CHAPTER 1 Permutations and combinations

1.1 Overview

1.1.1 Introduction

Ever since you were a small child, even before you started school, you have had the experience of counting. These simple counting techniques are built upon to develop very sophisticated ways of counting, in a field of mathematics known as combinatorics. Permutations and combinations allow us to calculate the number of ways objects belonging to a finite set can be arranged.

Have you encountered the pigeon-hole principle before? It is a deceptively simple statement that can be used to identify patterns in huge amounts of data, such as in DNA analysis.

Have you encountered Pascal's triangle before? It spans the mathematical fields of combinations, probability, the binomial theorem, Fibonacci numbers and the bell-shaped normal distribution. In developing his triangle, Blaise Pascal (1623–1662) made a fundamental contribution to the field of combinatorics.

In this chapter you will apply combinatorics to determine, for example, the number of ways a team of 5 players can be chosen from a group of 10. Consider its usefulness in developing rosters for staff or flow charts for projects. Just as Pascal's triangle spans mathematical fields, combinatorics spans industries as varied as gambling, internet information transfer and security, communication networks, computer chip architecture, logistics and DNA modelling. It has applications in any field where different choices mean different efficiencies.

LEARNING SEQUENCE

1.1 Overview
1.2 Counting techniques
1.3 Factorials and permutations
1.4 Permutations with restrictions
1.5 Combinations
1.6 Applications of permutations and combinations
1.7 Pascal's triangle and the pigeon-hole principle
1.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Counting techniques

1.2.1 Review of set notation

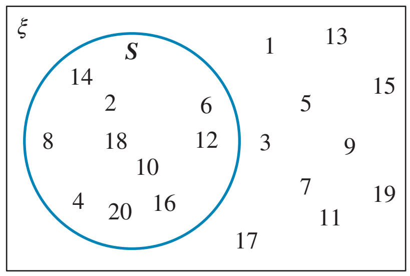
A **set**, S, is a collection of objects. The objects in a set are referred to as the **elements** of the set.

A set can be written in a variety of ways. Consider the following. Let the sample space be the set of numbers between 1 and 20, that is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Let the set S be the set of even numbers between 1 and 20 inclusive.

S can be:

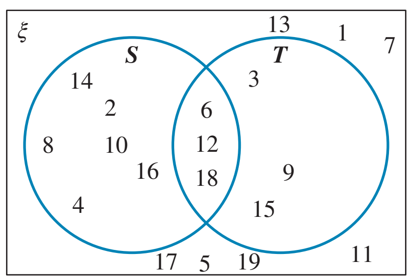
- written as a list: $S = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ (in any order)
- written as a rule: $S = \{n : n = 2r \text{ for } 1 \le r \le 10\}$
- shown in a Venn diagram.



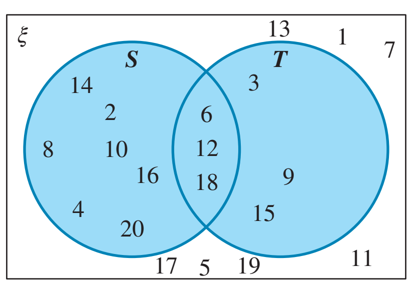
The complement of S, written as S', is the set of all things NOT in S. In this example, $S' = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$.

The complete set of objects being considered is called the **universal set**, ξ. It is represented by the rectangle in the Venn diagram, and is abbreviated with Greek letter ξ (pronounced 'ksi'). In this example, $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Now consider a second set, T, which is the set of numbers that are multiples of 3 between 1 and 20; that is, $T = \{3, 6, 9, 12, 15, 18\}$. The sets S and T can be combined in various ways.

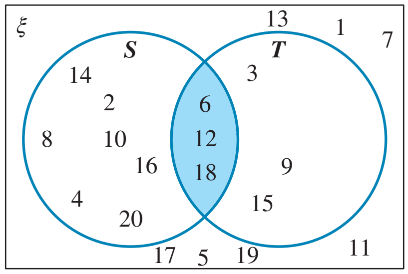


The **union** of S and T is all the elements in either S or T or both. It is shown as follows.



$$S \cup T = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$$

The **intersection** of S and T is all the elements that are in both S and T.



$S \cap T = \{6, 12, 18\}$

1.2.2 The inclusion–exclusion principle

Continuing the example from subsection 1.2.1, we have the two sets $S = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and $T = \{3, 6, 9, 12, 15, 18\}$. If we find the number of elements in the two sets, we obtain $n(S) = 10$ and $n(T) = 6$.

In general, for two sets S and T, $n(S \cup T) = n(S) + n(T) - n(S \cap T)$.

This is known as the **inclusion–exclusion principle**.

$n(S \cup T)$ is the number of elements in the union of S and T (i.e. the number of elements in S or T or both). This is equal to $n(S)$ (the number of elements in S) plus $n(T)$ (the number of elements in T) minus $n(S \cap T)$ (the number of elements in both S and T, as these have already been counted in sets S and T).

