Squaring the Circle

"There is geometry in the humming of the strings . . .

. . . there is music in the spacing of the spheres."

PYTHAGORAS

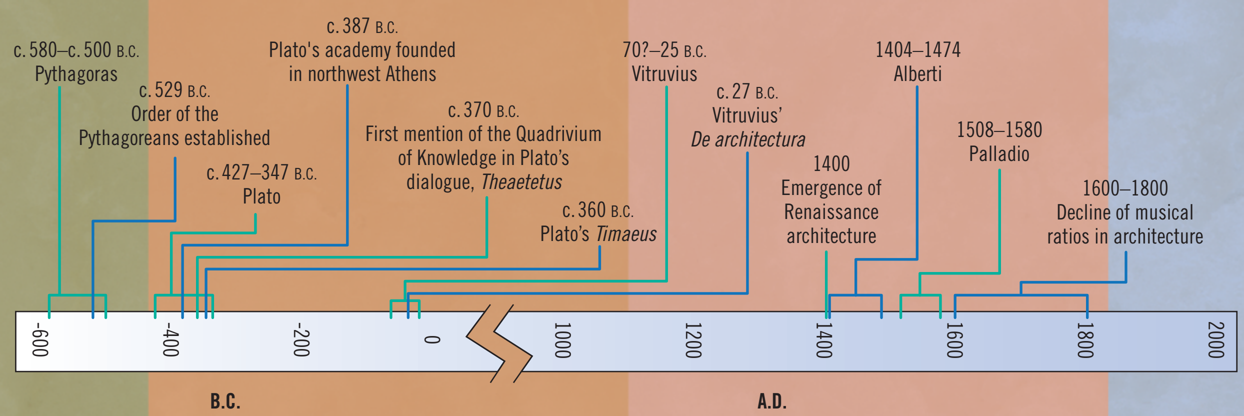
FIGURE 1.1 ▪ Pythagoras in Raphael's *School of Athens* (detail)

1

Music of the Spheres

We're about to set out on a wonderful journey. We will follow parallel paths through time, from ancient Egypt to the twentieth century; through foreign lands, their art and architecture and their people who have shaped mathematics throughout the centuries; through the geometric figures, from the humble zero-dimensional point to the fractionally dimensioned Sierpinski tetrahedron; and through the geometries, from ancient plane and solid Euclidean geometry to modern fractal geometry. Although the paths are different, we'll try to weave them together into one coherent journey. Along the way, you will be invited to perform simple mathematics exercises, create two- and three-dimensional artworks, write various papers, use a computer, and make a model. At times, you will be encouraged to work in a team or give a presentation to your class. Pick any of these activities that match your own interests.

Our journey begins with Pythagoras and his followers, the Pythagoreans (Figure 1.1), whose ideas dominate much of the material in this book. A great deal of what has been written about Pythagoras and his followers is more myth and legend than historical fact, and we will not try to separate fact from legend



here. Instead, we will introduce several discoveries and contributions the Pythagoreans made to present-day mathematics, keeping in mind that we are talking about what *possibly* happened in the fifth century B.C. and what is sometimes called the Pythagorean tradition. We will present a few mathematical ideas from what commentators have described as a much larger philosophical system encompassing ethics, logic, politics, religion, medicine, and so forth.

In this chapter, we will explore the Pythagorean idea of the music of the spheres, which is later used by Plato to describe the creation of the world and appears again in Renaissance architecture. In preparation for the study of the music of the spheres, we will introduce the mathematical concepts of ratio and proportion, which will be essential in demonstrating the origin of the musical ratios, and further, the mathematical construction of the musical scale. We will also explore sequences, means, and some basic geometric constructions that will help us understand how the same mathematical concepts were later used in Renaissance architecture. So let's begin our journey and make some fascinating mathematical discoveries along the way.

PYTHAGORAS
(c. 580–c. 500 B.C.)

Pythagoras was born in Ionia on the island of Samos. One of his teachers was Thales, who is called the father of Greek mathematics, astronomy, and philosophy and who was one of the Seven Sages of Greece.

Following Thales' advice, Pythagoras went to Egypt in his early twenties. According to Iamblichus, "he frequented all the temples . . . , [did] the most studious research . . . [and] he did not neglect any contemporary celebrity, whether a sage renowned for wisdom, or a peculiarly performed mystery. He did not fail to visit any place where he thought he might discover something worthwhile . . . [and] visited all the Egyptian priests, acquiring all the wisdom each possessed. Thus he passed twenty-two years in the sanctuaries of temples, studying astronomy and geometry, and being initiated . . . in all the mysteries of the gods."[1]

He spent another 12 years in Babylonia where he was "associated with the Magi . . . [and where] he studied and completed arithmetic, music, and all the other sciences." He returned to Samos at the age of 56.

PYTHAGORAS AND THE PYTHAGOREANS

According to Pythagorean legend, Pythagoras was one of the greatest philosophers and mathematicians of his time. He is said to have been driven from the island of Samos by his disgust for Polycrates, the tyrant of Samos. In 529 B.C., Pythagoras made the relatively short trip from Samos to Crotone, a Dorian Greek colony in southern Italy. There he gained a large following of students and disciples. He started an academy that gradually formed into a society or brotherhood called the *Order of the Pythagoreans*. The Pythagorean disciplines were said to include silence, music, incenses, physical and moral purification, rigid cleanliness, vegetarianism, pure linen clothes, self-denial, utter loyalty, common possessions, and secrecy, to prevent the Pythagoreans' knowledge from coming into the possession of the profane. These disciplines may have been the roots of disciplines for later monastic orders, such as the Jesuit *Rules of St. Ignatius*.

The works of Pythagoras are known only through the work of his disciples. The Pythagoreans relied on oral teaching, and perhaps their pledge of secrecy accounts for the lack of documents. The oral teachings were eventually committed to writing, but knowing just how many of the "Pythagorean" discoveries were made by Pythagoras himself is impossible because the tradition of later Pythagoreans ascribed everything to the Master.

Pythagorean Number Symbolism

Aristotle is perhaps the main source of information about the Pythagoreans. In his *Metaphysica*, he sums up the Pythagoreans' attitude toward numbers. "The [Pythagoreans were] . . . the first to take up mathematics . . . [and] thought its principles were the principles of all things. Since, of these principles, numbers . . . are the first, . . . in numbers they seemed to see many resemblances to the things that exist . . . more than [just] air, fire and earth and water, [but things such as] justice, soul, reason, opportunity. . . ."

The Pythagoreans did not recognize all the numbers we use today; they recognized only the positive whole numbers. Zero, negative numbers, and irrational numbers didn't exist in their system. Moreover, the Pythagoreans' idea of a number was different from the quantitative one we have today. Now we use a number to indicate a quantity, amount, or magnitude of something, but for the Pythagoreans, each number had its own particular attribute. For example, the number one, or unity, which they called the *monad*, was seen as the source of all numbers. "Unity is the principle of all things and the most dominant of all that is: All things emanate from it and it emanates from nothing. It is indivisible and . . . it is immutable and never departs from its own nature through multiplication ($1 \times 1 = 1$). Everything that is intelligible and not yet created exists in it. . . ."[2]

The number two, the *dyad*, represented duality, subject and object. The Pythagoreans believed the world to be composed of pairs of opposites, as given by Aristotle in this famous table.[3]

The Pythagorean Table of Opposites				
Limit, Unlimited	Odd, Even	One, Plurality	Right, Left	Male, Female
At Rest, Moving	Straight, Crooked	Light, Darkness	Good, Bad	Square, Oblong

With three, the *triad*, that dualism was resolved. The two extremes were united, giving *Harmonia*. This idea of *reconciliation of opposites* will appear again in the discussions of the golden mean and in Chapter 8, "Squaring the Circle." Below is a summary of some of the sometimes fanciful attributes the Pythagoreans gave to numbers:

1. *Monad*. Point. One is the source of all numbers. It is good, desirable, essential, and indivisible.
2. *Dyad*. Line. Two represents diversity, duality, a loss of unity, the number of excess and defect. It is the first feminine number.
3. *Triad*. Plane. By virtue of the triad, unity and diversity of which it is composed are restored to harmony. Three is the first odd, masculine number.
4. *Tetrad*. Solid. This is the first feminine square. It represents justice, and it is steadfast and square. Four is the number of the square, the elements, the seasons, ages of man, lunar phases, and virtues.
5. *Pentad.* This is the masculine marriage number, uniting the first female number and the first male number by addition. It is the number of fingers or toes on each limb and the number of regular solids or polyhedra. It is considered incorruptible because multiples of 5 end in 5.
6. *Hexad.* The first feminine marriage number, uniting 2 and 3 by multiplication. It is the area of a 3-4-5 triangle. It is the first *perfect number*; a number equal to the sum of its exact divisors or factors, except itself. Therefore, $1 + 2 + 3 = 6$.
7. *Heptad*. Seven is referred to as the virgin number, because 7 alone has no factors, and 7 is not a factor of any number within the Decad. Also, a circle cannot be divided into seven parts by any known construction.
8. *Ogdoad.* The first cube.
9. *Ennead.* The first masculine square. Nine is incorruptible, because when it is multiplied by any number it "reproduces" itself. For example, $9 \times 6 = 54$ and $5 + 4 = 9$. Try this again by multiplying 9 by any number, however large.
10. *Decad*. Ten is the number of fingers or toes on a human. It contains all of the numbers; after 10, the numbers merely repeat themselves. It is the sum of the archetypal numbers ($1 + 2 + 3 + 4 = 10$).

Odd numbers were considered masculine; even numbers were feminine. Odds were considered stronger than evens because (a) unlike an odd number, when an even number is halved it has nothing in the center; (b) odd plus even always gives odd; and (c) two evens can never produce an odd, while two odds produce an even. Because the birth of a son was considered more fortunate than the birth of a daughter, odd numbers became associated with good luck. "The gods delight in odd numbers," wrote Virgil in his *Eclogue* viii.

The Renaissance architect Leon Battista Alberti later wrote about odd and even numbers in architecture. "[The Ancients] observed, as to Number, was that it was of two Sorts, even and uneven, and they made use of both . . . for they never made the Ribs of their Structure, that is to say, the Columns, Angles and the like, in uneven Numbers; as you shall not find any Animal that stands or moves about upon an odd Number of Feet. On the contrary, they made their Apertures always in uneven Numbers, as Nature herself has done in some Instances . . . the great Aperture, the Mouth, she has set singly in the Middle."[4] We will have more to say about Alberti later in this chapter.

Figured Numbers

The Pythagoreans represented numbers by using patterns of dots, possibly as a result of arranging pebbles into patterns, as shown in Figure 1.2. There were square numbers, rectangular numbers, and triangular numbers.

Associated with figured numbers is the idea of the gnomon. *Gnomon* means "carpenter's square" in Greek. It is the name given to the upright stick on a sundial. For now, we will use the word *gnomon* as the Pythagoreans did, to refer to an L-shaped border appended to a figured number.

In Figure 1.3a successive L-shaped borders, or gnomons, are added to the monad. To get some idea of Pythagorean thinking, first note that for the monad, the number of points in each gnomon is odd and each successive figure formed is a square. In Figure 1.3b, successive gnomons are added to the dyad. The number of points in each gnomon added to the dyad is even, and each successive figure is a rectangle or oblong. Therefore, the Pythagoreans, in their Table of Opposites, associated odd and even with square and oblong, respectively. Further, each gnomon about the monad forms a square, a stable form whose ratio of width to height never changes. By contrast, each gnomon about the dyad forms a rectangle whose ratio of width to height changes each time. As such, the Pythagoreans associated *limited* with odd and *unlimited* with even.

