1

STRESS AND STRAIN

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1.1 INTRODUCTION

Quantitative treatment of the mechanical behavior of ceramics (or any solid) requires the mathematical description of stress and strain. Each of these quantities is a second-rank tensor. The full three-dimensional treatment of stress and strain will be presented, but it is convenient to begin with a simple two-dimensional treatment and discuss types of mechanical behavior in terms of these. This approach permits qualitative ideas about the mechanical behavior of ceramics to be explained without obscuring them within the complexity of a full three-dimensional treatment. The scheme followed here is to introduce stress and strain in terms of an easily visualized picture of deforming a bar.

Consider a rectangular bar [as shown in Figure 1.1(a)] of length L, height h, and width w with a force F acting parallel to the length on each end (i.e., uniaxial loading). (The force is denoted by an arrow; however, it is distributed uniformly over the surface to which it is applied.) The bar will deform under the action of the forces. Accordingly, the bar is taken to deform by an amount δL

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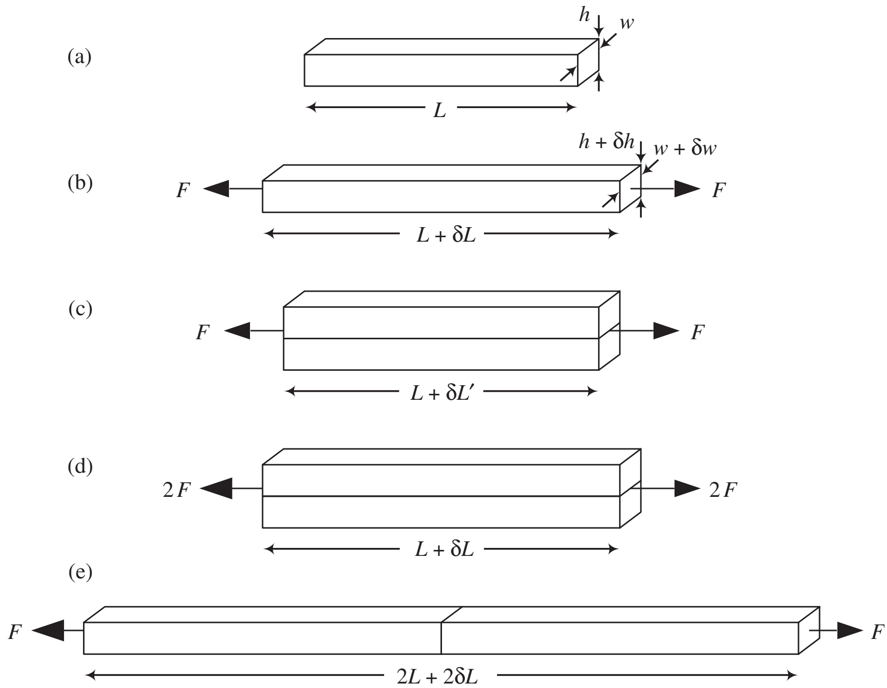


FIGURE 1.1 Bars subjected to tensile force.

in the direction of the force F and to deform an amount δh in the direction of the height h and δw in the direction of the width w, as shown in Figure 1.1(b). For a tensile force F as shown, the deformation δL is an extension, but the deformation at right angles to F is usually a contraction; δw and δh are generally negative for an applied tensile force. For certain directions in certain single crystals, the deformation δw or δh can actually be positive for an applied tensile stress. These are exceptional and rather rare cases.

Consider the two identical bars shown in Figure 1.1(c) subjected to the same force F. These bars are connected side by side in such a way that the load is shared equally between them. Each bar therefore supports only $F/2$ so that the deformations will be smaller in magnitude: that is, $\delta L' < \delta L$, $|\delta h'| < |\delta h|$, and $|\delta w'| < |\delta w|$. Absolute values are used for δh and δw since they are usually negative. For a linear elastic material (Chapter 3) these deformations will be exactly one-half of those for the single bar because the force supported by each bar is halved. However, if the force applied to the composite bars is $2F$, as shown in Figure 1.1(d), it is intuitively obvious that the deformations of each bar will be the same as the single bar in Figure 1.1(b).

Comparing the deformations in Figure 1.1(b)–(d) shows that the applied force is not a particularly useful way of quantifying the driving force for the deformation: for a fixed applied force the resulting deformation changes if the cross-sectional area of the bar is changed. However, we see that doubling the cross-sectional area while at the same time doubling the force does result in the same deformation. This suggests that the variable controlling the deformation behavior is not total load but load per unit area. Accordingly, the **tensile stress** or **normal stress** σ is defined as the force F divided by the cross-sectional area A, so that

$$\sigma = \frac{F}{A} = \frac{F}{wh} \tag{1.1}$$

This is the definition of **engineering stress**, in contrast to the quantity termed **true stress**, which will be discussed in a later section. However, ceramics usually will fail at small strain when the difference between engineering stress and true stress is not significant. Unless otherwise stated, the term ''stress'' will refer to engineering stress in this book.

Comparing now the deformation of two bars connected end to end [Figure 1.1(e)] with the deformation of a single bar, the force F produces twice the extension of the latter case. The deformation inside the material is accommodated by stretching and bending of the interatomic bonds; comparing Figures 1.1(b) and (e) suggests that the deformation at the atomic level is the same in both cases. Doubling the length of the specimen results in double the extension. The deformation is therefore specified by the strain, which is defined as the ratio of the extension to the original length:

$$\varepsilon = \frac{\delta L}{L} \tag{1.2}$$

This is the definition of **engineering strain**, in contrast to the quantity termed **true strain**, which will be discussed in a later section. Unless otherwise stated, the term ''strain'' will refer to engineering strain in this book.

By using stress and strain instead of force and deformation, for the bar of Figure 1.1 we find that for a given applied stress the strain will always be the same, irrespective of the length or cross-sectional area of the bar.

In addition to forces normal to the end faces of the bar, surface tractions or shear forces might also be applied. Figure 1.2 shows an initially rectangular body subjected to tractions T applied to its upper and lower surfaces. The tractions tend to cause the body to deform into a parallelepiped whose adjacent sides have rotated by an angle ϕ with respect to each other. The shear forces give rise to a **shear stress** τ, which is defined in an analogous fashion to the tensile stress; that is, the stress is the force divided by the area over which the force is applied:

$$\tau = \frac{T}{A} = \frac{T}{wL} \tag{1.3}$$

but in this case the force is applied parallel to the area. Similarly to tensile strain, the shear strain γ is defined as the ratio of the deformation to the original

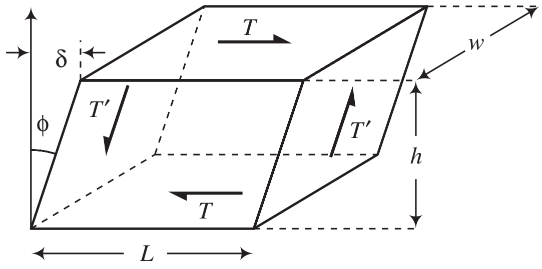


FIGURE 1.2 Rectangular body subjected to surface shear tractions.

dimension, which in this case is

$$\gamma = \frac{\delta}{h} = \tan\phi \approx \phi \tag{1.4}$$

It will be seen later that there are two different definitions of shear strain. Equation (1.4) defines the **engineering shear strain**. Note that this use of the term engineering strain is different from the context of using "engineering strain" to distinguish from "true strain"; see Section 1.7 for a discussion this topic. The engineering shear strain is related to the relative rotation angle ϕ and in the case of small strain (which is normally the case for deformation of ceramics) equals the angle in radians. Additional shear forces T' must be applied to the left and right sides of the body in Figure 1.2 in order to maintain rotational stability. They result in a shear stress τ that is equal and opposite to the shear stress on the top and bottom faces. The shear strain γ is the result of both shear stresses.