For our example above:

$$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$

$$n(S) = 10, n(T) = 6, n(S \cap T) = 3$$

So, $n(S \cup T) = 13$, which can be confirmed by counting all the elements which occur in S, T or both.

If three sets are involved, the inclusion–exclusion principle becomes:

$$n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T) - n(T \cap R) - n(S \cap R) + n(S \cap T \cap R)$$

An example of the inclusion–exclusion principle with three sets is shown in Worked example 1.

WORKED EXAMPLE 1

Let R be the set of natural numbers between 15 and 30 inclusive that are divisible by 2.
Let S be the set of natural numbers between 15 and 30 inclusive that are divisible by 3.
Let T be the set of natural numbers between 15 and 30 inclusive that are divisible by 5.

a. **Construct a Venn diagram to represent R, S and T.**
b. **Use this diagram to evaluate $n(R \cup S \cup T)$.**
c. **Recall the inclusion–exclusion principle to compute $n(R \cup S \cup T)$.**

THINK

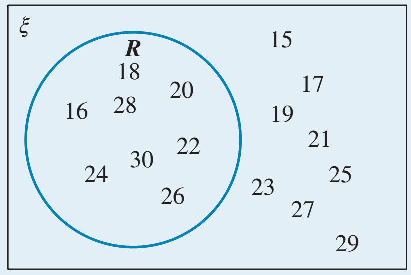
a. 1. R is the set of natural numbers between 15 and 30 divisible by 2, so:
$R = \{16, 18, 20, \ldots 30\}$

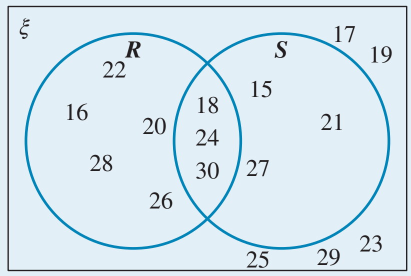
2. S is the set of natural numbers between 15 and 30 divisible by 3, so:
$S = \{15, 18, 21, 24, 27, 30\}$
$R \cap S$ are the numbers which occur in both S and R, so:
$R \cap S = \{18, 24, 30\}$

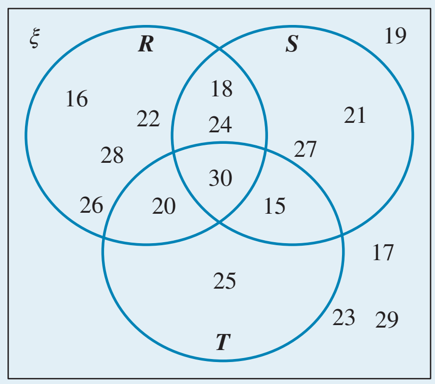
3. T is the set of natural numbers between 15 and 30 divisible by 5, so:
$T = \{15, 20, 25, 30\}$
$R \cap T = \{20, 30\}$
$S \cap T = \{15, 30\}$
$R \cap S \cap T = \{30\}$

WRITE

a.







b. $n(R \cup S \cup T)$ is the number of elements in sets R, S and T. Count the number of elements within each portion of each circle.

$$n(R \cup S \cup T) = 4 + 2 + 1 + 2 + 1 + 1 + 1 = 12$$

c. 1. Recall the inclusion-exclusion principle formula.

$$n(S \cup T \cup R) = n(S) + n(T) + n(R) - n(S \cap T) - n(T \cap R) - n(S \cap R) + n(S \cap T \cap R)$$

2. The number of elements in each set can be substituted into the formula.

$$n(S \cup T \cup R) = 6 + 4 + 8 - 2 - 2 - 3 + 1 = 12$$

This agrees with the calculation from the Venn diagram in part **b**.

1.2.3 Types of counting techniques

Counting techniques allow us to determine the number of ways an activity can occur. This in turn allows us to calculate the **probability** of an event. Recall from your earlier probability studies that the probability of event A, P(A), can be determined by counting the number of elements in A and dividing by the total number in the sample space, ξ, according to the formula

$$\mathrm{P}(A) = \frac{n(A)}{n(\xi)}.$$

Different types of counting techniques are employed depending on whether order is important. When order is important, this is called an **arrangement** or a **permutation**; when it is not important, it is called a **selection** or a **combination**. Permutations and combinations are defined more formally in sections 1.3 and 1.5.

1.2.4 The addition and multiplication principles

To count the number of ways in which an activity can occur, first make a list. Let each outcome be represented by a letter and then systematically list all the possibilities.

Consider the following question:

In driving from Brisbane to Rockhampton I can take any one of 4 different roads *and* in driving from Rockhampton to Townsville there are 3 different roads I can take. How many different routes can I take in driving from Brisbane to Townsville?

To answer this, let R_1, R_2, R_3, R_4 stand for the 4 roads from Brisbane to Rockhampton and T_1, T_2, T_3 stand for the 3 roads from Rockhampton to Townsville.