From these patterns, the Pythagoreans derived relationships between numbers that may have led to the discovery of geometrical theorems. For example, noting that a square number can be subdivided by a diagonal line into two triangular numbers, we can say that a square number is always the sum of two triangular numbers. For example,

$$25 = 10 + 15.$$

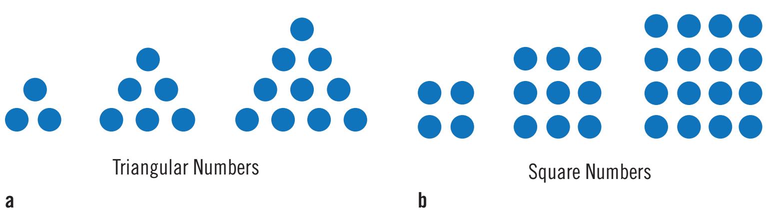


FIGURE 1.2 ▪ Some Figured Numbers

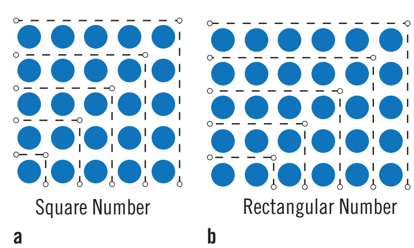


FIGURE 1.3 ▪ Gnomons

The observation that figured numbers follow certain patterns may have furthered the Pythagorean belief that the study of numbers would lead to the discovery of universal laws.

The Sacred Tetraktys

The triangular number ten, or decad, was especially important to the Pythagoreans and was called the *Sacred Tetraktys*. The prefix *tetra-* means four, and the word *Tetraktys* means a "set of four things." Ten dots form a neat equilateral triangle with four dots on each side, as shown in Figure 1.4.

"I swear by him who brought us the Tetraktys, which is the source and root of everlasting nature."
Pythagorean Oath

Ten is important because it is, of course, the number of fingers and the base of the decimal number system. The Pythagoreans also saw significance in ten being the sum of the first four integers:

$$1 + 2 + 3 + 4 = 10.$$

Recall that in Pythagorean number symbolism, the number *one* represented the point, *two* the line, *three* the surface, and *four* the solid. To the Pythagoreans, the Tetraktys represented the continuity linking the dimensionless point with the solid body.

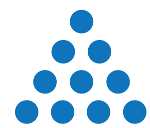


FIGURE 1.4 ■ **The Sacred Tetraktys**

In addition, the Tetraktys, like other triangular numbers, is composed of both odd and even integers (1, 2, 3, and 4). This is in contrast to the square, which is composed of consecutive odd integers only, and the rectangle, which is composed of consecutive even integers only. Because the Pythagoreans felt that the universe was composed of an interweaving of both odd and even, limited and unlimited, they associated the Tetraktys with the cosmos.

An important discovery attributed to the Pythagoreans is that the side and diagonal of a square are *incommensurable* (having no common measure). This means that there is no measure that is contained an integral number of times in both the side and the diagonal.

The word *Tetraktys* is attributed to Theon of Smyrna (A.D. 100), a Greek mathematician and astronomer. The Sacred Tetraktys was not the only interesting set of four. Here are the ten sets of four given by Theon:

Numbers	1	2	3	4
Geometry	point	line	surface	solid
Elements	fire	air	water	earth
Solids	pyramid	octahedron	icosahedron	cube
Living things	seed	growth in length	growth in breadth	growth in thickness
Societies	man	village	city	nation
Faculties	reason	knowledge	opinion	sensation
Seasons	spring	summer	autumn	winter
Ages of a person	infancy	youth	adulthood	old age
Parts of living things	body	rationality	emotion	willfulness

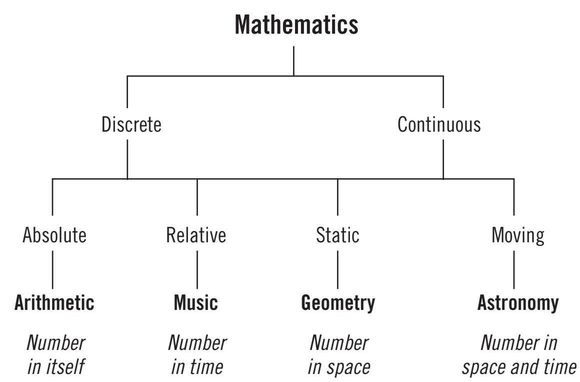
Whenever we have two or more groups containing the same number of things (say, four), the notion of connections between the groups naturally arises. *Correspondence* refers to the idea that any groups defined by the same number are somehow related. For example, Plato associated the four elements with the four solids (fire with the pyramid, earth with the cube, and so forth). According to Vincent Hopper, the notion of correspondences, such as the relating of the seven planets to the seven days of the week, originated in astrology.[5] That notion was a persistent idea throughout history and will be a recurring theme here. For example, Chapter 4, "Ad Quadratum and the Sacred Cut," will discuss more correspondences between groups of four.

The Quadrivium

Another group of four attributed to the Pythagoreans is the division of mathematics into four groups. This is the famous *Quadrivium of Knowledge*, the four subjects needed for a bachelor's degree in the Middle Ages. The first mention of the quadrivium may have been Plato's dialogue *Theaetetus*, c. 370 B.C.

> "Is Theodorus an expert in geometry?
> Of course he is, Socrates, very much so.
> And also in astronomy and arithmetic and music and in all the liberal arts?
> I am sure he is."

Later, we will discuss the *trivium*, which when combined with the quadrivium comprises the *seven liberal arts*. Each subject is represented in art by an allegorical



figure, and each is often personified by a particular person. Not surprisingly, arithmetic is personified by Pythagoras and geometry by Euclid.

Our journey will include all the subjects of the quadrivium. Arithmetic, geometry, and music will be included in this chapter; more arithmetic and geometry will be discussed throughout; and astronomy will be included in Chapter 11, "The Sphere and Celestial Themes in Art and Architecture."

RATIO AND PROPORTION

"Remove number from everything, and all will come to nothing."

ISODORE OF SEVILLE (560–636)

The geometric concepts of ratio and proportion and their occurrence in art and architecture appear throughout history. The Pythagoreans' musical ratios are discussed at length in this chapter, and the golden ratio, which we treat in detail in Chapter 2, "The Golden Ratio," recurs throughout our studies. In later

SCHOOL OF ATHENS
Raphael, 1510–1511

Although Raphael did not indicate the identities of the figures in this painting (Figure 1.5), scholars agree about whom most of them are. Other identities, based on a new analysis of the painting,[6] are controversial.

Socrates sprawls on the steps, hemlock cup nearby. His student Plato, the idealist, is at center left, pointing upward to divine inspiration. He holds his *Timaeus,* a book that will be discussed in Chapter 2, "The Golden Ratio," and Chapter 10, "The Solids." Beyond him are those philosophers that appealed to intuition and emotions.

Plato's student Aristotle, the man of good sense, is on Plato's left, holding his *Ethics* in one hand and holding out the other hand in a gesture of moderation. Beyond him are representatives of rational activities (logic, grammar, and geometry). As such, Raphael placed the big three of Greek philosophy center stage.

Crito and Apollodorus, Socrates' students, are to his right. They are displaying shock and disbelief at his death. Euclid is shown with a compass, lower right. Beyond Euclid, Ptolemy wears a crown and holds a terrestrial globe, while Zoroaster nearby holds a celestial globe. Diogenes, in black, upper right, stands beneath the statue of Athena, and Pericles, with helmet, stands upper left.

Of course, Raphael did not know what these philosophers looked like. He gave Euclid the face of the architect Bramante, and Plato bears a strong resemblance to Leonardo da Vinci (see Chapter 12, "Brunelleschi's Peepshow and the Origins of Perspective"). Raphael also put himself into the painting, facing the viewer on the right. In front of him is a likeness of Perugino, his friend and teacher.

The figure sitting on the front step, with an elbow on a stone block, resembles Michelangelo. Kenneth Clark points out that he seems to be painted in a different style, that of the Sistine Chapel ceiling. Michelangelo wouldn't let anyone into the Chapel; however, Bramante, who was architect of St. Peters at the time, had the key and he and Raphael entered when Michelangelo was away.[7]

FIGURE 1.5 ■ ***School of Athens*, Raphael, 1510–1511**

chapters, we'll also explore the ratios found in geometric shapes like triangles and rectangles, and we'll document their appearances in art and architecture as well. In this section, we'll provide some mathematical background for calculating ratio and proportion, which will be essential in preparing us for what lies ahead. We'll start our parallel journey through the geometric figures with the simplest, the point and the line.

A line subdivided into numbered increments, like a ruler, is called a *number line.*

Points and Lines

A *point* is a geometric element that has position but no dimensions. If the point P is thought of as moving to a new position Q, its path, called a *locus*, generates a *curve*, as shown in Figure 1.6a. If the point moves without changing direction, it generates a *straight line* (Figure 1.6b). We will simply say *line* when we mean a straight line and use *curve* for other lines.

The straight line has no thickness and extends infinitely far in both directions. A line is considered to have one *dimension*, while a point has zero dimensions. A point on a line divides the line into two half-lines or *rays*, as shown in Figure 1.7a. A *line segment* is the portion of the line between (and including) two points, called *endpoints*. The line segment in Figure 1.7b has endpoints P and Q. We will refer to this segment as line PQ or line QP.

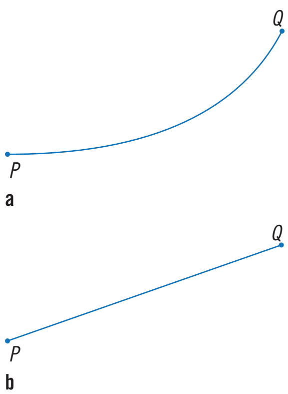


FIGURE 1.6 ■ A Moving Point

Ratios

A *ratio* is the quotient of two quantities (that is, one quantity divided by the other). Therefore, the ratio of P to Q is P/Q. A ratio can also be written using the colon (:), as shown here:

$$P : Q.$$

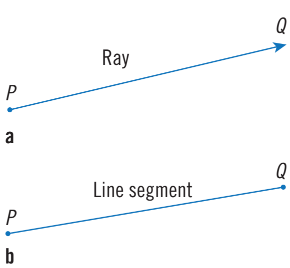


FIGURE 1.7 ■ Ray and Line Segment

■ **EXAMPLE:** What is the ratio of the smaller segment to larger segment in Figure 1.8?

● **SOLUTION:** The ratio is

$$\frac{ab}{bc}. \quad ■$$

FIGURE 1.8 ■ Division of a Line Segment

■ **EXAMPLE:** If a certain floor plan has a width of 7 yd (yards) and a length of 45 ft (feet), what is the ratio of width to length?

● **SOLUTION:**

$$\frac{7 \text{ yd}}{45 \text{ ft}}$$

When writing the ratio of two physical quantities, you can express both quantities using the *same* units, so that they cancel and leave the ratio *dimensionless*. In the previous example, both dimensions can be converted to yards. The units cancel and the result is a simpler dimensionless ratio:

$$\frac{7 \text{ yd}}{15 \text{ yd}} = \frac{7}{15}. \quad ■$$

In the following chapter, we will show how to divide a line segment by the golden ratio.

Proportions

When two ratios are set equal to each other, the result is a *proportion*. If the ratio of a to b equals the ratio of c to d, we have the proportion

$$\frac{a}{b} = \frac{c}{d}.$$

This is sometimes written using double colons:

$$a : b :: c : d.$$

Therefore, a proportion relates four quantities, a, b, c, and d. These quantities are called the *terms* of the proportion.

■ **EXAMPLE:** What is the proportion if the ratio of the height of a certain building to its width equals the ratio of 3 to 4?

● **SOLUTION:** The proportion is

$$\frac{\text{height}}{\text{width}} = \frac{3}{4}.$$ ■

You solve a proportion just as you would any other equation.

■ **EXAMPLE:** Find x in the proportion

$$\frac{3}{x} = \frac{7}{9}.$$

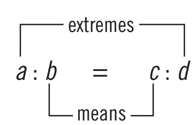
● **SOLUTION:** Clear fractions by multiplying by the common denominator $9x$:

$$27 = 7x$$

from which,

$$x = \frac{27}{7}.$$ ■

The two inside terms of a proportion are called the *means*, and the two outside terms are called the *extremes*.