The example of Figure 1.1 is particularly simple because only one force is applied. Tensile forces could also be applied perpendicular to the length of the bar leading to additional tensile stress and strain components. Additionally, shear forces can be applied to the faces of the bar, leading to shear stresses and strains. For a complete three-dimensional description of stress and strain, each must be represented by second-rank tensors, that is, 3×3 matrices with some special tensor properties.

The example of Figure 1.1 is also simple because the stress and strain are uniform throughout the bar. In more complex problems the stress and strain vary with position inside a body and the definitions need to be modified. For example, if the stress is nonuniform, it is inappropriate to define the strain as the change in overall length divided by the original overall length. This introduces the concept of **stress at a point** and **strain at a point**. However, the definitions of stress and strain at a point are essentially the same except that the definitions involve forces and deformations in an infinitesimally small element instead of the overall body.

1.2 TENSOR NOTATION FOR STRESS

Figure 1.3 shows a small element of material inside a body. Each face of the element is acted upon by forces from the surrounding material. The total vector force $\underline{\mathbf{F}}$ on the face of constant x can be resolved into three mutually orthogonal components: a force F_x perpendicular to the face and two surface tractions (shear forces) parallel to the face acting in the y and z directions, T_y and T_z, respectively. Surface area is a vector quantity of magnitude equal to the area and direction normal to the area acting out of the body. For the x face under consideration the vector area is A_x acting in the positive x direction, as shown in the figure. The force components on this face therefore represent three components of stress, a tensile stress and two shear stresses, all acting at mutual right angles.

The components of stress are generally written (Sines, 1969)

$$\sigma_{\text{on } x \text{ plane in } y \text{ direction}} = \sigma_{xy} = \frac{\text{force in } y \text{ direction acting on } x \text{ plane}}{\text{area of plane perpendicular to } x} \tag{1.5}$$

Here "x plane" means "plane perpendicular to the x axis" or "plane of constant x." In terms of indices denoting xyz by x_1, x_2, x_3, the above stress component is written

$$\sigma_{\text{on } x_i \text{ plane in } x_j \text{ direction}} = \sigma_{ij} \tag{1.6}$$

There are thus nine possible stress components in three dimensions. However, it will be shown that $\sigma_{ij} = \sigma_{ji}$ for $i \neq j$, so that there are only six

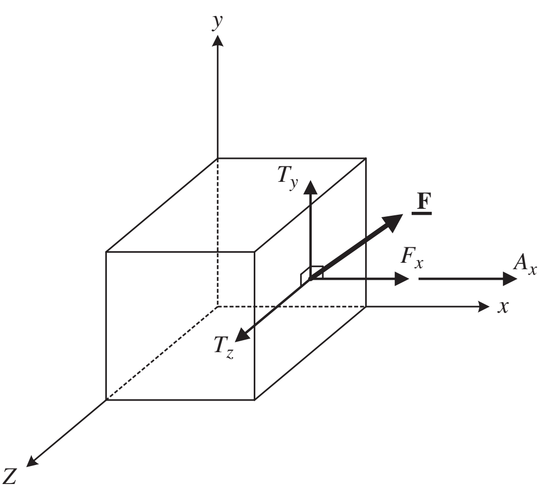


FIGURE 1.3 Forces acting on the face of a small element of material.

independent components. The components with $i = j$ are termed **normal stresses** or **tensile stresses** (with a compressive stress considered a negative tensile stress). For normal stresses the force acts in a direction perpendicular (normal) to the area to which it is applied (parallel to the area vector). The components with $i \neq j$ are termed **shear stresses** for which the force acts in a direction parallel to the area over which it is applied (perpendicular to the area vector). The shear stresses are sometimes written τ_{ij} instead of σ_{ij} to emphasize their nature as shear stresses.

Written as a matrix the components of stress in three dimensions are

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \text{or} \quad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{1.7}$$

with

$$\sigma_{12} = \sigma_{21} \qquad \sigma_{23} = \sigma_{32} \qquad \sigma_{31} = \sigma_{13} \tag{1.8}$$

The double underlines for $\underline{\underline{\sigma}}$ signify that it is a second-rank tensor. It is easier to visualize matters in two dimensions and attention will be restricted accordingly in the next few pages. For two dimensions there are a total of four stress components of which only three are independent. Written as the matrix, the stresses in two dimensions are

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \quad \text{or} \quad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \tag{1.9}$$

with

$$\sigma_{12} = \sigma_{21} \tag{1.10}$$

The stress components in two dimensions are shown in Figure 1.4, which shows an infinitesimal square element acted upon by forces both normal and parallel to each surface. The arrows for the stresses show the direction of action of the forces represented by the stresses—the stresses themselves act in both directions simultaneously. The forces normal to the faces produce normal stresses. The force F_x on the right face (the $+x$ face) is balanced by an equal and opposite force $-F_x$ on the left face (the $-x$ face) to maintain stability. Similar results hold for the upper and lower faces. The forces parallel to the faces produce shear stresses; again each force on one face is balanced by an equal and opposite force on the opposite face. For the element to be under no net torque, the shear forces must balance such that $\sigma_{xy} = \sigma_{yx}$.

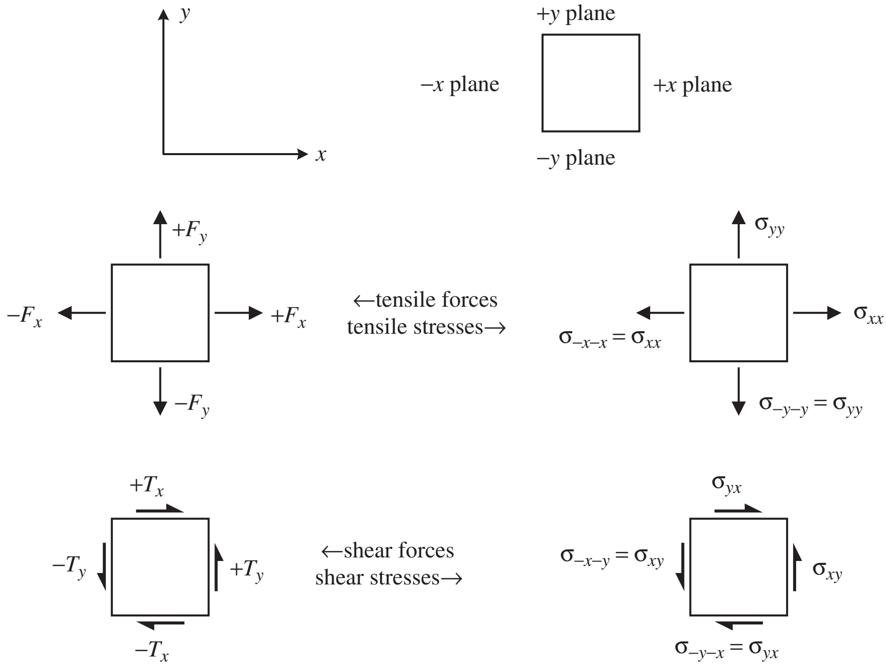


FIGURE 1.4 Components of stress in two dimensions.