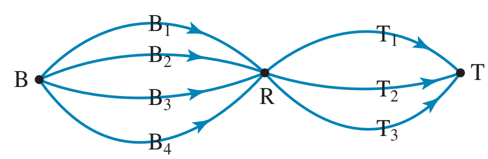
Use the figure to systematically list the roads:

R_1T_1, R_1T_2, R_1T_3

R_2T_1, R_2T_2, R_2T_3

R_3T_1, R_3T_2, R_3T_3

R_4T_1, R_4T_2, R_4T_3



Hence, there are 12 different ways I can drive from Brisbane to Townsville.

In the above example it can be argued logically that if there are 4 ways of getting from Brisbane to Rockhampton and 3 ways of getting from Rockhampton to Townsville then there are 4×3 ways of getting from Brisbane to Townsville.

This idea is formalised in the **multiplication principle**.

The multiplication principle should be used when there are operations or events (say, A and B), where one event is followed by the other – that is, when order is important.

The multiplication principle:
If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B in the order AB.

***Note:* In this case '*and*' means to multiply.**

A useful technique for solving problems based on the multiplication principle is to use boxes. In the example above we would write

1st	2nd
4	3

The value in the '1st' column represents the number of ways the first operation — the trip from Brisbane to Rockhampton — can be performed.

The value in the '2nd' column stands for the number of ways the second operation — the trip from Rockhampton to Townsville — can be performed.

To apply the multiplication principle you multiply the numbers in the lower row of boxes.

WORKED EXAMPLE 2

Two letters are to be chosen from the set of 5 letters. A, B, C, D and E, where order is important.

a. Recall how to list all the different ways that this may be done.

b. Use the multiplication principle to calculate the number of ways that this may be done.

c. Determine the probability the first letter will be a C.

THINK	WRITE
a 1. Begin with A in first place and make a list of each of the possible pairs.	**a** AB AC AD AE
2. Make a list of each of the possible pairs with B in the first position.	BA BC BD BE
3. Make a list of each of the possible pairs with C in the first position.	CA CB CD CE
4. Make a list of each of the possible pairs with D in the first position.	DA DB DC DE
5. Make a list of each of the possible pairs with E in the first position. *Note:* AB and BA need to be listed separately as order is important.	EA EB EC ED
b The multiplication principle could have been used to determine the number of ordered pairs.	**b** | 5 | 4 |
1. Rule up two boxes which represent the pair.	
2. Write down the number of letters which may be selected for the first box. That is, in first place any of the 5 letters may be used.	
3. Write down the number of letters which may be selected for the second box. That is, in second place, any of the 4 letters may be used. *Note:* One less letter is used to avoid repetition.	
4. Evaluate.	$5 \times 4 = 20$ ways
5. Answer the question.	There are 20 ways in which 2 letters may be selected from a group of 5 where order is important.

c 1. Recall the probability formula.The total number in the set $n(\xi)$ was determined in part **b**.

c $P(A) = \dfrac{n(A)}{n(\xi)}$

$n(\xi) = 20$

2. Let A be the event that the pair starts with a C. Draw a table showing the requirement imposed by the first letter to be C.

1	

3. Complete the table. Once the first letter has been completed, there are 4 choices for the second letter. Use the multiplication principle to determine the number of combinations starting with C.

1	4

There are $1 \times 4 = 4$ possible combinations beginning with C.
So, $n(A) = 4$.

4. Use the probability formula to answer the question.

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{4}{20} = \frac{1}{5}$$

This is confirmed by examining the answer to part **a**.

WORKED EXAMPLE 3

a. Use the multiplication principle to calculate how many ways an arrangement of 5 numbers can be chosen from {1, 2, 3, 4, 5, 6}.
b. Determine the probability of the number ending with 4.

THINK

WRITE

a 1. Instead of listing all possibilities, draw 5 boxes to represent the 5 numbers chosen.
Label each box on the top row as 1st, 2nd, 3rd, 4th and 5th.
Note: The word arrangement implies order is important.

2. Fill in each of the boxes showing the number of ways a number may be chosen.
- **a.** In the 1st box there are 6 choices for the first number.
- **b.** In the 2nd box there are 5 choices for the second number as 1 number has already been used.
- **c.** In the 3rd box there are 4 choices for the third number as 2 numbers have already been used.
- **d.** Continue this process until each of the 5 boxes is filled.

1st	2nd	3rd	4th	5th
6	5	4	3	2

3. Use the multiplication principle as this is an *'and'* situation.

$\text{No. of ways} = 6 \times 5 \times 4 \times 3 \times 2$
$= 720$

4. Answer the question.

An arrangement of 5 numbers may be chosen 720 ways.

b. 1. Recall the probability formula.
The total number of arrangements, $n(\xi)$, was determined in part **a**.

b $P(A) = \dfrac{n(A)}{n(\xi)}$

$n(\xi) = 720$

2. Let A be the event that the number ends with 4.
Draw a table showing the requirement imposed by the last letter to be 4.

1st	2nd	3rd	4th	5th
				1

3. Complete the table. Once the last number has been completed, there are 5 choices for the number in the first position, 4 choices for the next number. Continue this process until each of the 5 columns has been filled.
Use the multiplication principle to determine the number of combinations ending with 4.

