left(\dfrac{j}{K}\right)(1) + (1)\left(\dfrac{j}{K}\right) & \left(\dfrac{j}{K}\right)(jK) + (1)(1) \end{pmatrix} \\
&= \begin{pmatrix} 0 & jK \\ \dfrac{j}{K} & 0 \end{pmatrix}
\end{aligned}$$

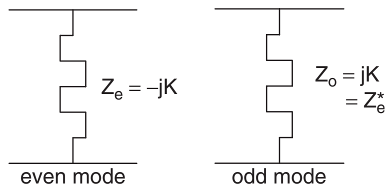

Figure 1.23 Even- and odd-mode models of an impedance inverter K.

The even-mode input impedance of the half network is given by $Z_e = A_1/C_1 = -jK$, and the odd-mode input impedance of the half network is given by $Z_o = B_1/D_1 = +jK$. The even- and odd-mode models of an impedance inverter are shown in Figure 1.23. For an admittance inverter J it can be shown that the even-mode input admittance of the half network is given by $Y_e = 1/Z_e = jJ$ and the odd-mode input admittance is given by $Y_o = Y_e^* = -jJ$.

1.6 CONCLUSIONS

In this chapter we recalled some of the fundamental relations for scattering and *ABCD* characterizations of two-port systems. We have also introduced the notion of impedance and admittance inverters, which are not easily realized using lumped elements but will be key in designing microwave filters. The bisection theorem has introduced the concept of even and odd modes that can be used to reduce the complexity of microwave filter design [1.5].

REFERENCES

[1.1] J. D. Rhodes, *Theory of Electrical Filters* Wiley, New York, 1976.

[1.2] H. Baher, *Synthesis of Electrical Networks*, Wiley, New York, 1984.

[1.3] J. S. Hong and M. J. Lancaster, *Microstrip Filters for RF/Microwave Applications*, Wiley, New York 2001.

[1.4] G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance-Matching Networks, and Coupling Structures*, McGraw-Hill, New York, 1964.

[1.5] P. Jarry, ed., Special Volume on Microwave Filters and Amplifiers, *Research Signpost*, 2005.