The sign convention for stress can be defined in terms of the directions of the force vectors and area vectors. If both the force and area vectors act in the positive direction or in the negative direction, the stress is positive; if the force acts in the negative direction and the area in the positive direction or vice versa, the stress is negative. For example, in Figure 1.4 the component of normal stress on the right-hand side of the element, σ_{xx}, represents a force acting in the positive x direction and the area normal is also in the positive x direction, giving a positive stress. The component on the left, σ_{-x-x}, is a force acting in the negative x direction with an area normal acting in the negative x direction and so is again positive and therefore equals σ_{xx}. This sign convention coincides with that stated earlier, that tensile stresses (i.e., stresses tending to cause extensions) are positive while compressive stresses (tending to cause contractions) are negative. Consider now that the shear stress σ_{xy} as drawn in Figure 1.4 is positive since it is a force acting in the positive y direction is acting on an area whose normal is in the positive x direction. The reader should verify that σ_{yx} as drawn in Figure 1.4 is also acting in the positive sense. Using this sign convention, σ_{xy} and σ_{yx} tend to cause rotations in the opposite sense, but since $\sigma_{xy} = \sigma_{yx}$, there is no net rotational moment.

The dimensions of stress are force divided by area and so the unit is newtons per square meter or the preferred unit the pascal (Pa). Other units that might be

TABLE 1.1 Units of Stress

$1\ \mathrm{N/m^2} = 1\ \mathrm{Pa}$
$1\ \mathrm{kg/m^2} = 9.81\ \mathrm{Pa}$
$1\ \mathrm{dyn/cm^2} = 0.1\ \mathrm{Pa}$
1 psi (pound per square inch) = 6.89476 kPa
1 bar = 0.1 MPa
1 atm = 0.101325 MPa

encountered are listed in Table 1.1 together with their conversion factors to pascals. Stresses encountered while working with ceramics range roughly from 1 to 1000 MPa (1 GPa)—most ceramics survive a normal stress of 10 MPa while few can withstand a stress of a few gigapascals.

1.3 STRESS IN ROTATED COORDINATE SYSTEM

We now examine how a stress tensor is expressed in a rotated coordinate system. Considering the bar of Figure 1.1 and taking the x axis along the long axis of the bar, the stress caused by a load in this direction acting on the cross-sectional area A_x of the bar is

$$\sigma = \frac{F_x}{A_x} = \sigma_{x_1 \text{ plane}, x_1 \text{ direction}} = \sigma_{11} = \sigma_{xx} \tag{1.11}$$

For this bar and this load the other stress components are zero so that the stress tensor anywhere in the bar is given by

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.12}$$

For this simple **state of stress**, known as **uniaxial tension**, it is obvious that the frame of reference for the stress tensor should be chosen with one axis parallel to the axis of the bar. However, while this is a convenient choice, the stress could be referred to any other system of axes. The question now arises, if one knows the components of stress referred to some set of axes xyz, what are the components of stress referred to some other set of axes $x'y'z'$ which are rotated with respect to the xyz axes? We will first consider the two-dimensional case in which the stress components to be determined are referenced to axes $x'y'$ which are rotated at an angle θ with respect to the xy axes.

Figure 1.5(a) defines the relationship between the xy and $x'y'$ axes and defines the sense of the rotation angle θ, which is the angle measured

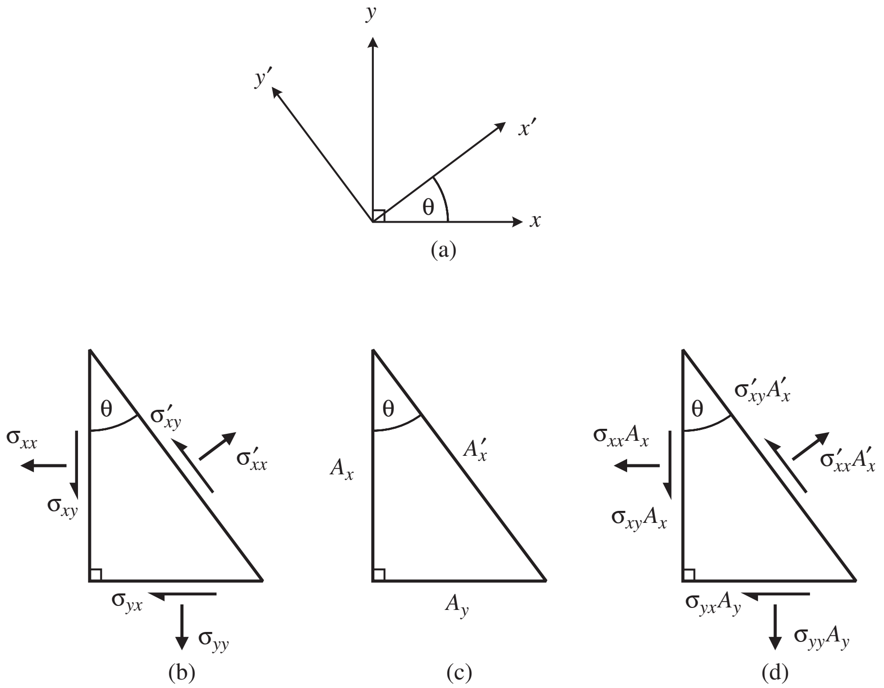


FIGURE 1.5 Stress in a rotated coordinate system: (a) axes; (b) stresses; (c) areas; (d) forces.

counterclockwise from the x axis to the x' axis. We consider the stability of a small element of the solid shown in the lower left portion of Figure 1.5, which has a triangular section whose three faces are perpendicular to the x, y, and x' axes. The stresses experienced by each face of the element do not depend on which set of axes are chosen to describe them and so we may choose any convenient axes provided we use the same set of axes consistently for each face. We therefore refer the two orthogonal faces to the xy axes and the hypotenuse to the $x'y'$ axes. Figure 1.5(b) shows the components of stress acting on each face using this choice of axes. The arrows point in the direction of the forces represented by the stress components. We use the notation here that components of stress referred to the $x'y'$ axes, $\sigma_{i'j'}$, may be rewritten as σ'_{ij} for clarity. The reader should verify that each component of stress points in the positive direction using the sign convention for stresses.