When the means of a proportion are equal, as in

$$a : b = b : c,$$

the term b is called the *geometric mean* or *mean proportional* between a and c. You can find the mean proportional by solving for b. Multiplying both sides by bc gives

$$b^2 = ac.$$

10 GEOMETRIC MEAN OR MEAN PROPORTIONAL

$$\text{Geometric mean } b = \pm\sqrt{ac}$$ ●

A white numeral in a green box indicates a formula or a statement that is listed in Appendix F of this text for your reference. Although the formulas and statements appear consecutively in Appendix F, they will not usually appear in numerical order throughout the text.

■ **EXAMPLE:** Find the geometric mean between 3 and 12.

● **SOLUTION:** Using Equation 10,

$$b = \pm\sqrt{3(12)} = \pm 6.$$

Therefore, you get *two* answers, +6 and −6. You can see that both are correct by placing each into the proportion:

$$\frac{3}{6} = \frac{6}{12}$$

and

$$\frac{3}{-6} = \frac{-6}{12}.$$

Therefore, both 6 and −6 are the geometric means between 3 and 12. ■

Subdividing a Line Segment by a Given Ratio

Later, we will subdivide lines and other geometric figures into various ratios, such as the golden ratio. Here we show how, by example.

■ **EXAMPLE:** Subdivide a line segment of length 148 cm into the ratio of 5 : 9. Work to the nearest tenth of a centimeter.

● **SOLUTION:** Set up the proportion:

$$\frac{\text{shorter segment}}{\text{whole line}} = \frac{5 \text{ parts}}{(5 + 9) \text{ parts}}$$

$$\frac{\text{shorter segment}}{148 \text{ cm}} = \frac{5}{14}$$

$$\text{shorter segment} = \frac{5(148) \text{ cm}}{14} = 52.9 \text{ cm}$$

$$\text{longer segment} = 148 - 52.9 = 95.1 \text{ cm}$$ ■

The Rule of Three

Solving proportion problems was an important subject during the Renaissance. This skill was crucial to merchants, who had to deal with problems of pasturage, brokerage, discount, tare allowance, adulteration of commodities, barter, and currency exchange. Not only did every city have its own currency, but each had its own weights and measures. Chapter 10, "The Solids," will discuss how a merchant's ability to gauge volumes of solids helped him to relate to Renaissance paintings.

The *Rule of Three*, also called the *Golden Rule* and the *Merchant's Key*, was the universal mathematical tool of the literate commercial populace during the Renaissance. In his *Del abaco*, Piero della Francesca explained how to solve a proportion: "Multiply the thing one wants to know about by the thing that is dissimilar to it, and divide by the remaining thing. The result is dissimilar to the thing we want to know about."

One *braccio* is about $\frac{1}{3}$ of a person's height, or about 23 in.

■ **EXAMPLE:** Seven bracci of cloth are worth nine lire; how much will five bracci of cloth be worth?

● **SOLUTION:** The thing we want to know about = 5 bracci of cloth.
The thing dissimilar to it = 9 lire.
The remaining thing = 7 bracci of cloth.
Therefore,

$$\frac{5 \times 9}{7} = \frac{45}{7} = 6\frac{3}{7} \text{ lire.}$$

The units are lire, because lire are dissimilar to bracci, the units of the thing about which we wanted to know. ■

SKILLS IN PROPORTION AFFECTED THE WAY PEOPLE VIEWED ART AND ARCHITECTURE

So what does ability to solve proportions have to do with art and architecture? The British historian Michael Baxandall claims that the skills used to solve exchange problems during the Renaissance were the same as those used to make or see pictures. He makes the following points in his text, *Painting and Experience in Fifteenth-Century Italy.*

- Renaissance education placed exceptional value on a few mathematical skills, such as the ability to compute proportions. People did not know more mathematics than we do; however, what they did know, they understood well. Mathematics was a relatively larger part of their intellectual equipment.
- The math skills used by merchants were the same as those used by the artist or architect.
- Due to the status of these math skills in his society, the artist was encouraged to use them in his work.

Because merchants were experienced in manipulating ratios, they were sensitive to pictures and buildings that carried marks of similar processes. The step from computing the proportions of a currency exchange to examining the proportions of a physical body (such as a human head or a building) was a small one.

■ EXERCISES • RATIO AND PROPORTION

1. Define or describe the following terms:

straight line	line segment	ray	half-line
ratio	proportion	dimensionless ratio	means
extremes	mean proportional	locus	

2. Find the value of x in each of the following proportions. Round your answers to the nearest tenth unit, if needed, or leave in fractional form.
   a. $x : 5 = 3 : 10$ b. $x : 4 = 4 : 6$
   c. $3 : x = 4 : 6$ d. $x : 2.75 = 114 : 226$
3. Find the mean proportional between the following quantities.
   a. 2 and 50 b. 4 and 16 c. 6 and 150 d. 5 and 45
4. Subdivide each line segment by the given ratio to the nearest tenth unit.
   a. 200 units long, in a ratio of 3 : 8 b. 385 units long, in a ratio of 2 : 7
   c. 285 units long, in a ratio of 2 : 3 d. 56.8 units long, in a ratio of 3 : 4
5. If 350 lb (pounds) of marble costs $75, how much will 435 lb of marble cost?
6. If three masons can lay a 15-foot section of brick wall, how many masons are needed to lay 25 feet of a similar wall in the same length of time?
7. If it costs $5843 to paint a particular house having a surface of 3755 sq. ft, how much will it cost to paint a house having a surface of 7325 sq. ft?
8. Prove or demonstrate that the statements listed below are true: In any proportion,

1 The product of means equals the product of extremes.

2 The extremes may be interchanged.

3 The means may be interchanged.

4 The means may be interchanged with the extremes.

FIGURE 1.9 ■ **Closeup of Tablet in Rafael's *School of Athens***

PYTHAGORAS AND THE MUSICAL RATIOS

According to legend, the Pythagoreans built an elaborate number lore, but perhaps the numbers that impressed them most were those found in the musical ratios or musical intervals.

The *frequency* of a tone is the rate of vibration (so many vibrations per second) of whatever is producing the tone, such as a vibrating string. A *musical ratio* is the ratio of the frequency of one tone to another, such as between two piano keys. For example, two notes an octave apart have a musical ratio of 2 : 1.

Look again at Pythagoras in the *School of Athens* (Figure 1.1), where we see him explaining the musical ratios to a pupil. A closeup of the tablet he is holding is shown in Figure 1.9 and reveals several elements key to the musical ratios. See if you can identify the following in the closeup:

- The Greek names for the musical ratios (*diatessaron*, *diapente*, and *diapason*)
- The Roman numerals for 6, 8, 9, and 12, which show the ratio of the musical intervals
- The word ΕΠΟΓΛΟΩΝ, the name of the tone, which represents the interval between any two consecutive notes
- The triangular number ten, the Sacred Tetraktys mentioned earlier, inscribed toward the bottom of the tablet

This tablet shows several musical ratios. They are listed in the following chart.

			Greek Term	**Latin Term**
6 : 12	octave	(1 : 2)	diapason	duplus
6 : 9 or 8 : 12	fifth	(2 : 3)	diapente	sesquialtera
6 : 8 or 9 : 12	fourth	(3 : 4)	diatessaron	sesquitertia
8 : 9	tone	(8 : 9)	tonus	sesquioctavus

Note that all the numbers in the musical intervals, (1 : 2), (2 : 3), and (3 : 4), are contained in the Sacred Tetraktys.

No one knows how the Pythagoreans discovered the musical ratios, but legend says that they found them by experimenting with the *monochord*, a device with a single string and a movable bridge. An illustration of the monochord is shown in Figure 1.10. They moved the bridge to different locations along the string, as shown in Figure 1.11, and noted where the tone produced by the shortened string was harmonious with the tone produced by the string in its original length.

Let's say that the string has a full length of 12 units. (Because we are interested in ratios, the length we choose doesn't matter, and 12 will make the numbers easier.) Placing the bridge so as to shorten the string to 9 units produced a tone harmonious with that of the original 12-unit string. This tone was higher than the original—higher by a ratio of 4 : 3. This musical ratio is called the *fourth*.

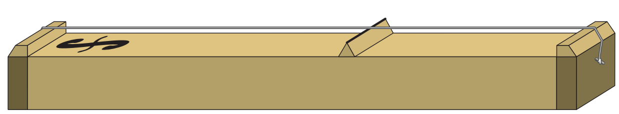


FIGURE 1.10 ■ **The Monochord**

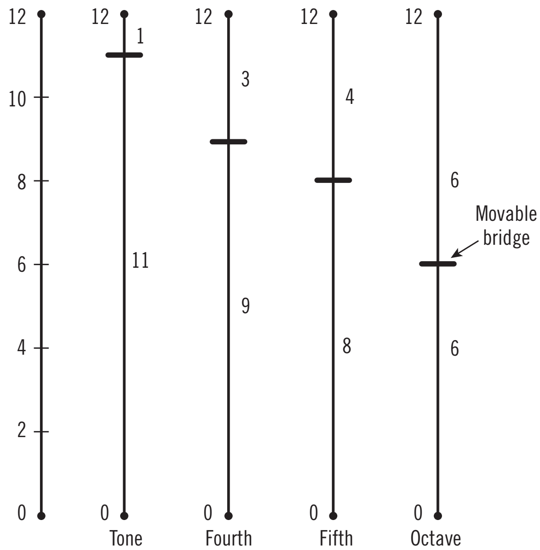


FIGURE 1.11 ■ Bridge Positions

Similarly, shortening the string to 8 units produced a harmonious tone higher than the original by a ratio of 3 : 2. This musical ratio is called the *fifth*. The fourth, the fifth, and the tone can be heard in the familiar song: "Here comes the bride, all dressed in white . . ." The intervals are "Here (fourth) comes the bride, all (fifth) dressed in (tone) white. . . ."

Shortening the string to half its original length, or 6 units, produced a harmonious tone twice as high as the original, or 2 : 1. This musical ratio is called the *octave*. Finally, shortening the string to one-twelfth its original length produced a tone called the *interval*. Therefore, they found that the only pleasant musical intervals could be expressed as the ratio of whole numbers (1 : 2, 2 : 3, and 3 : 4), as shown on the tablet in the *School of Athens* and graphically in Figure 1.12.

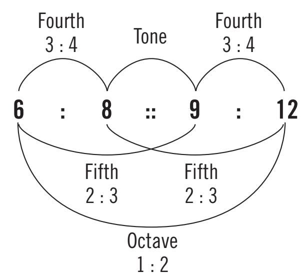


FIGURE 1.12 ■ The Musical Ratios

Why do some intervals sound pleasant while others sound discordant? To answer this, let's look at the physics of a vibrating string. A string secured at both ends, when plucked or bowed, vibrates as a whole and produces a tone that is called the *fundamental* (Figure 1.13a). Therefore, the D string on a violin will sound a D.

However, the motion of a vibrating string is much more complicated. While vibrating as a whole it will, at the same time, vibrate in halves (Figure 1.13b). Each half-string produces a tone an octave higher than the fundamental. A string will, in addition, vibrate in thirds, fourths, and so on (Figures 1.13c and 1.13d), and produce a whole series of *overtones* or *higher harmonics*. These harmonics are progressively weaker than the fundamental, and they vary from instrument to instrument. They occur naturally whenever a string is plucked or a horn is blown, and they add richness and variety to the tone.

When the fundamental or an overtone of one plucked string exactly matches the fundamental or an overtone of another plucked string, the two strings sounded in unison or in quick succession will sound harmonious. They sound "right." Note, however, that they are all integer ratios of the full string length, and it is these ratios that the Pythagoreans discovered with the monochord.