1st	2nd	3rd	4th	5th
5	4	3	2	1

There are
$5 \times 4 \times 3 \times 2 \times 1 = 120$ possible combinations ending with 4.
So, $n(A) = 120$.

4. Use the probability formula to answer the question.

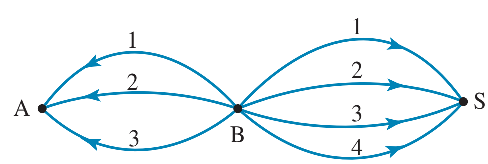
$$P(A) = \frac{n(A)}{n(\xi)} = \frac{120}{720} = \frac{1}{6}$$

This is confirmed by examining the answer to part **a**.

Now consider a different situation, one in which the two operations do not occur one after the other.

I am going to travel from Brisbane to either Sydney *or* Adelaide. There are 4 ways of travelling from Brisbane to Sydney and 3 ways of travelling from Brisbane to Adelaide.

How many different ways can I travel to either Sydney *or* Adelaide?



It can be seen from the figure that there are $4 + 3 = 7$ ways of completing the journey. This idea is summarised in the **addition principle**.

The addition principle should be used when two distinct operations or events occur in which one event is not followed by another — that is, when the events are mutually exclusive.

The addition principle:
If there are n ways of performing operation A and m ways of performing operation B, then there are $n + m$ ways of performing A *or* B.

***Note:* In this case '*and*' means to add.**

WORKED EXAMPLE 4

One or two letters are to be chosen from the set of 6 letters A, B, C, D, E, F. Assuming order is important, use the multiplication principle and the addition principle to calculate:

a. the number of ways to choose 2 letters.

b. the number of ways to choose 1or 2 letters.

THINK	WRITE
a **1.** Determine the number of ways of choosing 1 letter.	Number of ways of choosing 1 letter = 6.
2. Rule up two boxes for the first and second letters.	(1st: 6, 2nd: 5)
3 Determine the number of ways of choosing 2 letters from 6. In the 1st box there are 6 choices for the first letter. In the 2nd box there are 5 choices for the second letter as 1 letter has already been used.	
4. Use the multiplication principle (as this is an ‘*and*’ situation) to evaluate the number of ways of choosing 2 letters from 6.	Number of ways of choosing 2 letters $= 6 \times 5$ $= 30$
5. Answer the questions.	There are 30 ways of choosing 2 letters.
b **1.** Determine the number of ways of choosing 1 or 2 letters from 6 letters. Use the addition principle as this is an ‘*or*’ situation.	The number of ways of choosing 1 or 2 letters is $6 + 30 = 36$.
2. Answer the question.	There are 36 ways of choosing 1 or 2 letters from 6.

1st	2nd
6	5

The multiplication and addition principles can be used to count the number of elements in the union of two or three sets in the same way as for one set. Remember, the multiplication principle is used with ‘*and*’ situations, and the addition principle is used with ‘*or*’ situations.

WORKED EXAMPLE 5

Oscar’s cafe offers a choice of 3 starters, 9 main courses and 4 desserts.

a. How many choices of 3-course meals (starter, main, dessert) are available?

b. How many choices of starter and main course meals are offered?

c. How many choices of meals comprising a main course and dessert are offered?

d. How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?

e. If one of the starter options is chicken wings and one of the mains is grilled fish, determine the probability of choosing chicken wings and grilled fish in a 3-course meal.

THINK

a **1.** Consider each course as separate sets and rule up 3 boxes to represent each course — starter, main, dessert. Label each box on the top row as S, M and D.

2. Determine the number of ways of choosing each meal: starter = 3, main = 9, dessert = 4.

3. Use the multiplication principle (as this is an ‘*and*’ situation) to evaluate the number of choices of 3-course meals.

4. Answer the question.

b **1.** Rule up 2 boxes to represent each course — starter, main. Label each box on the top row as S and M.

2. Determine the number of ways of choosing each meal: starter = 3, main = 9.

3. Use the multiplication principle (as this is an ‘*and*’ situation) to evaluate the number of choices of starter and main courses.

4. Answer the question.

c **1.** Rule up 2 boxes to represent each course — main and dessert. Label each box on the top row as M and D.

2. Determine the number of ways of choosing each meal: main = 9, dessert = 4.

3. Use the multiplication principle (as this is an ‘*and*’ situation) to evaluate the number of choices of main course and dessert.

4. Answer the question.

d **1.** Determine the number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered.
Use the addition principle as this is an ‘*or*’ situation.

2. Answer the question.

WRITE

a

S	M	D
3	9	4

$$\begin{aligned}\text{Number of choices} &= 3 \times 9 \times 4\\ &= 108\end{aligned}$$

There are 108 choices of 3-course meals.

b

S	M
3	9

$$\begin{aligned}\text{Number of choices} &= 3 \times 9\\ &= 27\end{aligned}$$

There are 27 choices of starter and main course.

c

M	D
9	4

$$\begin{aligned}\text{Number of choices} &= 9 \times 4\\ &= 36\end{aligned}$$

There are 36 choices of main course and dessert.

d The number of ways of choosing 2- or 3-course meals, assuming that a main course is always ordered, is:
$108 + 27 + 36 = 171$

There are 171 ways of choosing 2- or 3-course meals, assuming that a main course is always ordered.

e **1.** Recall the probability formula.
The total number in the set $n(\xi)$ was determined in part **a**.