The area of each face is proportional to the length of the side of the triangular section and the three areas are defined in Figure 1.5(c). The forces applied to each face, shown in Figure 1.5(d), are calculated by multiplying each component of stress by the area of the face to which it is applied. The triangular element is stable (i.e., not accelerating) so the net force applied to it must equal zero. Equating the components of all the forces resolved in the x' direction gives

$$\sigma'_{xx}A'_x = \sigma_{xx}A_x\cos\theta + \sigma_{xy}A_x\sin\theta + \sigma_{yy}A_y\sin\theta + \sigma_{yx}A_y\cos\theta \tag{1.13}$$

The areas of the sides are related by

$$A_x = A'_x\cos\theta \qquad \text{and} \qquad A_y = A'_x\sin\theta \tag{1.14}$$

Substitution into Eq. (1.13) and using $\sigma_{xy} = \sigma_{yx}$ give

$$\sigma'_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \tag{1.15}$$

The shear stress in the $x'y'$ coordinate system is found by resolving components of force in the y' direction:

$$\sigma'_{xy}A'_x = -\sigma_{xx}A_x\sin\theta + \sigma_{xy}A_x\cos\theta + \sigma_{yy}A_y\cos\theta - \sigma_{yx}A_y\sin\theta \tag{1.16}$$

which gives

$$\sigma'_{xy} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) \tag{1.17}$$

It may be shown that Eqs. (1.15) and (1.16) also ensure rotational stability, namely that there is no net torque acting on the element. Similar considerations applied to a triangular element whose hypotenuse is perpendicular to the y' direction provides the final stress component σ'_{yy}:

$$\sigma'_{yy} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \tag{1.18}$$

Using the well-known trigonometric identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\sin\theta\cos\theta \tag{1.19}$$

the three components of stress can be expressed in terms of 2θ:

$$\begin{aligned}
\sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \\
\sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \sigma_{xy}\sin 2\theta \\
\sigma'_{xy} &= \tfrac{1}{2}(\sigma_{yy} - \sigma_{xx})\sin 2\theta + \sigma_{xy}\cos 2\theta
\end{aligned} \tag{1.20}$$

Turning now to the three-dimensional case, a general expression for the stresses on a plane of any orientation in three dimensions can be written in

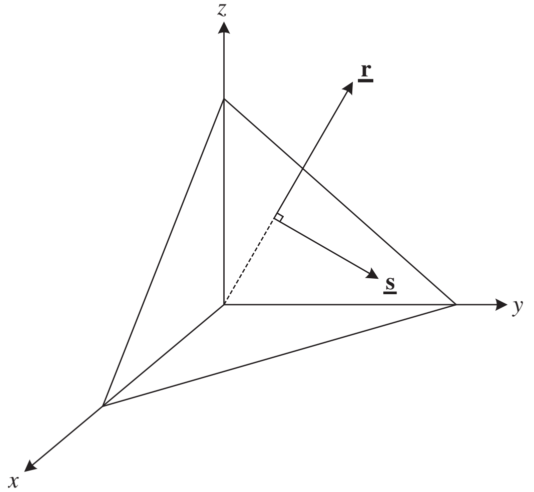


FIGURE 1.6 Plane with normal **r** and direction **s** in the plane.

terms of the direction cosines of the normal to the plane and the direction of action of the stress (Sines, 1969). Consider the plane in Figure 1.6 having a normal **r** with direction cosines a_{ir} to the axes $xyz = x_1x_2x_3$. The stress normal to this plane is

$$\sigma'_{rr} = \sum_i \sum_j a_{ir}\, a_{jr}\, \sigma_{ij} \tag{1.21}$$

The shear stress acting on this plane in the **s** direction with direction cosines a_{js} is

$$\sigma'_{rs} = \sum_i \sum_j a_{ir}\, a_{js}\, \sigma_{ij} \tag{1.22}$$

1.4 PRINCIPAL STRESS

Examination of Eqs. (1.20) shows that at a particular value of θ defined by

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \tag{1.23}$$

σ'_{xy} is zero. This means that for any general stress tensor in two dimensions a set of axes can be found for which the shear stresses vanish. These axes are called the **principal axes** and the directions of the principal axes are **principal directions**. Planes containing pairs of principal axes are **principal planes**. The normal stresses referred to the principal axes are the **principal stresses**. Referred to the principal axes, Eqs. (1.20) become

$$\begin{aligned} \sigma'_{xx} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + R \\ \sigma'_{yy} &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - R \\ \sigma'_{xy} &= 0 \end{aligned} \tag{1.24}$$

where

$$R = \sqrt{\left[\tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\right]^2 + \sigma_{xy}^2} \tag{1.25}$$

The principal stresses σ'_{xx} and σ'_{yy} may be renamed σ_1 and σ_2 where, by convention, $\sigma_1 \geq \sigma_2$.

Inverting Eqs. (1.24) using (1.23) permits calculation of the components of stress referenced to axes inclined at an angle θ to the principal axes in terms of the principal stresses:

$$\begin{aligned} \sigma_{xx} &= \tfrac{1}{2}(\sigma'_{xx} + \sigma'_{yy}) + \tfrac{1}{2}(\sigma'_{xx} - \sigma'_{yy})\cos 2\theta = \sigma'_{xx}\cos^2\theta + \sigma'_{yy}\sin^2\theta \\ \sigma_{yy} &= \tfrac{1}{2}(\sigma'_{xx} + \sigma'_{yy}) - \tfrac{1}{2}(\sigma'_{xx} - \sigma'_{yy})\cos 2\theta = \sigma'_{xx}\sin^2\theta + \sigma'_{yy}\cos^2\theta \\ \sigma_{xy} &= \sigma_{yx} = \tfrac{1}{2}(\sigma'_{xx} - \sigma'_{yy})\sin 2\theta = (\sigma'_{xx} - \sigma'_{yy})\sin\theta\cos\theta \end{aligned} \tag{1.26}$$

The Mohr circle construction (Sines, 1969; Courtney, 1990) is a graphical method for obtaining the principal stresses that gives useful insight into the properties of the stress tensor. In this construction the abscissa is normal stress σ and the ordinate is shear stress, usually written as τ for the Mohr circle. A different sign convention applies to shear stresses in the Mohr circle construction: Shear stresses causing a clockwise rotation are taken as positive and those causing a counterclockwise rotation are taken as negative (Sines, 1969). In our case with the z axis out of the paper in Figure 1.5, σ_{xy} is a positive rotation so $\tau_{xy} = \sigma_{xy}$ and σ_{yx} is a negative rotation giving $\tau_{yx} = -\sigma_{yx}$. For a given pair of principal stresses σ_1 and σ_2 the locus of Eqs. (1.26) as θ varies is a circle on a shear stress/normal stress plot, as shown in Figure 1.7. One can consider this circle in two ways: (1) for a situation in which the principal stresses and directions are known and (2) when the stress tensor is known and the principal stresses and directions are to be determined.