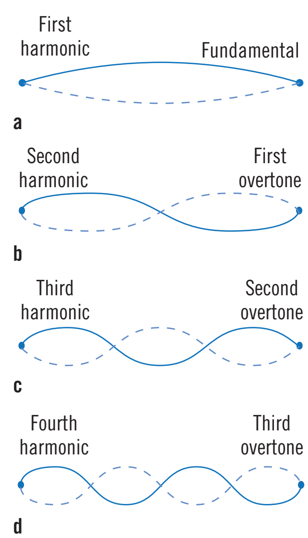


FIGURE 1.13 ■ Modes of Vibration of a String That Is Fastened at Both Ends

FIGURE 1.14 ■ **The Harmonic Scale from *Regule florum musices* (*Rules of Music's Flowers*), Pietro Cannuzio, Florence, 1510**

FIGURE 1.15 ■ **Pythagoras in *Theorica Musica*, F. Gaffurio, 1492**

As such, we have a few pleasant-sounding intervals, the tone, the fourth, the fifth, and the octave, but that is hardly a complete scale. Starting at C, these intervals would give us F, G, and C, an octave higher than where we started. What about the other notes of the scale, such as those depicted in the Italian music books shown in Figures 1.14 and 1.15?

The title page from the music book *Rules of Music's Flowers* (Figure 1.14) dates from nearly the same year as the *School of Athens*. It shows a pattern similar to that on Pythagoras' tablet and also features compasses, which acknowledge a connection between music and geometry. In the upper-left panel of Figure 1.15, we see Iubal (or Jubal) and men hitting an anvil with hammers numbered 4, 6, 8, 9, 12, and 16. (According to the Old Testament, *Iubal* is the "father of all who play the lyre and the pipe."[8]) Other frames show "Pithagoras" or "Pytagora" hitting different sized bells, plucking strings under different tensions, or tapping glasses filled to different depths with water; all of them are marked with those same numbers (4, 6, 8, 9, 12, and 16). In each frame, Pythagoras sounds the ones marked 8 and 16, which is an interval of 1 : 2, the octave. In the lower-right frame, he and Philolaus, another Pythagorean, blow pipes of lengths 8 and 16, again giving the octave. At the same time, Pythagoras holds pipes 9 and 12, giving the ratio 3 : 4, the fourth, while Philolaus holds 4 and 6, giving the ratio 2 : 3, the fifth.

The Greek philosopher Plato later elaborated on the Pythagorean idea of the musical ratios by using mathematical series, sequences, and means to fill the gaps between the known notes. Let's take a look at the mathematics now.

SEQUENCES, SERIES, AND MEANS

In the following section, we will see how Plato used arithmetic, geometric, and harmonic means to develop a musical scale, and later how Alberti and Palladio used these means to determine the proportions of rooms in a building. In the following chapter, we will learn about the Fibonacci sequence, and later the *ad quadratum* and *sacred cut* sequences. Here, we will briefly provide the mathematical background for understanding sequences, series, and means in this chapter, as well as later in our story. Let's start by defining some terms.

A *sequence* is a set of terms arranged in order, such as

$$1, 1, 2, 3, 5, 8, \ldots$$

A *series* is an expression for the sum or difference of the terms of a sequence. The terms are, therefore, connected by plus or minus signs, such as the series

$$1 + 1 + 2 + 3 + 5 + 8 \ldots$$

Arithmetic Progressions

An *arithmetic progression* is a sequence in which each term (after the first) equals the sum of the preceding term and a constant, called the *common difference*. To find the common difference, simply subtract any term from the one following it.

■ **EXAMPLE:** Find the common difference for the arithmetic progression

$$3, 8, 13, 18 \ldots$$

● **SOLUTION:**

Common difference $= 13 - 8 = 5$. ■

You can often determine a term of a sequence from those preceding it. The relationship between a term and those preceding it is called a *recursion relationship* or a *recursion formula*. For the arithmetic progression, you can find any term a_n simply by adding the common difference d to the term a_{n-1} immediately preceding it.

12 RECURSION FORMULA FOR AN ARITHMETIC PROGRESSION

$$a_n = a_{n-1} + d$$

A recursion formula is useful for generating a sequence by using a spreadsheet on a computer. See Problem 11 in the exercise set for this section.

■ **EXAMPLE:** Using the recursion formula, find the next term in the following sequence:

$$3, 8, 13, 18 \ldots$$

● **SOLUTION:** We get the common difference by subtracting any term from the one immediately following, so

$$d = 8 - 3 = 5.$$

The next term in the sequence is then 18 + 5 or 23. ■

The *arithmetic mean* between two numbers is what we commonly call the *average*, that is, the sum of the numbers divided by two. Therefore, to find the arithmetic mean b between two terms a and c, simply take half the sum of a and c.

9 ARITHMETIC MEAN

$$\text{Arithmetic mean } b = \frac{a + c}{2}$$

We'll see later how Palladio used the arithmetic mean to find room proportions.

■ **EXAMPLE:** Find the arithmetic mean between the numbers 5 and 9.

● **SOLUTION:**

$$\frac{5 + 9}{2} = \frac{14}{2} = 7$$ ■

Geometric Progressions

A *geometric progression* is a sequence in which each term (after the first) equals the preceding term times a constant called the *common ratio*. To find the common ratio, divide any term by the one preceding it.

■ **EXAMPLE:** Find the common ratio for the geometric progression

$$5, 20, 80, 320 \ldots$$

● **SOLUTION:**

$$\text{Common ratio} = \frac{80}{20} = 4$$ ■

To find any term a_n of a geometric progression, multiply the term a_{n-1} immediately preceding by the common ratio r. Therefore, the recursion formula is as follows.

14 RECURSION FORMULA FOR A GEOMETRIC PROGRESSION

$$a_n = ra_{n-1}$$

■ **EXAMPLE:** Find the next term in the geometric progression of the preceding example.

● **SOLUTION:** The common ratio was 4 and the last term given was 320, so the next term in the progression is

$$4(320) = 1280.$$

To find the geometric mean b between two numbers a and c, take the square root of the product of a and c.

10 GEOMETRIC MEAN OR MEAN PROPORTIONAL

$$\text{Geometric mean } b = \pm\sqrt{ac}$$

■ **EXAMPLE:** The geometric mean b between 3 and 12 is

$$b = \sqrt{3 \times 12} = \sqrt{36} = 6.$$

Harmonic Progressions

A sequence is called a *harmonic progression* if the reciprocals of its terms form an arithmetic progression. The name *harmonic* goes back to the Pythagoreans. Recall that when a stretched string is shortened to $\frac{1}{2}$ its original length, it produces a tone that is "harmonious" with the tone produced by the original length. The same is true when the string is shortened to $\frac{1}{3}$ the original length, and $\frac{1}{4}$, and so forth. The fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ form a harmonic progression.

The sequence

$$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \ldots$$

is a harmonic progression because the reciprocals of its terms, which are

$$1, 3, 5, 7, 9, 11, \ldots,$$

form an arithmetic progression.

The harmonic mean b between two numbers a and c is equal to twice the product of a and c, divided by their sum.

11 HARMONIC MEAN

$$\text{Harmonic mean } b = \frac{2ac}{a + c}$$

■ **EXAMPLE:** What is the harmonic mean b between 6 and 12?

● **SOLUTION:** By Equation 11,

$$b = \frac{2(6)(12)}{6 + 12} = \frac{144}{18} = 8.$$

The numbers 6, 8, and 12 now form a harmonic progression; their reciprocals form an arithmetic progression.

The Three Means by Geometric Construction

Another way to find arithmetic, geometric, and harmonic means is by *geometric construction*. Geometric constructions are an important part of learning geometry, in addition to being fun to do. *Constructions* are created using only a compass and a straightedge. Rulers, squares, and protractors are not allowed. Further, some constructions can be completed with a single setting of the compass; these are called *rusty compass constructions*.

We will add to these constructions as we go along. To get an idea of the full range of constructions given in this book, turn to the complete listing in Appendix C.

In preparation for finding the three means by geometric construction, let's begin by looking at a simple construction to draw a perpendicular line. Two lines are said to be *perpendicular* if they intersect to form four equal angles. Two lines are said to be *parallel* if they do not intersect, no matter how far they are extended. Many constructions, including that for the three means, require us to draw a perpendicular to a given line, so let's do that construction first. Figure 1.16 illustrates the construction for a perpendicular to a line or to a line segment (the portion of a line between two endpoints).

In Figure 1.16, we are shown the construction of a perpendicular to a line through a given point, which can be either on the line or off the line, as shown. Starting at the given point *P*, swing an arc cutting the given line at *A* and *B*. From *A* and *B*, draw arcs of equal radius that intersect at *C*. Finally, draw line *CP*, the perpendicular to the given line. Further, line *CP* is also the *perpendicular bisector* of line segment *AB*.

Let's proceed now to our constructions of the three means. Palladio explains how to find means in Book 1, Chapter XXIII of his *Four Books of Architecture*, although he does not call them by these names.

To find an arithmetic mean between two lengths, Palladio simply makes a line whose length is the sum of the two lengths, and bisects it. To find the geometric mean or mean proportional, he uses the same method described by Euclid in Book VI, Prop. 13. This method is demonstrated in Figure 1.17, in which we find the mean proportional (geometric mean) between *AB* and *BC*. Lay *AB* and *BC* end to end on the same straight line. Draw a semicircle on *AC*, and erect a perpendicular *BD* at *B*. *BD* is the mean proportional between *AB* and *BC*.

Palladio also gives the construction for finding the harmonic mean, and this is demonstrated in Figure 1.18. To find the harmonic mean between *AB* and *BC* in the rectangular floor plan *ABCD*, first find the arithmetic mean between *AB* and *BC*. This can be done by calculation or by construction. Next, extend *AB* by that amount to *E*. Draw *EC* and extend to where it intersects *AD* extended at *F*. Length *DF* is the harmonic mean proportional between *AB* and *BC*.

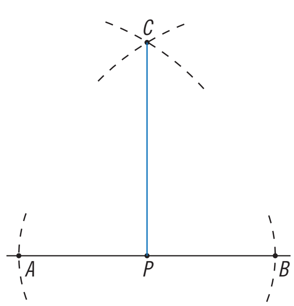


FIGURE 1.16 ■ Perpendicular to a Line through a Given Point Not on the Line

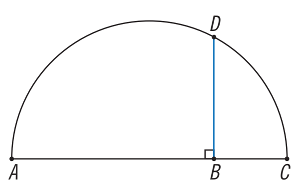


FIGURE 1.17 ■ Finding the Mean Proportional

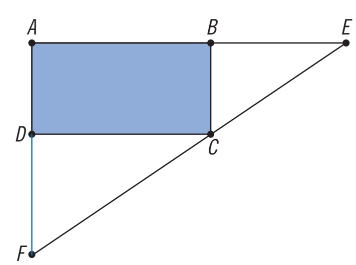


FIGURE 1.18 ■ Finding the Harmonic Mean

EXERCISES • SEQUENCES, SERIES, AND MEANS

1. Define the following terms:

sequence	series	arithmetic progression
geometric progression	harmonic progression	common difference
common ratio	arithmetic mean	geometric mean
harmonic mean	mean proportional	recursion formula
geometric construction	rusty compass construction	perpendicular
parallel		

2. Find the common difference in these arithmetic progressions:
   a. 125, 131, 137, 144 . . . b. 54, 46, 38, 30 . . .
3. Find the common ratio in these geometric progressions:
   a. 25, 75, 225, 675 . . . b. 1.54, 3.85, 9.625 . . .
4. Insert an arithmetic mean between these numbers:
   a. 12 and 18 b. 283 and 495
5. Insert a geometric mean between these numbers:
   a. 5 and 20 b. 4.68 and 8.26
6. Insert a harmonic mean between these numbers:
   a. 1 and 10 b. 3.92 and 8.83
7. Given the numbers 10 and 20, insert an arithmetic mean, a geometric mean, and a harmonic mean.
8. Why is the harmonic mean called "harmonic"?
9. Draw a sequence of squares, starting with side 1 and increasing by .5, to illustrate an arithmetic progression.

   $$1, 1.5, 2, 2.5, \ldots$$

10. Draw a sequence of squares on diagonals, that is, where the side of each successive square is equal to the diagonal of the preceding square. Use this to demonstrate a geometric progression with a common ratio equal to $\sqrt{2}$.

    $$1, \sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$$

11. Choosing any numbers you want, use the recursion formulas for the arithmetic and geometric progressions to generate progressions on a computer.
12. Using only a compass and a straightedge, repeat the constructions for the three means.
13. Using only the Line and Circle tools of a computer drafting program, repeat the constructions for the three means.
14. Try to devise ways to reproduce the construction of a perpendicular by paper folding. If you get stuck, see Olson's *Mathematics Through Paper Folding*.