$P(A) = \frac{n(A)}{n(\xi)}$

$n(\xi) = 108$

2. Let A be the event that the meal contains chicken wings and grilled fish.
Draw a table showing the requirement imposed that the starter and main be these 2 dishes.

S	M	D
1	1	

3. Complete the table — there are 4 dessert options. Use the multiplication principle to determine the number of combinations of the meal.

S	M	D
1	1	4

There are $1 \times 1 \times 4 = 4$ possible combinations for the meal.
So, $n(A) = 4$.

4. Use the probability formula to answer the question.

$$P(A) = \frac{n(A)}{n(\xi)} = \frac{4}{108} = \frac{1}{27}$$

The probability of choosing chicken wings and grilled fish as part of the 3-course meal is $\frac{1}{27}$.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concepts 1 & 2

Set notation Summary screen and practice questions
Counting techniques Summary screen and practice questions

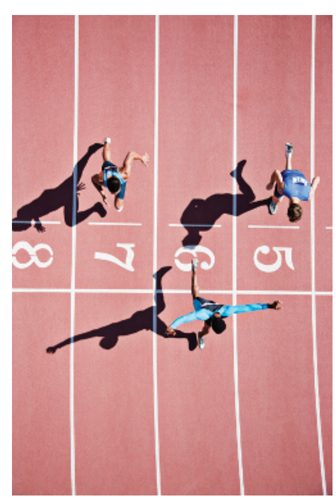
Exercise 1.2 Counting techniques

Technology free

1. **WE1** Let R be the set of natural numbers between 30 and 45 inclusive that are divisible by 2.
Let S be the set of natural numbers between 30 and 45 inclusive that are divisible by 3.
Let T be the set of natural numbers between 30 and 45 inclusive that are divisible by 5.
a. Construct a Venn diagram to represent R, S and T.
b. Use this diagram to evaluate $n(R \cup S \cup T)$.
c. Recall the inclusion–exclusion principle to compute $n(R \cup S \cup T)$.

2. Recall the inclusion–exclusion principle to calculate the number of cards in a deck of 52 that are either red or even or a 4.
3. Student Services has the following data on Year 11 students and their sport commitments:
   - 18 play no sport.
   - 16 play netball (and possibly other sports).
   - 24 play football.
   - 20 are involved in a gym program.
   - 7 play netball and football.
   - 6 play netball and are in the gym program.
   - 15 play football and are involved in the gym program.
   - 5 students do all three activities.

   How many students are there in Year 11?
4. **WE2** Two letters are to be chosen from A, B and C, where order is important.
   a. Recall how to list all the different ways that this may be done.
   b. Use multiplication principle to calculate the number of ways that this may be done.
   c. Determine the probability the last letter will be a B.
5. List all the different arrangements possible for a group of 2 colours to be chosen from B (blue), G (green), Y (yellow) and R (red).
6. List all the different arrangements possible for a group of 3 letters to be chosen from A, B and C.
7. a. **WE3** Use the multiplication principle to calculate how many ways can an arrangement of 2 letters be chosen from A, B, C, D, E, F and G?
   b. In how many ways can an arrangement of 3 letters be chosen from 7 different letters?
   c. In how many ways can an arrangement of 4 letters be chosen from 7 different letters?
   d. How many different arrangements of 5 letters can be made from 7 letters?
   e. Determine the probability of the letters starting with an E.
8. a. A teddy bear’s wardrobe consists of 3 different hats, 4 different shirts and 2 different trousers. How many different outfits can the teddy bear wear?
   b. A surfboard is to have 1 colour on its top and a different colour on its bottom. The 3 possible colours are red, blue and green. In how many different ways can the surfboard be coloured?
   c. A new phone comes with a choice of 3 cases, 2 different sized screens and 2 different storage capacities. With these choices, determine how many different arrangements are possible.
   d. Messages can be sent by placing 3 different coloured flags in order on a pole. If the flags come in 4 colours, determine how many different messages can be sent.
9. a. **WE4** One or 2 letters are to be chosen in order from the letters A, B, C, D, E, F and G. Use the multiplication principle and the addition principle to calculate the number of ways can this be done.
   b. Two or 3 letters are to be chosen in order from the letters A, B, C, D, E, F and G. In how many ways can this be done?



10. Manish is in a race with 7 other runners. If we are concerned only with the first, second and third placings, in how many ways can Manish finish first or second or third?
11. **WE5** Hani and Mary’s restaurant offers its patrons a choice of 4 entrees, 10 main courses and 5 desserts.
    a. How many choices of 3-course meals (entree, main, dessert) are available?
    b. How many choices of entree and main course are offered?

c. How many choices of meals comprising a main course and dessert are offered?
d. How many choices of 2- or 3-course meals are available (assuming that a main course is always ordered)?
e. Determine the probability of choosing vegetable soup for entree and roast for main in a 3-course meals.