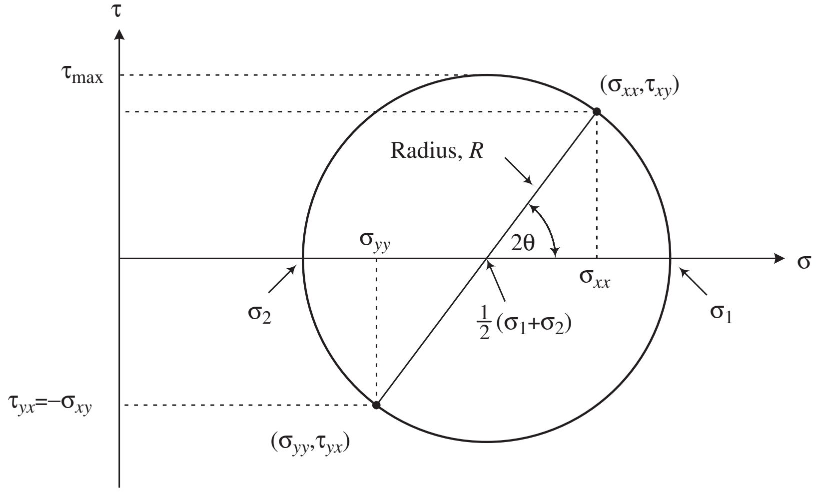


FIGURE 1.7 Mohr circle for biaxial tension.

In the first situation the initial axes are principal axes, $\sigma'_{xx} = \sigma_1$, $\sigma'_{yy} = \sigma_2$, and $\sigma'_{xy} = \sigma'_{yx} = 0$. The center of the circle is on the σ axis at $\frac{1}{2}(\sigma_1 + \sigma_2)$. The radius of the circle is $\frac{1}{2}(\sigma_1 - \sigma_2)$. The circle intersects the σ axis at σ_1 and σ_2. To obtain the stress in a system of coordinates rotated by an angle θ, a diameter of the circle is drawn rotated through an angle 2θ to the σ axis. This diameter intersects the circle at two points. The intersection point adjacent to the 2θ angle has abscissa σ_{xx} and ordinate $\tau_{xy} = \sigma_{xy}$. The intersection of the circle with the opposite end of the diameter has abscissa σ_{yy} and ordinate $\tau_{yx} = -\sigma_{yx} = -\sigma_{xy}$.

In the second situation the initial axes are not principal since the shear stress is not zero. Two points on the Mohr circle are constructed with coordinates (σ_{xx}, σ_{xy}) and (σ_{yy}, $-\sigma_{xy}$). Alternatively and equivalently, the Mohr convention for the sign of shear stress (clockwise rotation corresponds to positive shear) may be used and the two points are (σ_{xx}, τ_{xy}) and (σ_{yy}, τ_{yx}). The center of the Mohr circle is on the σ axis at $\frac{1}{2}(\sigma_{xx} + \sigma_{yy})$ and the radius is given by

$$\text{Radius} = R = \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2\right]^{1/2} \tag{1.27}$$

The principal stresses are the intersection of the circle with the σ axis, and the orientation of the principal axes is given by the angle θ:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}\right) \tag{1.28}$$

Examination of the Mohr circle illustrates two important results. For a given pair of principal stresses σ_1 and σ_2, the maximum normal stress that can be observed in any rotated coordinate system is on the extreme right-hand side of the circle; that is, the maximum normal stress is the bigger principal stress $\sigma_{max} = \sigma_1$. Similarly, the minimum normal stress (most negative stress) is at the extreme left of the circle and equals the smaller principal stress $\sigma_{min} = \sigma_2$. Further, the maximum shear stress occurs at the top and bottom of the Mohr circle, $2\theta = \pm 90^\circ$, $\theta = \pm 45^\circ$, and is given by

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) \tag{1.29}$$

These results show the importance of principal stress analysis in understanding the mechanical behavior of ceramics. Most ceramics fail by brittle fracture in tension: Failure is controlled by the biggest tensile stress σ_1. Ceramics at high temperature (as well as most metals and polymers) can deform in shear: This deformation is controlled by the maximum shear stress, which is itself related to the principal stresses. Principal stress analysis is therefore necessary in understanding the response of any material to a complex state of stress.

It is interesting to examine the stresses given by the Mohr circle construction (or equivalently by direct calculation from the above equations) for special situations. Consider the case of a bar under simple uniaxial tension σ such as in Figure 1.1. No other stresses are applied and in particular no shear stresses are applied, so the axis of the bar is principal and an axis perpendicular to the bar is also principal. The principal stresses are therefore $\sigma_1 = \sigma$ and $\sigma_2 = 0$. The Mohr circle for this case is shown in Figure 1.8(a). The center of the circle is at $\frac{1}{2}\sigma$ and the radius is also $\frac{1}{2}\sigma$. The shear stresses are a maximum at $2\theta = \pm 90^\circ$ with magnitude $\frac{1}{2}\sigma$. At this orientation the two normal stresses are both $\frac{1}{2}\sigma$. This example illustrates how a ductile material can fail in shear even though only a tensile stress is applied.

As a further example consider the application of two equal stresses $\sigma_1 = \sigma_2 = \sigma$, that is, equi-biaxial tension. The center of the Mohr circle is at σ and the radius is zero. The Mohr circle is just a point, as shown in Figure 1.8(b). As the coordinate system rotates, the values of σ_{xx} and σ_{yy} remain equal to σ and the shear stresses remain equal to zero for all values of θ.

A final example, equi-opposite biaxial tension, is shown in Figure 1.8(c), in which a tensile stress of magnitude σ is applied in one direction and a compressive stress of equal magnitude, $-\sigma$, is applied in a perpendicular direction. Since no other stresses are applied, these stresses are principal so $\sigma_1 = \sigma$ and $\sigma_2 = -\sigma$. The Mohr circle in this case is centered on the origin. At $\theta = \pm 45^\circ$ the normal stresses are zero and the shear stress is the maximum, $\tau_{max} = \sigma$, so in this case axes can be found for which the stress tensor is purely shear.

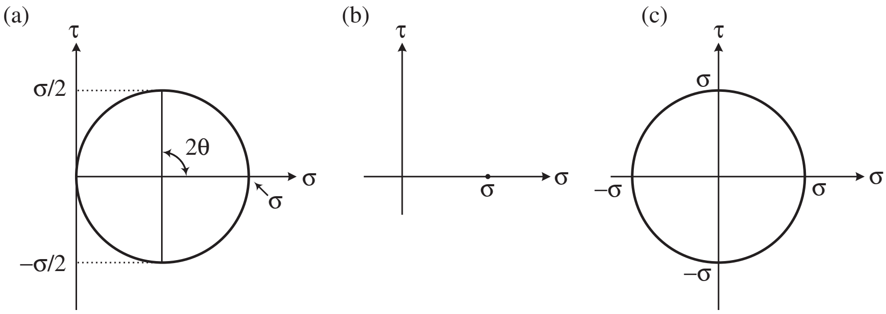


FIGURE 1.8 Mohr circle for (a) uniaxial tension, (b) equibiaxial tension, and (c) equi-opposite biaxial tension.