PLATO AND THE MUSICAL RATIOS

"*. . . the safest general characterization of the European philosophical tradition is that it consists of a series of footnotes to Plato.*"

ALFRED NORTH WHITEHEAD

Plato acquired property about 387 B.C. in the northwestern outskirts of Athens, a site that had been an olive grove, park, and gymnasium sacred to the legendary

PLATO (c. 427–347 B.C.)

Plato was born in Athens, and in his youth he was interested in politics. He eventually became a disciple of Socrates and adopted Socrates' style of debate, in which truth was pursued through questions and answers. Plato saw Socrates die in 399 B.C. at the hands of the Athenian rulers. In Raphael's *School of Athens* (Figure 1.5), Socrates is prone, with a cup nearby. Plato's final years were spent lecturing at his Academy and writing. He died at about the age of 80 in Athens.

Plato shared the Pythagoreans' love of numbers. In his *Epinomis,* Plato wrote, "Numbers are the highest degree of knowledge, and Number is Knowledge itself." Plato loved geometry. When asked "What does God do?" Plato supposedly replied, "God geometrizes . . . [Geometry is] pursued for the sake of the knowledge of what eternally exists, and not of what comes for a moment into existence, and then perishes, . . . [it] must draw the soul towards truth and give the finishing touch to the philosophic spirit."[9]

FIGURE 1.19 ▪ **Plato and Aristotle in Raphael's *School of Athens***

Plato even defined beauty in geometric terms. "I do not mean by beauty of form such beauty as that of animals or pictures, . . . but . . . understand me to mean straight lines and circles, and planes and solid figures which are formed out of them by turning lathes and rulers and measures of angles; for them I affirm to be not only relatively beautiful, like other things, but eternally beautiful. . . ."[10]

Although Plato loved geometry, he had a low opinion of art. He taught that since our world is a copy or image of reality, then a work of art is a copy of a copy, three steps away from reality. He wrote that "painting and . . . the whole art of imitation, is busy about a work which is far removed from the truth . . . and is its mistress and friend for no wholesome or true purpose . . . it is the worthless mistress of a worthless friend, and the parent of a worthless progeny."[11] Raphael had placed Plato and his student Aristotle in the center of his *School of Athens* (Figure 1.19).

hero Academus. The school he started there, often described as the first university, offered courses in astronomy, biology, mathematics, political theory, and philosophy. Over the doors to his academy were the words

αγεωμετρητοζ μηδειζ εισιτω

which mean, "Let no one destitute of geometry enter my doors." Compare this with what Leonardo later wrote in one of his notebooks, "Let no one read me who is not a mathematician." Plato's Academy was the inspiration for a Platonic academy in Renaissance Florence, started by Lorenzo de Medici and led by the neo-Platonist Marsilio Ficino. It included Pico della Mirandola, Cristoforo Landino, Angelo Poliziano, and sometimes Michelangelo. It was also the inspiration for another academy near Vicenza, started by Count Trissino and attended by Palladio. The ruins of the academy still exist.

Plato left many writings, but his love of geometry is especially evident in his *Timaeus.* Written toward the end of Plato's life, the *Timaeus* describes a conversation among Socrates, Plato's teacher; Critias, Plato's great grandfather; Hermocrates, a Sicilian statesman and soldier; and Timaeus, a Pythagorean, philosopher, scientist, general, contemporary of Plato, inventor of the pulley, and the first to distinguish among the harmonic, arithmetic, and geometric progressions. In the book, Timaeus pays homage to Pythagoras and describes the geometric creation of the universe.

Creation of the World by the Musical Ratios

> The *Timaeus* also mentions *Atlantis*, a superior civilization that was submerged in one day and night by floods and earthquakes. Only Egypt, being dry, survived to preserve and hand down the old traditions.[12]

In his *Timaeus*, Plato says that the creator made the *world soul* out of various ingredients and formed it into a long strip.[13] This strip was made of a flexible material, which would later be cut lengthwise and bent into celestial circles. The strip was then marked at intervals as follows:

- First, the creator took one portion from the whole, (1 unit)
- next, a portion double the first, (2 units)
- a third portion half again as much as the second, (3 units)
- the fourth portion double the second, (4 units)
- the fifth three times the third, (9 units)
- the sixth eight times the first, (8 units)
- and the seventh 27 times the first (27 units)

Therefore, we get the seven integers (1, 2, 3, 4, 8, 9, 27) that contain the Monad, source of all numbers, the first even and first odd, and their squares and cubes.

The seven integers Plato used to describe the creation of the world by musical ratios are often shown arranged into two series called Plato's lambda, after the Greek letter lambda (λ).

```
      1
    2   3
  4       9
8           27
```

FIGURE 1.20 ■ Arithmetic Portrayed as a Woman

After the Monad, the left branch contains the first even number, its square, and its cube; the right branch contains the first odd number, its square, and its cube. These seven numbers contain all the musical consonances. Plato also believed that they "embrace the secret rhythm in the macrocosm and the microcosm alike."[14] Plato's lambda appears in the allegory to arithmetic shown in Figure 1.20.

If you mark off Plato's seven numbers on a musical staff, starting, say, at low C, you get four octaves and a bit more as shown in Figure 1.21. It is the beginning of a musical scale, but it has many gaps.

However, Plato goes on to *fill* each interval with an arithmetic mean and a harmonic mean, as follows. Let's apply Equations 9 and 10 to the first octave,

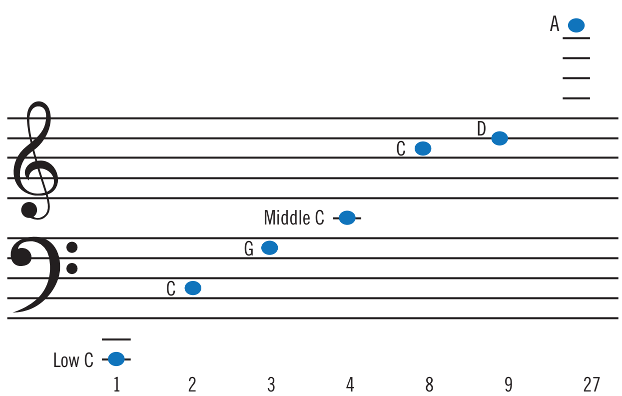


FIGURE 1.21 ■ The Seven Numbers of Plato's Lambda as a Musical Scale

starting from low C in Figure 1.21. The means between 1 and 2 are

$$\text{Arithmetic mean} = \frac{1 + 2}{2} = \frac{3}{2}$$

$$\text{Harmonic mean} = \frac{2(1)(2)}{1 + 2} = \frac{4}{3}$$

So the numbers in the first octave are

$$1, \frac{4}{3}, \frac{3}{2}, \text{ and } 2.$$

The ratio between the first and second numbers (1 and 4/3) is 3 : 4, the fourth. We get the same ratio between the third and fourth numbers (3/2 and 2). The ratio between the first and third numbers (1 and 3/2) is 2 : 3, the fifth. We get the same ratio between the second and fourth numbers (4/3 and 2). *These are the same intervals that the Pythagoreans found pleasing, but Plato found them from arithmetic calculations alone*, and not by experimenting with stretched strings to determine which ratios sounded best. The English philosopher and Plato scholar Francis Cornford wrote, " . . . Plato has constructed a section of the diatonic scale, whose range is fixed by considerations extraneous to music."[15]

If we now add these arithmetic and harmonic means on our original seven-note scale, and similarly insert arithmetic means between higher octaves, we get the scale shown in Figure 1.22.

However, there were still gaps. Plato took the geometric interval between the fourth and the fifth as a full tone. It is found by dividing 3/2 by 4/3.

$$\frac{3}{2} \div \frac{4}{3} = \frac{3}{2} \times \frac{3}{4} = \frac{9}{8}$$

Plato then filled up the scale with intervals of 9/8, as shown in Figure 1.23. Starting, say, at middle C, multiplying by 9/8 takes us to D, and multiplying D by 9/8 gives E. Multiplying E by 9/8 would overshoot F, so Plato stopped at F. This left an interval of

$$\frac{4}{3} \div \frac{81}{64} = \frac{4}{3} \times \frac{64}{81} = \frac{256}{243}$$

between E and F. This ratio is approximately equal to half that of the full tone, and so is called a *semitone*. Note that two semitones approximately equal one full tone.

$$\text{Two semitones: } \frac{256}{243} \times \frac{256}{243} \cong 1.110$$

$$\text{One full tone: } \frac{9}{8} = 1.125$$

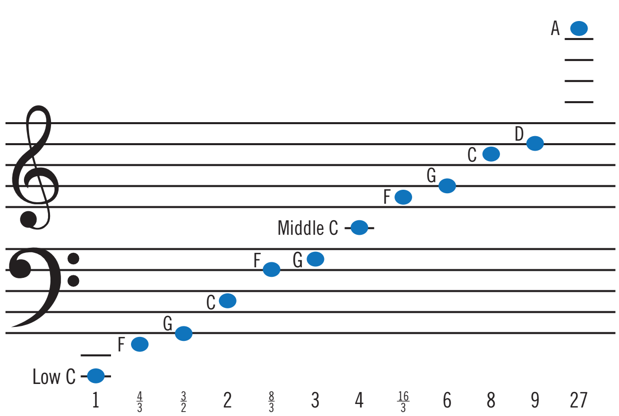


FIGURE 1.22 ■ Musical Scale Showing Fourths and Fifths

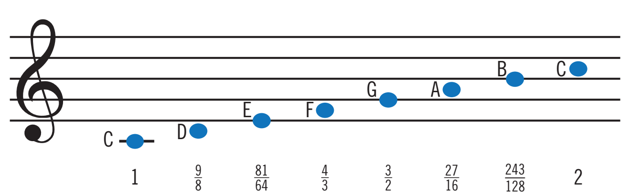


FIGURE 1.23 ■ One Octave of the Musical Scale Showing Tones and Semitones
Only one octave is shown here, but the procedure can be repeated for the remaining octaves.

Ascending from F to B, we proceed again by factors of 9/8. The interval from B to C is once more the semitone 256/243.

Note that while the fourths and fifths were found using arithmetic and harmonic means, the whole tone intervals were found by geometric means. As such, the intervals C to D to E and F to G to A to B form two *geometric progressions*.

This mathematically constructed scale is close to a modern scale, but there are differences. For comparison, Table 1.1 lists the major diatonic scale, starting from middle C. The table also shows the frequency of each note, in hertz (Hz), cycles per second, with A above middle C equal to 440 Hz, by international agreement.

TABLE 1.1 ■ The Major Diatonic Scale

Note		Frequency, Hz, Diatonic Scale in Key of C	Interval between Notes	Interval from C	Frequency, Hz, Equally Tempered Scale
Do	C	264		1 : 1, unison	261.6
Re	D	297	9/8	9 : 8, major second	293.7
Mi	E	330	10/9	5 : 4, major third	329.6
Fa	F	352	16/15	4 : 3, perfect fourth	349.2
So	G	396	9/8	3 : 2, perfect fifth	392.0
La	A	440	10/9	5 : 3, major sixth	440.0
Ti	B	495	9/8	15 : 8, major seventh	493.9
Do	C	528	16/15	2 : 1, perfect octave	523.3

This too, is not the end of the story. With the diatonic scale, the twelve tones within an octave, including the semitones, are not equally spaced. This would require that a piano, for example, be retuned so that it could play in each key. To avoid this, instruments are now tuned to the *equally tempered scale*, popularized by J. S. Bach. Here, the twelve semitones are equally spaced by an interval of $2^{1/12}$, or the twelfth root of two. Therefore, multiplying $2^{1/12}$ by itself twelve times gives 2, or an octave. The frequency of each note, in the equally tempered scale, is also given in the table. As with the diatonic scale, A = 440 Hz is taken as standard.