12. Jake is able to choose his work outfits from the following items of clothing: 3 jackets, 7 shirts, 6 ties, 5 pairs of trousers, 7 pairs of socks and 3 pairs of shoes.
a. Calculate how many different outfits are possible if he wears one of each of the above items. (He wears matching socks and matching shoes.)
b. If Jake has the option of wearing a jacket or not, but he must wear one of each of the above items, determine how many different outfits are possible. Justify your answer.

13. **MC** There are 12 people on the committee at the local football club. In how many ways can a president and a secretary be chosen from this committee?
A. 2 **B.** 23 **C.** 132 **D.** 144

14. **MC** A TV station runs a cricket competition called *Classic Catches*. Six catches, A to F, are chosen and viewers are asked to rank them in the same order as the judges. The number of ways in which the six catches can be ranked:
A. 1 **B.** 6 **C.** 30 **D.** 720

Technology active

15. The local soccer team sells ‘doubles’ at each of their games to raise money. A ‘double’ is a card with 2 digits on it representing the score at full time. The card with the actual full time score on it wins a prize. If the digits on the cards run from 00 to 99, how many different tickets are there?

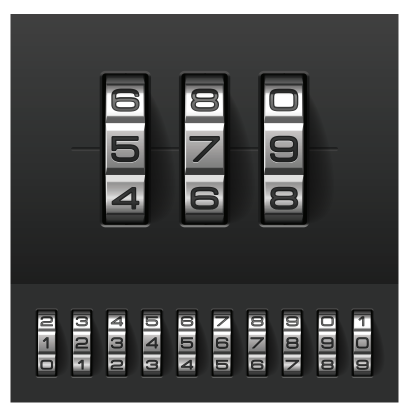
16. Jasmin has a phone that has a 4-digit security code. She remembers that the first number in the code was 9 and that the others were 3, 4 and 7 but forgets the order of the last 3 digits. How many different trials must she make to be sure of unlocking the phone?

17. Julia has a banking app that has two 4-digit codes. She remembers that she used the digits 1, 3, 5 and 7 on the first code and 2, 4, 6 and 8 on the second code, but cannot remember the order. What is the maximum number of trials she would need to make before she has opened both codes? (Assume that she can try an unlimited number of times and once the first code is correct, she can try the second code.)

18. How many different 4-digit numbers can be made from the numbers 1, 3, 5 and 7 if the numbers can be repeated (that is 3355 and 7777 are valid)?

19. How many 4-digit numbers can be made from the numbers 1, 3, 5, 7, 9 and 0 if the numbers can be repeated? (Remember — a number cannot start with 0.)

20. A combination lock has 3 digits each from 0 to 9.
a. How many combinations are possible?
The lock mechanism becomes loose and will open if the digits are within one either side of the correct digit. For example if the true combination is 382 then the lock will open on 271, 272, 371, 493 and so on.
b. How many combinations would unlock the safe?
c. List the possible combinations that would open the lock if the true combination is 382.



1.3 Factorials and permutations

1.3.1 Factorials

The Physical Education department is to display 5 new trophies along a shelf in the school foyer and wishes to know in how many ways this can be done.

Using the multiplication principle from the previous section, the display may be done in the following way:

Position 1	Position 2	Position 3	Position 4	Position 5
5	4	3	2	1

That is, there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

Depending on the number of items we have, this method could become quite time consuming.

In general when we need to multiply each of the integers from a particular number, n, down to 1, we write $n!$, which is read as n **factorial**.

Hence:

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \\ 8! &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40\,320 \end{aligned}$$

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

The number of ways n distinct objects may be arranged is $n!$(n factorial) where:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

That is, $n!$ is the product of each of the integers from n down to 1.
A special case of the factorial function is: $0! = 1$.

WORKED EXAMPLE 6

Evaluate the following factorials.

a. $7!$ **b.** $13!$ **c.** $\dfrac{8!}{5!}$ **d.** $\dfrac{(n-1)!}{(n-3)!}$

THINK	WRITE
a. 1. Write 7! in its expanded form and evaluate.	**a.** $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 5040$
2. Verify the answer obtained using the factorial function on a calculator.	
b. 1. Write 13! in its expanded form and evaluate.	**b.** $13! = 13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 6\,227\,020\,800$

2. Verify the answer obtained using the factorial function on a calculator.

c. 1. Write each factorial term in its expanded form.

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1}$$

2. Cancel down like terms.

$$= 8 \times 7 \times 6$$

3. Evaluate.

$$= 336$$

4. Verify the answer obtained using the factorial function on a calculator.

d. 1. Write each factorial term in its expanded form.