1.4.1 Principal Stresses in Three Dimensions

The principal stresses in three dimensions are found by solving the eigenequation for the stress tensor:

$$\left|\underline{\underline{\boldsymbol{\sigma}}} - \sigma \underline{\underline{\mathbf{I}}}\right| = 0 \qquad \text{or} \qquad \left|\underline{\underline{\boldsymbol{\sigma}}} - \sigma \underline{\underline{\mathbf{I}}}\right| = \det\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \quad (1.30)$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix and σ is a scalar. The eigenequation represents a cubic equation in σ (Courtney, 1990):

$$\sigma^3 - J_1\sigma^2 - J_2\sigma - J_3 = 0 \quad (1.31)$$

where the coefficients J_1, J_2, and J_3 are the **stress invariants** (see below). This equation is solved to find the three eigenvalues σ_1, σ_2, and σ_3. The eigenvectors point in the principal directions and so coincide with the principal axes. Unlike the two-dimensional case, there are no conveniently simple equations for the principal stresses in three dimensions, nor is there a simple geometric solution equivalent to the Mohr circle. However, eigenanalysis is readily performed using a wide variety of computational tools. By convention the eigenvalues are chosen so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$; σ_1 is therefore the largest normal stress in any direction and σ_3 is the smallest (most negative). Analogous to the two-dimensional case, the shear stress has maximum values on planes inclined at $\pm 45°$ to the principal planes. The largest shear stress acting on any plane is therefore

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) \quad (1.32)$$

with locally maximum values of $\frac{1}{2}(\sigma_1 - \sigma_2)$ and $\frac{1}{2}(\sigma_2 - \sigma_3)$.

Since we exist in a three-dimensional world, it is clear in retrospect that the two-dimensional analysis presented earlier contains the implicit assumption that the z plane is principal. In many practical situations one of the principal planes is known and the two-dimensional treatment is appropriate. For example, a traction-free surface is principal. Most mechanical testing techniques (Chapter 6) apply a simple stress tensor to samples and at least one principal plane is readily identifiable.

1.5 STRESS INVARIANTS

When stresses are transformed from one coordinate system to a rotated system, there are three properties of the stress tensor that remain constant. These three stress invariants are

$$\begin{aligned} J_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 \\ J_2 &= -\sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \\ &= -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \\ J_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \\ &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{1.33}$$

The existence of these quantities that are invariant under rotations means that the stress tensor is indeed a tensor and not simply a matrix. The **hydrostatic stress** is the mean normal stress:

$$\sigma_h = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = \tfrac{1}{3}J_1 \tag{1.34}$$

and is clearly invariant, as is the **hydrostatic pressure**, which, taking a compressive stress as a positive pressure, is minus the hydrostatic stress:

$$p = -\sigma_h = -\tfrac{1}{3}J_1 \tag{1.35}$$

1.6 STRESS DEVIATOR

It is well known that the effect of multiple forces can be combined by vector addition in which corresponding components of forces are added. It is a general property of vectors such as force and area. A similar result holds for the stress tensor since the components of stress represent components of forces and areas. If multiple stresses are applied to a body, the stress tensors for each stress can be added together element by element to obtain the overall stress tensor. This is known as the **principal of superposition**—different stress tensors are superimposed by simple tensor addition. This result is useful in a wide variety of situations. Conversely, a total stress tensor can be decomposed into two or more tensors.

It is useful to separate the stress into the components causing dilation without change of shape (pressure) and components causing distortion without change in volume. For example, to a first approximation hydrostatic pressure alone causes transformation in transformation-toughened zirconia. Plastic deformation is not caused by hydrostatic pressure so that it is sometimes useful to subtract the hydrostatic stress and consider only the remaining stresses. The **stress deviator** $\underline{\underline{\boldsymbol{\sigma}}}^*$ is defined by

$$\begin{aligned} \sigma_{ij}^* &= \sigma_{ij} - \tfrac{1}{3} J_1 \\ &= \sigma_{ij} + p \qquad i = j \\ \sigma_{ij}^* &= \sigma_{ij} \qquad\quad\;\; i \neq j \end{aligned} \tag{1.36}$$

or explicitly

$$\underline{\underline{\boldsymbol{\sigma}}} = \underline{\underline{\boldsymbol{\sigma}}}^* - p\,\underline{\underline{\mathbf{I}}} = \begin{pmatrix} \sigma_{11}+p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}+p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}+p \end{pmatrix} + \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{1.37}$$

1.7 STRAIN

When forces are applied to a body, it deforms. Every point in the body is displaced from its original position by an amount that can be represented by a vector. This results in the concept of the **vector displacement field**—at every point in the body the deformation is represented by a vector $\underline{\mathbf{v}}$. The three components of $\underline{\mathbf{v}}$ in the x, y, and z directions are u, v, and w or, equivalently, u_1, u_2, and u_3. The components of $\underline{\mathbf{v}}$ are all functions of position:

$$\underline{\mathbf{v}} = \begin{cases} (u, v, w) = (u(x,y,z), v(x,y,z), w(x,y,z)) \\ (u_1, u_2, u_3) = (u_1(x_1,x_2,x_3), u_2(x_1,x_2,x_3), u_3(x_1,x_2,x_3)) \end{cases} \tag{1.38}$$

The strains within the body can be expressed in terms of this vector displacement field. Considering first the two-dimensional case for simplicity, Figure 1.9 shows a small rectangular element $ABCD$ with sides dx and dy which is displaced to $A'B'C'D'$ when forces are applied. The element is deformed by the

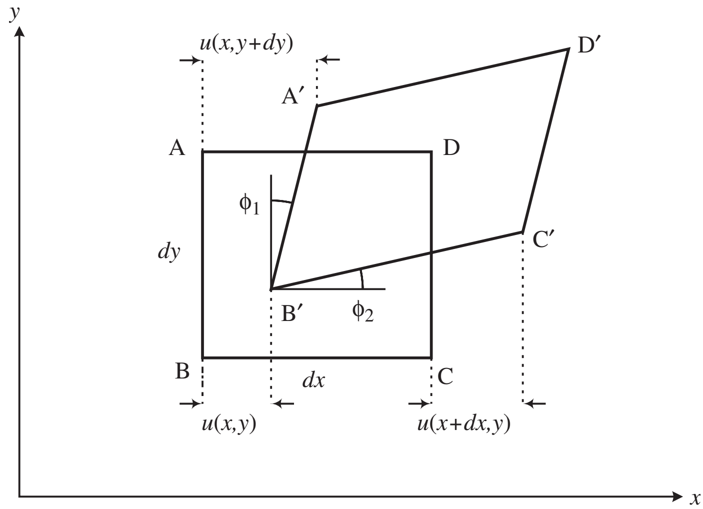


FIGURE 1.9 Definition of strain.

forces so that the sides AB and BC are rotated by angles ϕ_1 and ϕ_2, respectively, from their original orientation. The coordinates of the vertex B are (x, y) so that the displacement BB' in the x direction is $u(x, y)$ and the displacement CC' is $u(x + dx, y)$. The change in length of the side BC measured in the x direction is therefore $u(x + dx, y) - u(x, y)$. The normal strain measured in the x direction in the side BC as it deforms to $B'C'$ [from Eq. (1.2)] is

$$\varepsilon_{xx} = \frac{u(x + dx, y) - u(x, y)}{dx} = \frac{\partial u}{\partial x} \qquad (1.39)$$