Music of the Spheres

By experimenting with plucked strings, the Pythagoreans discovered that the intervals that pleased people's ears were

- Octave 1 : 2
- Fifth 2 : 3
- Fourth 3 : 4

All of them are contained in the simple numbers 1, 2, 3, and 4, the very numbers in their beloved Sacred Tetraktys, which added up to the number of fingers. This staggered the Pythagoreans, who felt they had discovered some basic law of the universe.

Quoting Aristotle's *Metaphysics* again, "[the Pythagoreans] saw that the . . . ratios of musical scales were expressible in numbers . . . and that all things seemed to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of number to be the elements of all things, and the whole heaven to be a musical scale and a number."

It seemed clear to the Pythagoreans that the distances between the planets would have the same ratios as the ones that produced harmonious sounds in a plucked string. Their solar system consisted of spheres rotating in circular orbits, each sphere giving off a sound, the way a projectile makes a sound as it swishes through the air. The closer spheres gave lower tones, while the farther spheres moved faster and gave higher-pitched sounds. All combined into a beautiful harmony, *the music of the spheres*. This idea that creation was closely linked to music was continued by many writers over the centuries.

Plato told how the universe was created according to the musical ratios. The Cambridge scholar E. M. W. Tillyard wrote, “But there was the further notion that the created universe was itself in a state of music, that it was one perpetual dance.”[16] In his *Republic*, Plato says of the cosmos, “Upon each of its circles stood a siren who was carried round with its movements, uttering the concords of a single scale.”

The historian of art and architecture Rudolph Wittkower wrote, “the doctrine of a mathematical universe which . . . was subject to harmonic ratios, was triumphantly reasserted by a number of great thinkers. . . .”[17] Among them, Isidore of Seville (560?–636), a medieval encyclopedist, wrote, “Nothing exists without music; for the universe itself is said to have been framed by a kind of harmony of sounds, and the heaven itself revolves under the tones of that harmony.” William Shakespeare (1564–1616), in *The Merchant of Venice*, wrote,

> “There’s not the smallest orb which thou behold’st
> But in his motion like an angel sings . . .”[18]

The astronomer Johannes Kepler (1571–1630) wrote that he wished “to erect the magnificent edifice of the harmonic system of the musical scale . . . as God, the Creator Himself, has expressed it in harmonizing the heavenly motions.”[19] Later he wrote, “I grant you that no sounds are given forth, but I affirm . . . that the movements of the planets are modulated according to harmonic proportions.” Quoting the English poet John Dryden (1631–1700),

> “From harmony, from Heav’nly harmony
> This universal frame began;
> From harmony to harmony
> Thro’ all the compass of the notes it ran,
> The diapason closing full in man.”[20]

Finally, the English philosopher Anthony Shaftsbury (1671–1713) related human nature to the musical ratios.

> “Virtue has the same fix’d Standard.
> The same Numbers, Harmony, and Proportion will have place in Morals;
> and are discoverable in the Characters and Affections of Mankind.”[21]

THE MUSICAL RATIOS IN ARCHITECTURE

> *“Geometry . . . is of much assistance in architecture . . . it teaches us the use of the rule and compasses . . . and rightly apply the square, the level, and the plummet.”*[22]
>
> VITRUVIUS

A major figure in our study of geometry in architecture is the Roman architect, Vitruvius. We provide a brief biographical sketch here, and refer to his writings throughout this text.

VITRUVIUS (70–25 B.C.)

Vitruvius, whose full name is Marcus Vitruvius Pollio, was probably born in Formiae (now Formie), Italy. He was an architect and an artillery engineer, probably in the service of the first emperor Augustus. Little is known of Vitruvius' life, but he is best known as the author of the famous treatise *De architectura* (*Ten Books on Architecture*), a handbook for architects and the oldest surviving work on the subject.

De architectura is above all a practical guide to materials, methods, and design. It covers almost every aspect of architecture: city planning, building materials, fortifications, temples, colonnades, private houses, public buildings, baths, basilicas, theaters, gymnasiums, harbors, farmhouses, military engines, public spaces such as the forum, floor, and stucco decoration, hydraulics, clocks, mensuration, and astronomy. Vitruvius was the first to discuss the classical orders of architecture: Doric, Ionic, and Corinthian. Interestingly, he urges the architect to understand music.

Much of the material appears to be based on his own experience, as well as on works by Greek architects such as Hermogenes. Apparently, Vitruvius seeks to preserve the classical Greek tradition in architectural design. Vitruvius' stated wish was that his name be honored long after his death. This was certainly the case, for the *Ten Books on Architecture* became the chief authority on ancient classical architecture, and his writings have been studied ever since the Renaissance.

Systems of Architectural Proportions

"Throughout the history of architecture there has been a quest for a system of proportion that would facilitate the technical and aesthetic requirements of a design."[23]

KAPPRAFF

Architects and builders have always sought systems of proportions to create buildings that were visually pleasing. One of the best-known authorities on the subject was Roman architect Vitruvius' famous treatise *De architectura*, or *Ten Books on Architecture*, in which he compares proportion in architecture to proportion in the greatest work of art, the human body. In *De architectura* or *Ten Books on Architecture*, he wrote, "Symmetry is a proper agreement between the members of the work itself, and relation between the different parts and the whole. . . ."[24] Later he added, "Therefore since nature has proportioned the human body so that its members are duly proportioned to the frame as a whole . . . in perfect buildings the different members must be in exact symmetrical relations to the whole general scheme."[25] In this instance, Vitruvius uses "symmetrical relations" to mean "having the same proportions," rather than some kind of mirror symmetry.

The key elements required in such a system of proportions were later defined by American mathematician Jay Kappraff. He wrote, "Such a system [of proportions] would have to ensure a repetition of a few key ratios throughout the design, have additive properties that enable the whole to equal the sum of its parts, and . . . be adaptable to the architect's technical means. The repetition of ratios enables a design to exhibit a sense of unity and harmony of its parts. Additive properties enable the whole to equal the sum of its parts in a variety of different ways, giving the designer flexibility to choose a design that offers the greatest aesthetic appeal while satisfying the practical considerations of the design. Architects and designers are most comfortable within the realm of integers, so any system based on irrational dimensions or incommensurable proportions should also be expressible in terms of integers to make it computationally

The repetition of ratios gives a structure a degree of self-similarity, a property of fractals that will be discussed in Chapter 13, "Fractals."

acceptable."[26] In short, such a system of proportions would use a repetition of a few key ratios, have additive properties, and use ratios of whole numbers.

There are three main systems of proportion in architecture, and in this section we'll explore a system based on the Pythagorean musical ratios. This system was primarily used by Renaissance architects like Alberti and Palladio. In Chapter 2, we'll explore a system based on the golden ratio, such as Le Corbusier's Modulor, and in Chapter 4 you will be introduced to a system based on the square, which was apparently used by the Romans.

AMPHION

The notion of architecture as *frozen music* is generally attributed to the German poet Goethe in a letter dated 1829. The German philosopher Friedrich Schelling, in his *Philosophie der Kunst,* wrote that "architecture in general is frozen music." Earlier, a Greek myth tells of Amphion (Figure 1.24), son of Zeus and Antiope, who played so exquisitely that stones moved and arranged themselves into structures.

FIGURE 1.24 ■ **Amphion**

The Musical Ratios in Renaissance Architecture

According to Rudolph Wittkower, "The conviction that . . . each part of a building . . . has to be integrated into one and the same system of mathematical ratios may be called the basic axiom of Renaissance architects . . . the architect is by no means free to apply . . . a system of ratios of his own choosing . . . and proportions in architecture have to embrace and express the cosmic order. But . . . what are the mathematical ratios that determine the harmony in this macrocosm and microcosm? They had already been revealed by Pythagoras and Plato. . . ."[27]

Wittkower further observes that Renaissance architects were convinced of the Pythagorean vision of the harmonic structure of the universe, and that some inborn sense makes us aware of that harmony. Therefore, if a building contains the same mathematical proportions present in the musical ratios, this "inner sense tells us . . . when the building we are in partakes of the vital force which lies behind all matter and binds the universe together."[28]

As you saw earlier, inserting arithmetic and harmonic means in a geometric progression enabled Plato to mathematically construct a musical scale. These means were also used in Renaissance architecture, but as Kappraff suggests, in ratios of whole numbers. Take a series of numbers in a geometric progression, such as

$$1, 2, 4, 8 \ldots$$

Between each pair of numbers, insert an arithmetic mean and a harmonic mean. First, however, multiply by six to avoid fractional quantities:

$$6, 12, 24, 48 \ldots$$

Inserting arithmetic and harmonic means between each pair of numbers gives:

$$6, 8, 9, 12, 16, 18, 24, 32, 36, 48 \ldots$$

- The geometric progression (6, 12, 24. . .) determines each octave [1 : 2].
- The harmonic mean within each octave, say 6 : 8, determines the fourth [3 : 4].
- The arithmetic mean within each octave, say 6 : 9, determines the fifth [2 : 3].
- The harmonic and arithmetic means within each octave determine the tone [8 : 9].

According to Wittkower, "Whenever one meets ratios of [this series] it is safe to presume that this is not casual but are the result of reflections which depend directly or indirectly on the Pythagoreo-Platonic division of the musical scale."[29] We'll now see how two Renaissance architects, Alberti and Palladio, used the three means and the musical ratios.

Two strings tuned to the same pitch will both vibrate even if only one is plucked, a phenomenon of *resonance.* In the case of strings, the air between the strings transmits the vibrations from one to another. However, some have speculated that a form of resonance might be the reason that we relate so intensely to the archetypal musical ratios, even for visual phenomena.

LEON BATTISTA ALBERTI (1404–1474)

Alberti has been called the prototype of the "Renaissance Man." He was born into a wealthy merchant-banker family, received classical Latin training at a boarding school in Padua, and studied law at the University of Bologna. He obtained a position at the Vatican and eventually took holy orders, although his later interests and activities were entirely secular.

Alberti seems to have collaborated with the Florentine cosmographer Paolo Toscanelli, who provided Columbus with the map that guided him on his first voyage, which led to Alberti's treatise on geography. After this, he wrote the first Italian grammar book, hoping to promote the Tuscan vernacular, and wrote a work on cryptography.

His study of ancient architectural practices resulted in his *De re aedificatoria* (*Ten Books of Architecture*) of 1452. It became a bible of Renaissance architecture and won him his reputation as the "Florentine Vitruvius." His architectural works include the façades of S. Maria Novella, home of Masaccio's *Trinity*, and the Palazzo Rucellai, both in Florence; Tempio Malatestiano in Rimini; and in Mantua, the Churches San Sebastiano and Sant' Andrea.

Alberti and His *Ten Books of Architecture*

The idea that the same ratios that are pleasing to the ear would also be pleasing to the eye appears in the writings of Plato, Plotinus, St. Augustine, and St. Aquinas, but the most direct statement comes from Alberti. In his *Ten Books of Architecture*, he wrote, "indeed I am every day more and more convinced of the truth of Pythagoras' saying, that Nature is sure to act consistently . . . I conclude that the same numbers by means of which the agreement of sounds affect our ears with delight are the very same which please our eyes and our mind. We shall therefore borrow all our rules for finding our proportions from the musicians."[30] In other words, what sounds good also looks good.

In Book IX, Chapter V of his *Ten Books of Architecture*, Alberti gives definitions of the octave, the fifth, the fourth, and the tone, as found by Pythagoras. In Chapter VI, he defines the three means: arithmetical, geometrical, and *musical*, what we call the harmonic mean. In this same chapter, he recommends proportions for floor plans based on the musical ratios. Here Alberti distinguishes between small, medium, and large floor plans. For each, he recommends particular musical ratios for length to width.