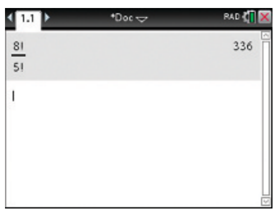
$$\frac{(n-1)!}{(n-3)!} = \frac{(n-1)(n-2)(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}{(n-3)(n-4) \times \ldots \times 3 \times 2 \times 1}$$

2. Cancel like terms.

$$= (n-1)(n-2)$$

TI | THINK — **WRITE**

c.1. On a Calculator page, complete the entry line as: $\frac{8!}{5!}$ then press ENTER.
Note: The factorial symbol can be found by pressing CTRL, then the Catalogue button.



2. The answer appears on the screen.

$$\frac{8!}{5!} = 336$$

CASIO | THINK — **WRITE**

c.1. On a Run-Matrix screen, complete the entry line as: $\frac{8!}{5!}$ then press EXE.
Note: To find the factorial symbol, press OPTN, then press F6, and then select PROB and x! by pressing F3, then F1.

2. The answer appears on the screen.

$$\frac{8!}{5!} = 336$$

In parts **c** and **d** of Worked example 6, there was no need to fully expand each factorial term.

The factorial $\frac{8!}{5!}$ could have first been simplified to $\frac{8 \times 7 \times 6 \times 5!}{5!}$ and then the 5! terms cancelled.

The factorial $\frac{(n-1)!}{(n-3)!}$ could have first been simplified to $\frac{(n-1)(n-2)(n-3)!}{(n-3)!}$ and then the $(n-3)!$ terms cancelled.

1.3.2 Permutations

The term permutation is often used instead of the term arrangement, and in this section we begin by giving a formal definition of permutation.

Previously, we learned that if you select 3 letters from 7 where *order is important*, the number of possible arrangement is:

1st	2nd	3rd
7	6	5

$$\begin{aligned}\text{The number of arrangements} &= 7 \times 6 \times 5\\ &= 210\end{aligned}$$

This value may also be expressed in factorial form: $7 \times 6 \times 5 = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = \dfrac{7!}{4!}$

Using more formal terminology we say that in choosing 3 things from 7 things where order is important, the number of permutations is ${}^7P_3 = 7 \times 6 \times 5$. The letter P is used to remind us that we are finding permutations.

The number of ways of choosing r things from n distinct things is given by the rule:

$$\begin{aligned}{}^nP_r &= n \times (n-1) \times \ldots \times (n-r+1)\\ &= \frac{n \times (n-1) \times \ldots \times (n-r+1)(n-r)!}{(n-r)!}\\ {}^nP_r &= \frac{n!}{(n-r)!}\end{aligned}$$

The definition of ${}^nP_r$ may be extended to the cases of ${}^nP_n$ and ${}^nP_0$.
${}^nP_n$ represents the number of ways of choosing n objects from n distinct things.

$$\begin{aligned}{}^nP_n &= n \times (n-1) \times (n-2) \times \ldots \times (n-n+1)\\ &= n \times (n-1) \times (n-2) \times \ldots \times 1\\ &= n!\end{aligned}$$

From the definition:

$$\begin{aligned}{}^nP_n &= \frac{n!}{(n-n)!}\\ &= \frac{n!}{0!}\end{aligned}$$

Therefore, equating both sides, we obtain: $n! = \dfrac{n!}{0!}$.
This can occur only if $0! = 1$.

$$\begin{aligned}{}^nP_0 &= \frac{n!}{(n-0)!}\\ &= \frac{n!}{n!}\\ &= 1\end{aligned}$$

The two special cases are:

$$\mathbf{{}^nP_n = n!}$$

$$\mathbf{{}^nP_0 = 1}$$

WORKED EXAMPLE 7

a. Calculate the number of permutations for $^{6}P_{4}$ by expressing it in expanded form.
b. Write $^{8}P_{3}$ as a quotient of factorials and hence evaluate.

THINK	WRITE
a. 1. Write down the first 4 terms beginning with 6.	**a.** $^{6}P_{4} = 6 \times 5 \times 4 \times 3$
2. Evaluate.	$= 360$
b. 1. Recall the rule for permutations.	**b.** $^{n}P_{r} = \dfrac{n!}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	$^{8}P_{3} = \dfrac{8!}{(8-3)!}$ $= \dfrac{8!}{5!}$
3. Use a calculator to evaluate 8! and 5!	$= \dfrac{40\,320}{120}$
4. Evaluate.	$= 336$

TI | THINK

a.1. On a Calculator page, press MENU, then select:
5: Probability
2: Permutations.
Complete the entry line as:
nPr (6, 4)
then press ENTER.

WRITE

2. The answer appears on the screen.

$^{6}P_{4} = 360$

CASIO | THINK

a.1. On a Run-Matrix screen, press OPTN, then F6. Select PROB by pressing F3, then select nPr by pressing F2. Complete the entry line as:
6P4
then press EXE.

WRITE

2. The answer appears on the screen.

$^{6}P_{4} = 360$

WORKED EXAMPLE 8

The netball club needs to appoint a president, secretary and treasurer. From the committee 7 people have volunteered for these positions. Each of the 7 nominees is happy to fill any one of the 3 positions.
a. Determine how many different ways these positions can be filled.
b. For three years, the same 7 people volunteer for these positions. Determine the probability one of them is president 3 years in a row.