Similar considerations can be used to obtain the results for the three-dimensional case, giving

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad (1.40)$$

By changing the coordinate notation to (x_1, x_2, x_3), we find the general form for the three components of normal strain:

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i} \qquad i = 1, 2, 3 \qquad (1.41)$$

The vertex AA' is displaced by a distance $u(x, y + dy)$ in the x direction so that the distance $B'A'$ measured in the x direction is $u(x, y + dy) - u(x, y)$. In the limit of small strain the angle ϕ_1 is given by

$$\phi_1 \simeq \tan\phi_1 = \frac{u(x, y+dy) - u(x, y)}{dy} = \frac{\partial u}{\partial y} \tag{1.42}$$

Similarly the angle ϕ_2 is given by

$$\phi_2 = \frac{\partial v}{\partial x} \tag{1.43}$$

The sides *AB* and *BC* are rotated relative to each other by a total angle $\phi = \phi_1 + \phi_2$. Using the definition for engineering shear strain, Eq. (1.4), the shear strain γ_{xy} is

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{1.44}$$

In general, for three dimensions we have

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad i,j = 1,2,3 \qquad i \neq j \tag{1.45}$$

Examination of (1.45) shows that $\gamma_{ij} = \gamma_{ji}$ so that while there are six components of shear strain there are only three independent components. If i is set equal to j in (1.45), the result differs from the definition for normal strain [Eq. (1.41)] by a factor of 2. Therefore the definitions of normal strain and engineering shear strain are incompatible so that they cannot be grouped together into a tensor and manipulated as a whole. This difficulty is overcome if we define the **simple shear strain** ε_{ij} to be one-half of the engineering shear strain. The general form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i,j = 1,2,3 \tag{1.46}$$

is applicable to both normal and shear strains so that they can be grouped together to form the strain tensor:

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \qquad \text{or} \qquad \underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{1.47}$$

The stress and strain tensors are both symmetric matrices, which means that many of the properties of the strain tensor are analogous to those of the stress tensor. In particular, it was noted earlier that for any set of stress components there are three orthogonal directions, the principal directions for stress, for which the shear stresses are all zero. An analogous result holds for strain. For any set of strain components there are three orthogonal directions, called the principal directions for strain, for which the shear strains are zero; the corresponding normal strains are called the principal strains. For an elastically isotropic body the principal directions for stress and the principal directions for strain coincide. For an elastically anisotropic body this is not necessarily so.

The methods described earlier for determining principal stresses can all be used for determining principal strains; in all equations components of stress are replaced by the equivalent components of strain. In particular, the Mohr circle construction can be used for two-dimensional cases (i.e., where the third direction is known to be principal). Another set of results for stress that can be adopted for strain is the stress invariants and the definition of deviatoric stress. The corresponding strain invariants, hydrostatic strain and strain deviator are given by substituting strain for stress in Eqs. (1.33) and (1.36). A parameter of interest is the **volumetric strain** ε_V, which is the fractional change in volume caused by the deformation. It equals the first strain invariant obtained from Eq. (1.33):

$$J_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_V = 3\varepsilon_h \tag{1.48}$$

1.8 TRUE STRESS AND TRUE STRAIN

The stress defined in Eq. (1.1) and the strain defined in Eq. (1.2) both contain factors which involve the size of the specimen, namely the cross-sectional area A and the length L. However, both of these parameters are changing as deformation takes place. We define now two types of stress and strain: **engineering stress** and **engineering strain**, where the cross-sectional area and length have the original values before the start of the deformation, and **true stress** and **true strain**, where the instantaneous cross-sectional area and length are used (Table 1.2). They are equivalent for small stresses and strains.

TABLE 1.2 Engineering and True Stress and Strain

Engineering	True
$\sigma = \dfrac{F}{A_{\text{original}}} = \dfrac{F}{A_0}$	$\sigma^t = \dfrac{F}{A_{\text{instantaneous}}} = \dfrac{F}{A}$
$\varepsilon = \dfrac{\delta L}{L_{\text{original}}} = \dfrac{\delta L}{L_0}$	$\varepsilon^t = \displaystyle\int_{L_0}^{L_0+\delta L} \frac{dL}{L} = \ln\left(\frac{L_0 + \delta L}{L_0}\right)$

Engineering stress and strain are convenient to use because in performing a test with uniaxial stress the original length and cross-sectional area are easily measured but it is not as convenient to continually measure their instantaneous values.

1.8.1 True Strain

To consider why we call stresses and strains "true" stresses and strains, we consider strains using the illustration of Figure 1.1. It should be true for a self-consistent definition of strain that, if the rectangular parallelepiped is strained in two steps by extending it first from L_0 to $L_1 = L_0 + \delta L_1$ and then extending again from L_1 to $L_2 = L_1 + \delta L_2$, the two strains should add to the total strain if the deformation were performed all at once, that is, from a length L_0 to a length L_2. That is, the final strain should not depend on how the strain is formed. Consider first the engineering strain:

$$\text{Increment 1}: \qquad \varepsilon_{0-1} = \frac{L_1 - L_0}{L_0} = \frac{\delta L_1}{L_0} \tag{1.49}$$

$$\text{Increment 2}: \qquad \varepsilon_{1-2} = \frac{L_2 - L_1}{L_1} = \frac{\delta L_2}{L_0 + \delta L_1} \tag{1.50}$$

If the deformation is performed in one step,

$$\text{Total strain}: \qquad \varepsilon_{0-2} = \frac{L_2 - L_0}{L_0} = \frac{\delta L_1 + \delta L_2}{L_0} \neq \varepsilon_{0-1} + \varepsilon_{1-2} \tag{1.51}$$

Clearly this is not equal to the sum of the partial strains so engineering strain is not additive and does not give a self-consistent measure of strain. However, if we use the instantaneous length to define an infinitesimal increment in true strain,

$$d\varepsilon^t = \frac{dL}{L} \tag{1.52}$$

then the total true strain associated with a change in length from L_{initial} to L_{final} is

$$\varepsilon^t = \int_{L_{\text{initial}}}^{L_{\text{final}}} \frac{dL}{L} = \ln\left(\frac{L_{\text{final}}}{L_{\text{initial}}}\right) = \ln\left(1 + \frac{\delta L}{L_{\text{initial}}}\right) \tag{1.53}$$

Consider now the two-step deformation discussed above. The associated components of true strain are

$$\text{Increment 1}: \qquad \varepsilon^t_{0-1} = \ln\left(\frac{L_1}{L_0}\right) \tag{1.54}$$

$$\text{Increment 2}: \qquad \varepsilon^t_{1-2} = \ln\left(\frac{L_2}{L_1}\right) \tag{1.55}$$

If the deformation is performed in one step,

$$\text{Total strain}: \qquad \varepsilon^t_{0-2} = \ln\left(\frac{L_2}{L_0}\right) = \ln\left(\frac{L_2}{L_1}\frac{L_1}{L_0}\right) = \varepsilon^t_{0-1} + \varepsilon^t_{1-2} \tag{1.56}$$

showing that true strain is additive and so independent of how the deformation is performed. A series expansion of the logarithm in Eq. (1.53) shows that for small strain the engineering and true strains have nearly the same value but at large strains they differ significantly. As an example, if a specimen were strained to twice its length or one-half its length, the engineering strain would be $\varepsilon = 1$ and $\varepsilon = -\frac{1}{2}$ respectively, but the true strains would be $\varepsilon^t = 0.69$ and $\varepsilon^t = -0.69$, respectively.