- For the smallest, he recommends the square (1 : 1), the sesquitertia (6 : 8), or the sesquialtera (6 : 9).
- For medium plans, he recommends either (a) the double square or octave (1 : 2) or (b) the sesquialtera doubled. In other words, the ratio 6 : 9 applied twice. For a plan whose width is 4 units, applying the ratio 6 : 9 gives 6, and then applying that ratio to 6 gives 9. As such, the final plan would be 4×9 units. This method is called *generation of ratios*. It is a means by which Alberti arrives at a ratio that is not one of the simple ratios (1 : 2, 2 : 3, 3 : 4) but is generated from those simple ratios.
- For the longest floor plans, he suggests (a) a triple proportion, two octaves; (b) a quadruple proportion, three octaves; or (c) a ratio of 3 to 8 (1 : 2 followed by 3 : 4).

Wittkower has written, "Proportions recommended by Alberti are the simple relations of one to one, one to two, one to three, two to three, three to four, etc., which are the elements of musical harmony. . . ."[31] He goes on to give as an example the façade of S. Maria Novella in Florence (Figures 1.25 and 1.26). In Figure 1.26, the façade is shown divided into rectangles with simple musical ratios. The whole façade is inscribed in a square of, say, 2 units. The lower story is comprised of two squares, and the upper square fits into one square. Therefore, the main parts are in the ratio of 1 : 2, the octave. The central bay of the upper story is a square of $\frac{1}{2}$ unit. Two squares of $\frac{1}{2}$ unit enclose the pediment and entablature. The entrance bay is $\frac{1}{2}$ unit by $\frac{3}{4}$ unit, for a ratio of 2 : 3.

Although Alberti's system of proportions met two of the three requirements mentioned earlier, the repetition of a few key ratios and the use of whole number ratios, it lacked the third. According to Kappraff, "Although the Renaissance system of Alberti succeeded in creating harmonic relationships in which key proportions were repeated in a design, it did not have the additive properties necessary for a successful system. It is fascinating that a system of proportions used by the Romans and the system . . . developed by Le Corbusier, known as the Modulor, both conform to the relationships inherent in the system of musical proportions . . . with the advantage of having additive properties."[32]

FIGURE 1.25 ■ Façade of S. Maria Novella, Florence

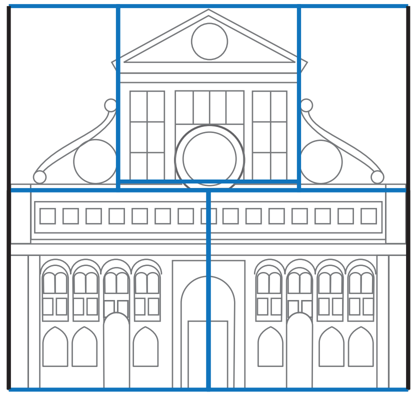


FIGURE 1.26 ■ Wittkower's Analysis of the Façade of S. Maria Novella

Palladio and the *Four Books of Architecture*

". . . in all fabrics is it requisite that their parts should correspond together, and have such proportions, that there may be none whereby the whole cannot be measured, and likewise all the other parts."[33]

PALLADIO

At the end of 20 years of intensive building, Palladio published *I quattro libri dell' architettura* in 1570. This work was a summary of his studies of classical architecture and, as examples, he included a number of his own designs. In the preface, Palladio cites Vitruvius as his "master and guide," acknowledges his debt to Alberti, and calls his patron, Trissino, "the Splendor of our times." The title page depicts allegories of Geometry and Architecture holding tools of the trade. Some of these same tools appear on the base of the statue of Palladio in Vicenza.

The first book contains studies of materials, the classical orders, and decorative ornaments. The second contains designs for domestic structures like town and country houses. The third has designs for public buildings, bridges, and town planning. The fourth book deals with temples.

Although Palladio does not mention the musical ratios by name, he does advocate them in several places. For example, in Book 1, Chapter XXII, he recommends seven shapes for the floor plans of rooms:[34]

- Round (rarely)
- Square
- Length equal to the diagonal of a square (the root-2 rectangle)
- Square and a third (3 : 4 or the fourth)
- Square and a half (2 : 3 or the fifth)
- Square and two-thirds (3 : 5 or the square plus a fifth)
- Two squares (1 : 2 or the octave)

All the rectangular rooms (except the root-2 rectangle, which we will discuss in Chapter 4) are given simple whole-number ratios that are the musical ratios or derived from them.

ANDREA PALLADIO (1508–1580)

Born Andrea Di Pietro Della Gondola, Andrea was given the name Palladio by a patron (Count Gian Giorgio Trissino, a humanist poet and scholar) as an allusion to Pallas Athena of Greek mythology.

Palladio is regarded as the greatest Italian architect of the sixteenth century. He is noted for his treatise, *The Four Books of Architecture* (1570), and for his many palaces and villas still standing in northeastern Italy. His name appears today in the popular *Palladian window* (one composed of a large central section surmounted by a semicircular window, flanked by two narrower, rectangular windows).

Palladio was born in Padua, was apprenticed to a sculptor in his youth, and was later a stone mason. His projects included the villa of Count Trissino, at Cricoli near Vicenza, intended as an academy for Trissino's pupils. This *Academia Trissiniana* was an echo of the Medici Platonic academy in Florence and a distant echo of the Pythagorean and Platonic academies. Trissino took an interest in Palladio and strove to provide him with a humanist education.

FIGURE 1.27 ■ Statue of Palladio in Vicenza

At the Villa Trissino, Palladio made contacts and may have met the architect Sebastiano Serlio, whose books on architecture were to be an inspiration to him. On three visits to Rome with Trissino, he saw works by the architects Bramante, Peruzzi, and Raphael, and he took measurements of ancient Roman structures. One result of these trips was the publication of his small book *The Antiquities of Rome* (1554), which remained a popular tourist guide for 200 years.

In about 1540, Palladio designed his first villa and his first palace, to be followed by a long and intensive career of building design. Trissino, a great advocate of Vitruvius, had introduced the writings of that architect to Palladio, who later made the plates for Daniele Barbaro's edition of Vitruvius' treatise *On Architecture*, published in 1556. A statue of Palladio (Figure 1.27) stands in Vicenza, the home of many of Palladio's buildings.

Palladio goes on to recommend proportions for room heights in Book I, Chapter XXIII. If a room is 6×12 ft, its height should be 9 ft (the arithmetic mean between 6 and 12), or its height should be 8 ft (the harmonic mean). If a room is 4×9 ft, its height should be 6 ft (the geometric mean between 4 and 9). Also in Chapter XXIII are the geometric constructions for finding the three means that were presented earlier.

An example of Palladio's use of means to find the height of a room occurs in his Villa Rotonda, pictured in Figure 1.28. Each large corner room measures 26×15 ft. The arithmetic mean of these two dimensions gives the height of the room.

$$\frac{26 + 15}{2} = 20.5 \text{ ft}$$

Decline of the Musical Ratios in Architecture

The use of the musical ratios in art and architecture did not end with Palladio. Wittkower mentions their advocacy by Francesco Giorgi, Inigo Jones, Henry Wotton, Joshua Reynolds, Francois Blondel, and Bernardo Antonio Vittone. However, the end of their use was in sight. Wittkower goes on to chart the decline.

© Luke Daniek

FIGURE 1.28 Villa Rotonda in Vicenza

In 1663, the French architect and physician Claude Perrault rejected the notion that certain ratios were in themselves beautiful, but only because people were used to them. He wrote that musical ratios cannot be translated into visual ratios. In his *Vita di Andrea Palladio* (1762), the Italian architect Tomasso Temanza wrote that proportion in music is completely different from proportion in architecture. He maintained that (a) the eye is not capable of judging simultaneously the various ratios in a room, length to width, length to height, and width to height; and (b) the proportions perceived will vary with the angle of vision under which a building is viewed. He also wrote that the use of harmonic proportions would lead to sterility. The Italian architect Guarino Guarini (c. 1670) wrote, "To please the eye one must take away from or add to the proportions, because one object is placed at eye level, another at great height, another in an enclosed space, and yet another in the open air."

In his *Analysis of Beauty* (1753), the English painter and engraver William Hogarth rejected any connection between mathematics and beauty. He refers to "the strange notion that if certain divisions of a string produce harmony to the ear . . . similar distances in . . . form . . . would delight the eye." The Scottish philosopher and historian David Hume, in his *Of the Standard of Taste* (1757), even rejected the notion that beauty is inherent in object, such as a building. He said, "beauty and deformity . . . are not qualities in objects, but belong entirely to the sentiment. . . ." In other words, beauty is in the eye of the beholder. The British statesman and orator Edmund Burke, in *On the Sublime and Beautiful* of 1757, denied that beauty had "anything to do with calculation and geometry." The English philosopher Lord Kames (Henry Home), in his *Elements of Criticism* (1761), noted that as we move about a room, the proportions of length to breadth vary continuously.

The Scottish clergyman and author Archibald Alison wrote *Essays on the Nature and Principles of Taste* (1790). For him, the trains of thought and the associations produced by artworks make them beautiful, and abstract or ideal standards destroy their function. Richard Payne Knight, in his *Analytical Inquiry into the Principles of Taste* (1805), wrote that the same spatial proportions that make one animal beautiful make another absolutely ugly, but the same ratios that produce harmony in a fiddle also produce harmony in a flute or a harp. Therefore, musical harmony and spatial proportions can have nothing in common. Finally, the English art critic John Ruskin, in *The Seven Lamps of Architecture* (1849), declared that possible proportions are as infinite as possible airs in music, and inventing beautiful proportions must be left to the inspiration of the artist.

SUMMARY

Now that we have completed the first leg of our journey, let us, like the two-faced god Janus who could see both past and future, look back at what we've covered and peek ahead to what is to come.

In this chapter, we explored the mathematical concepts of ratio and proportion, followed by sequences, series, and means. The three means were later used to describe how Plato constructed the musical scales, and how Renaissance architects computed building proportions. Sequences and series will be covered again in our discussions of the Fibonacci sequence, Chapter 2, and again in Roman architecture, Chapter 4. We also studied the first boxed and numbered formulas. More of them will be added in the coming chapters. For a look at the entire list of formulas, turn to Appendix F.

We also introduced Pythagoras, some Pythagorean number lore and musical discoveries, and the quadrivium. We touched on three subjects of the quadrivium (music, geometry, and arithmetic). We will discuss astronomy in Chapter 11, "The Spheres and Celestial Themes in Art and Architecture," where we expand on the idea of the music of spheres and celestial themes in art. Geometry will, of course, be the backbone of every chapter. In this chapter, we discussed plane Euclidean geometry, showing geometric figures of one and two dimensions, and we will stay with two-dimensional figures through Chapter 9. We will cover solid Euclidean geometry and three-dimensional figures in Chapters 10 and 11, touch on projective geometry in Chapter 12, and introduce fractal geometry and fractal dimensions in Chapter 13.

We'll add to Pythagorean number symbolism by talking about the symbolism of particular numbers as we get to them in the course of the book, and provide a complete summary in Appendix E. We will note geometric symbols, such as the Tetraktys and pentagram of the Pythagoreans. More of these symbols are listed in Appendix D. For recurring ideas and art motifs, we have discussed correspondences and reconciliation of opposites. We will add the trivium to the quadrivium, which together comprise the recurring motif of the seven liberal arts.

We performed some basic geometric constructions, followed by constructions for the three means. We will have a great many more constructions in later chapters. You can see a listing of them in Appendix C.

The geographical locations given here were ancient Crotone and Renaissance Italy. The other places we will visit are listed in Appendix A. In this chapter, Pythagoras, Plato, Vitruvius, Alberti, and Palladio, all of whom we will encounter again, were introduced. Our entire cast of characters can be found in Appendix B.

The main examples of art and architecture used in this chapter were the proportions of buildings. We discussed the need for systems of proportions in architecture, and cited some examples from Alberti and Palladio. We saw that the Renaissance analogy between audible and visible proportions echoed, for them, the harmonic structure of all creation. Music had a particular attraction; it was one of the quadrivium, while art and architecture were not. Its study and use helped to raise the status of art, which was considered a manual occupation. We will see in Chapter 12 that this was one of the reasons why artists studied perspective.