1.8.2 True Stress

The need for true stress is clear. If a body is subjected to large strains, the cross section can be considerably reduced and the engineering stress badly underestimates the local stress. This is particularly true for some ductile materials which can strain nonuniformly under a tensile load forming local "necks" where the cross section is considerably smaller than the original or the average cross section. In such a case the engineering stress is uniform along the length, but the true stress reaches a high value in the neck region and is a superior representation of stress.

For a uniaxial tension or compression test it is possible to convert engineering strain to true strain and to determine the true stress at a given strain provided the relationship between volume of the sample and the axial strain is known. The analysis is simplified if, as is usually the case, it is assumed that the volume of the sample does not change during plastic deformation. The volume V can be related to the initial (L_0 and A_0) and instantaneous (L_i and A_i) length and cross-sectional area:

$$V = A_0 L_0 = A_i L_i \tag{1.57}$$

Thus

$$\frac{L_i}{L_0} = \frac{A_0}{A_i} \tag{1.58}$$

The engineering strain is given by

$$\varepsilon = \frac{L_i - L_0}{L_0} = \frac{L_i}{L_0} - 1 \qquad \text{or} \qquad \frac{L_i}{L_0} = 1 + \varepsilon \tag{1.59}$$

while the true strain is

$$\varepsilon^t = \ln\left(\frac{L_i}{L_0}\right) = \ln(1 + \varepsilon) \tag{1.60}$$

To determine the true stress at a give strain,

$$\sigma^t = \frac{F}{A_i} = \frac{FL_i}{A_0 L_0} = \sigma(1 + \varepsilon) \tag{1.61}$$

True stress and true strain are only used where relatively large plastic strains are observed since elastic strains are usually so low that engineering and true stresses and strains are the same. For ceramics only elastic deformation is observed except under high temperatures, and so generally engineering stresses and strains are sufficient.

PROBLEMS

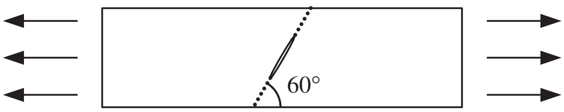
1. An elastic ceramic body is placed under a hydrostatic compression of 200 MPa. Additional stresses of $\sigma_{xx} = 400$ MPa, $\sigma_{yy} = 100$ MPa, and $\sigma_{xy} = \sigma_{yx} = 50$ MPa are then superimposed. No other stresses are applied. What is the total stress tensor? What are the principal stresses and what angles do the principal axes make to the x axis? What is the maximum shear stress anywhere in the body and what is its orientation?

2. The stress tensor at a point $P(x,y,z)$ is

$$\sigma = \begin{pmatrix} 300 & 100 & 100 \\ 100 & 0 & 200 \\ 100 & 200 & 0 \end{pmatrix} \qquad \text{MPa}$$

Determine the principal stresses and unit vectors in the principal directions. Determine the magnitude and directions of the maximum shear stress at P.

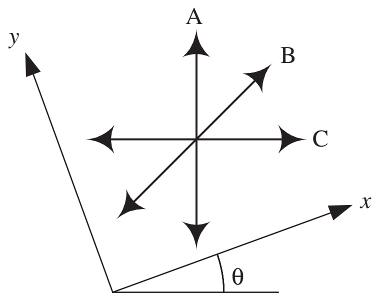
3. A tensile force of 50 N is applied uniformly over the end faces (measuring 10 mm by 100 μm) of a thin ceramic sheet. The sheet contains a small crack whose plane is inclined at 60° to the direction of the force. Find the tensile stress acting normal to the crack and the shear stress acting in the plane of the crack.



4. Prove by differentiating Eqs. (1.20) with respect to θ that (i) the maximum and minimum normal stresses are the principal stresses and that (ii) the maximum shear stress is one-half of the difference between the principal stresses and acts on planes inclined at $\pm 45^\circ$ to the principal axes.

5. The strains (and hence stresses) in the surface of a solid can be measured using strain gauges. A strain gauge is a thin layer of resistive metal printed in a zigzag pattern on a thin polymer substrate. When the metal pattern is subjected to strain, its resistance changes; the zigzag pattern is chosen so that the resistance change is proportional to the normal strain in only one direction. The substrate is glued to the surface of interest to sense changes in the strain. Strain gauges are not sensitive to shear strains, but the shear strain can be calculated from the three normal strains measured by three gauges arranged in what is called a "strain gauge rosette." The schematic below shows such a rosette with three gauges with their sensitive directions shown by the arrows. Gauges A and C are perpendicular and B lies at 45° between them.

A strain gauge rosette like the one in the sketch is glued to the surface of a specimen while the specimen is under zero stress. When stresses are applied to the specimen, it is determined that the principal strains in the surface are inclined at an angle to the gauges $\theta = 20^\circ$, as shown in the sketch. If the principal strains are $\varepsilon_{xx} = 0.03\%$ and $\varepsilon_{yy} = 0.01\%$, what strains will the three gauges measure?



6. Strain gauges attached to the surface of a ceramic body record the following strain components: $\varepsilon_{xx} = 2.00 \times 10^{-4}$, $\varepsilon_{yy} = 1.50 \times 10^{-4}$, and $\varepsilon_{xy} = -1.00 \times 10^{-4}$. Find the principal strains ε_1, ε_2 and their inclination to the

x axis. Also find the maximum shear strain ε_{max} and its angles of inclination to the x axis.

7. The components of a displacement field are given by (units are meters)

$$u_x = (x^2 + 20) \times 10^{-3} \qquad u_y = 2yz \times 10^{-3} \qquad u_z = (z^2 - xy) \times 10^{-3}$$

Find the displacements at the point (2,5,7) and the point (3,8,9). Find the change in the distance between these two points. Determine expressions for the total, hydrostatic, and deviatoric strain tensors. Calculate the strain tensor explicitly at the point (2,−1,3). What are the principal strains at this point?

8. Prove that the two-dimensional invariant $\sigma_{xx} + \sigma_{yy}$ is independent of the rotation angle θ. Furthermore, prove that in three-dimensions the invariant $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ is independent of the direction cosines. The following relationships may be needed for the proof:

$$a_{xx}^2 + a_{xy}^2 + a_{xz}^2 = 1 \qquad a_{yx}^2 + a_{yy}^2 + a_{yz}^2 = 1 \qquad a_{zx}^2 + a_{zy}^2 + a_{zz}^2 = 1$$

$$a_{xx}a_{yx} + a_{xy}a_{yy} + a_{xz}a_{yz} = 0$$
$$a_{yx}a_{zx} + a_{yy}a_{zy} + a_{yz}a_{zz} = 0$$
$$a_{zx}a_{xx} + a_{zy}a_{xy} + a_{zz}a_{xz} = 0$$