We then traced the decline in the use of the musical ratios in architecture, one of the main objections to it being that these proportions are not even visible to the viewer, given different angles of vision, limited fields of view, perspective, foreshortening, obstacles blocking vision, and so forth. It is hard to imagine that Renaissance architects such as Alberti, who developed the science of perspective, did not realize this at the time. Their point, however, was that church proportions should reflect the perfection of heaven even if they couldn't be seen. Returning to Alberti's statement, "I conclude that the same numbers by means of which the agreement of sounds affect our ears with delight are the very same which please our eyes *and our mind*." (Emphasis mine.) Wittkower adds, "It follows that perfect proportions must be applied to churches, whether or not the exact relationships are manifest to the 'outward' eye."[35]

The musical ratios were the basis for only one system of proportions. We will describe another system (the modulor system) based on the golden ratio in Chapter 2, and the Roman system of proportions (ad quadratum) based on the square in Chapter 4. All of that is ahead of us. For now, we'll proceed to the second leg of our journey, an investigation of the fascinating golden ratio.

EXERCISES AND PROJECTS

1. Define or describe the following terms:

figured numbers	gnomon	Sacred Tetraktys	correspondences
quadrivium	monochord	Plato's lambda	musical ratio

2. What are the three main systems of architectural proportions?
3. Name three requirements of an ideal system of proportions.
4. Given any two numbers an octave apart, say a and $2a$, prove algebraically that their arithmetic mean will give an interval of a fifth, that the harmonic mean will give an interval of a fourth, and that the interval between the two means gives the tone.
5. Reconstruct the harmonic scale using Plato's method. You can find his method in *Timaeus* para. 35–37. You may choose to crunch the numbers using a spreadsheet.
6. Make a painting, graphic design, poster, screensaver, or quilt in which the dimensions or locations of most of the elements form an arithmetic, geometric, or harmonic series. Write a few paragraphs explaining your design.
7. Make a design, as in the preceding project, based on the musical ratios.
8. Design a small building in which the façade and floor plan are based on the musical ratios.
9. Make a working model of Pythagoras' monochord. Devise and conduct some experiments with it, and summarize your results in a short paper.

10. Instead of making a monochord, do the previous project using one string of a violin or guitar. Use a pencil underneath the string as the movable bridge. Make sure the string does not touch the frets of the guitar during your experiments.
11. A *sonometer* is a physics laboratory apparatus that can be used as a monochord. If you can find one at your school, use it to do the previous project.

The next three projects will add to our growing list of geometric constructions. See Appendix C for the complete listing.

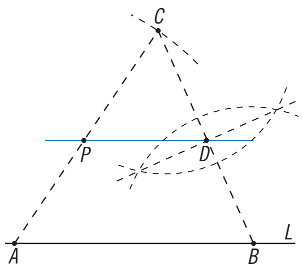


FIGURE 1.29 ■ Drawing a Line through a Given Point *P* Parallel to a Given Line *L*

12. *Drawing a Parallel.* Reproduce the construction given in Figure 1.29, which shows one way to construct a parallel line through a given point. Starting at the given point *P*, draw a line that intersects the given line *L* at some point *A*. Using a compass setting *AP*, draw an arc from point *P* to locate point *C*. Draw a line from *C* to some point *B* on *L*. Bisect *BC*, locating midpoint *D*. Finally, draw line *PD*, the parallel to *L*.
13. *Subdividing a Line Segment into Equal Parts.* Do the construction for subdividing a line segment *AB* into three equal parts shown in Figure 1.30. First draw any line *L* through *A*. Using a compass, lay off three equally spaced points *C*, *D*, and *E* on *L*. Draw *BE*, and then draw lines through *C* and *D* parallel to *BE*. Points *F* and *G* will subdivide *AB* into three equal parts. This construction can be easily modified to subdivide *AB* into any number of equal parts.

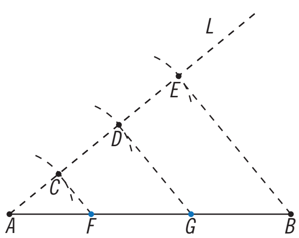


FIGURE 1.30 ■ Division of a Line Segment into Equal Parts

14. *Proportional Division of a Line Segment.* Do the construction shown in Figure 1.31, which demonstrates the proportional division of a line segment into a ratio of 2 : 3. To begin, draw any line *L* through *A*. Using a compass, lay off five equally spaced points on *L*. Draw *BD*, and from the 2/5 point *C*, draw *CE* parallel to *BD*. Point *E* will subdivide *AB* in a ratio of 2 : 3. Other ratios can be constructed in a similar manner.
15. Write a report about any book in the list of sources at the end of this chapter.
16. Write a short research paper or an in-depth term paper. Feel free to come up with your own topic, using the following suggestions to jog your imagination.
    - How proportions were solved during the Renaissance using the so-called *rule of three*. Research the rule of three, describe it in a paper, and give examples of its use.
    - The Pythagorean legend
    - Pythagorean arithmetic, outlined in Heath, Chapter III (see Sources)
    - Pythagorean number lore
    - Pythagorean geometry, as in Heath, Chapter V
    - The quadrivium and the trivium, the seven liberal arts
    - The history of the gnomon, outlined in Heath, p. 79
    - The physics of the vibrating string
    - The various kinds of means, including the three given in this chapter, and seven more, as described in Heath, p. 87
    - Raphael's *School of Athens*

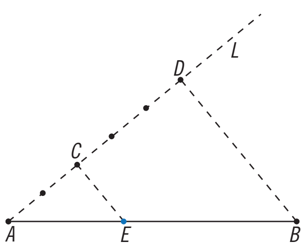


FIGURE 1.31 ■ Proportional Division of a Line Segment

- Plato, as outlined in Heath, p. 285
- Plato's Academy
- Plato's *Timaeus*
- The historical development of musical scales
- Vitruvius' *Ten Books on Architecture* and the impact it has had on the field
- The three classic orders of architecture
- Palladio's *Fugal System of Proportions*, described in Wittkower, p. 126 (see Sources), showing how Palladio coordinated the proportions of each room with *other* rooms in a building
- Leonardo's advocacy of the musical ratios, illustrated by his statement that "music is the sister of painting." Both convey harmonies, music by chords and painting by proportions.
- Leonardo's musical abilities
- How musical intervals and linear perspective are subject to the same numerical ratios, for objects of equal size receding at equal intervals diminish in harmonic progression
- Literary references to the *Music of the Spheres*
- Guido Monaco and the invention of musical notation
- Music as one of the subjects of the quadrivium (Figure 1.32)

17. Make an oral presentation to your class for any of the previous projects or papers.
18. If you play an instrument or sing, make a short musical presentation to your class in which you demonstrate the musical ratios mentioned in this chapter.

FIGURE 1.32 ■ Relief Carving of *Musica* at Pisa Duomo

Mathematical Challenges

The following problems require more mathematics than is presented in the text. Try some of them if you have the mathematical background.

19. Given an arithmetic mean A, a geometric mean G, and a harmonic mean H between the positive numbers a and b, where $a \neq b$, show that
$$A \geq G \geq H.$$
20. A driver goes a certain distance at a speed of 50 mi/h and returns to the starting point at 60 mi/h.
    a. Show that the arithmetic mean $(50 + 60)/2$ does *not* give the average speed for the entire round trip.
    b. Show that the harmonic mean *does* give the correct average speed for the round trip.
21. Prove that Euclid's construction in *Elements*, Book VI, Prop. 13, does indeed give the mean proportional.
22. The construction for the mean proportional can be used to geometrically find the square root of a number. Use it to find the square root of 7.64. Check your answer by calculator.
23. Prove that Palladio's construction does give the harmonic mean.

SOURCES

Aaboe, Asger. *Episodes from the Early History of Mathematics*. New York: Random House, 1964.
Alberti, Leon Battista. *The Ten Books of Architecture*. New York: Dover, 1986. A 1775 edition of the work written in 1452.
Cornford, Francis. *Plato's Cosmology*. New York: Harcourt, 1937.
De Vogel, C. J. *Pythagoras and Early Pythagoreanism*. Assen, the Netherlands: Van Gorcum, 1966.
Euclid. *The Thirteen Books of the Elements*. New York: Dover, 1956.
Grabow, Stephen. "Frozen Music: The Bridge Between Art and Science," article in *Companion to Contemporary Architectural Thought*, Farmer et al., eds. New York: Routledge, 1993, pp. 438–443.
Guthrie, Kenneth S. *The Pythagorean Sourcebook and Library*. Grand Rapids, MI: Phanes,1987.
Heath, Sir Thomas. *A History of Greek Mathematics*. New York: Dover, 1981. First published in 1921.
Hersey, George L. *Architecture and Geometry in the Age of the Baroque*. Chicago: University of Chicago, 2000.
Hersey, John. *Possible Palladian Villas*. Cambridge, MA: MIT Press, 1922.
Hersey, John. *Pythagorean Palaces*. Ithaca, New York: Cornell University Press, 1976.
Kappraff, Jay. *Connections: The Geometric Bridge Between Art and Science*. New York: McGraw-Hill, 1990.
March, Lionel. *Architectonics of Humanism*. Chichester, UK: Academy, 1998.
Newman, James R., ed. *The World of Mathematics*. New York: Simon and Shuster, 1956.
Palladio, Andrea. *The Four Books of Architecture*. New York: Dover, 1965.
Papadopoulos, Athanase. "Mathematics and Music Theory: From Pythagoras to Rameau." *The Mathematical Intelligencer*, Winter 2002, pp. 65–73.
Philip, J. A. *Pythagoras and Early Pythagoreanism*. Toronto: University of Toronto, 1966.
Plato. *Timaeus*. Edited and translated by John Warrington. London: Dent, 1965. Original c. 360 B.C.
Rowe, Colin. *The Mathematics of the Ideal Villa and Other Essays*. Cambridge, MA: MIT Press, 1976.
Vitruvius. *The Ten Books on Architecture*. New York: Dover, 1960.
Wittkower, Rudolf. *Architectural Principles in the Age of Humanism*. New York: Random House, 1965.

NOTES

1. Guthrie, p. 61.
2. Theon of Smyrna, *Mathematics Useful for Understanding Plato*, p. 66.
3. Aristotle, *Metaphysics* (I. 5 986 a 23).
4. Alberti, Book IX, Chapter V.
5. Hopper, p. 90.
6. Daniel Orth Bell. "New Identifications in Raphael's School of Athens." *Art Bulletin*, Dec. 1995, p. 639.
7. Clark, *Civilization*, p. 132.
8. Genesis 4:21.
9. *Republic*, p. 527.
10. Philebos, p. 51.

11. *Republic*, p. 603.
12. *Timaeus*, pp. 37–38.
13. *Timeaus*, p. 35 B, C.
14. Wittkower, p. 104.
15. Cornford, p. 72.
16. Tillyard, p. 101.
17. Wittkower, p. 142.
18. *Merchant of Venice*, V, I, 57.
19. *Harmonice Munde* (1619).
20. *A Song for St. Cecilia's Day*. 1687.
21. In "Advice to an Author," *Characteristicks*, 1737, I. p. 353.
22. Vitruvius I, I, 4.
23. *Proceedings of Nexus '96*, p. 115.
24. Vitruvius I, II, 4.
25. Vitruvius III, I, 4.
26. Kappraff in *Proceedings of Nexus '96*, p. 115.
27. Wittkower, p. 101.
28. Wittkower, p. 27.
29. Wittkower, p. 112.
30. Alberti, Book IX, Chapter V.
31. Wittkower, p. 45.
32. Kappraff in *Proceedings of Nexus '96*.
33. Palladio, Book IV, Chap. V.
34. See March, pp. 277–278, for an extensive list of ratios used by Palladio.
35. Wittkower, p. 27.