THINK	WRITE
a. 1. Recall the rule for permutations. *Note:* Order is important, so use permutations.	**a.** $^{n}P_{r} = \dfrac{n!}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	$^{7}P_{3} = \dfrac{7!}{(7-3)!}$ $= \dfrac{7!}{4!}$

3. Use a calculator to evaluate 7! and 4!

$= \dfrac{5040}{24}$

4. Evaluate.

$= 210$

5. Answer the question.

There are 210 different ways of filling the positions of president, secretary and treasurer.

b. 1. In three years of the president’s position, each year this could be awarded to one of the 7 people.

b.

7	7	7

The total number of ways the president’s position could be filled is $7 \times 7 \times 7$.

$n(\xi) = 7 \times 7 \times 7$
$= 343$

2. Let A be the event that the same person fills the position three years in a row.

There are 7 choices of the same person to fill the positions three years in a row.

$n(A) = 7.$

3. Calculate the probability that the same person fills the president’s position three years in a row.

$P(A) = \dfrac{n(A)}{n(\xi)}$

$= \dfrac{7}{7 \times 7 \times 7}$

$= \dfrac{1}{49}$

The probability that one of the 7 volunteers, fills the president’s position three years in a row is $\dfrac{1}{49}$.

1.3.3 Arrangements in a circle

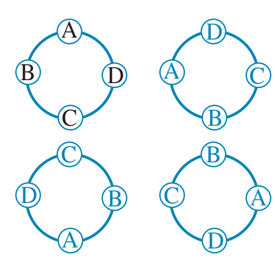
Consider this problem: In how many different ways can 7 people be seated, 4 at a time, on a bench?

By now you should quickly see the answer: ${}^7P_4 = 840$.

Let us change the problem slightly: In how many different ways can 7 people be seated, 4 at a time, at a circular table?

The solution must recognise that when people are seated on a bench, each of the following represents a different arrangement:

ABCD BCDA CDAB DABC



However, when sitting in a circle, each represents the *same* arrangement. It is important to note that in a circle arrangement we do not consider positions on the circle as different — it does not matter where the circle starts.

In each case B has A on the left and C on the right.

We conclude that the number ${}^7P_4$ gives 4 times the number of arrangements of 7 people in a circle 4 at a time. Therefore, the number of arrangements is $\frac{{}^7P_4}{4} = 210$.

In general, the number of different ways n objects can be arranged, r at a time, in a circle is:

$$\frac{{}^nP_r}{r}$$

WORKED EXAMPLE 9

a. **By recalling the appropriate formula, give an expression for the number of different arrangements if, from a group of 8 people, 5 are to be seated at a round table.**

b. **Evaluate this expression.**

c. **Each table receives one lucky door prize and one lucky seat prize. Determine the probability of the same person at one table winning both.**

THINK	WRITE
a. 1. Write down the rule for the number of arrangements in a circle.	**a.** $\frac{{}^nP_r}{r}$
2. Substitute the given values of n and r into the formula.	$= \frac{{}^8P_5}{5}$
3. Answer the question.	The number of ways of seating 5 people from a group of 8 people at a round table is given by the expression $\frac{{}^8P_5}{5}$.
b. 1. Use a calculator to evaluate ${}^8P_5$.	**b.** ${}^8P_5 = \frac{6720}{5}$
2. Evaluate.	$= 1344$
3. Answer the question.	The number of ways of seating 5 from a group of 8 people at a round table is 1344.
c. 1. There are 2 prizes, and each prize can be won by any one of the 5 people.	**c.** | 5 | 5 | The total number of ways the prizes could be won is 5×5. $n(\xi) = 5 \times 5$ $= 25$
2. Let A be the event that the same person wins both prizes.	There are 5 choices of the same person to win both prizes. $n(A) = 5$
3. Calculate the probability that the same person wins both.	$P(A) = \frac{n(A)}{n(\xi)}$ $= \frac{5}{5 \times 5}$ $= \frac{1}{5}$ The probability of the same person winning both prizes is $\frac{1}{5}$.

on Resources

Digital document: SkillSHEET Calculating ^nP_r (doc-26824)
Digital document: SpreadSHEET Permutations (doc-26825)

study on

Units 1 & 2 > Area 1 > Sequence 1 > Concept 3

Factorials Summary screen and practice questions

Exercise 1.3 Factorials and permutations

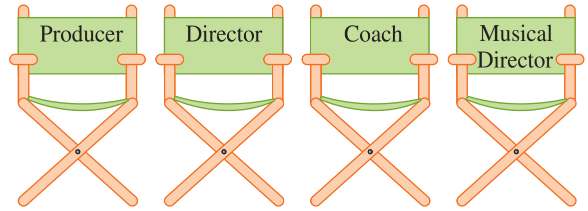
Technology free

1. Recall the definition of $n!$ and write each of the following in expanded form.
   a. $4!$ b. $5!$ c. $6!$ d. $7!$
2. **WE6c** Evaluate the following factorials.
   a. $\frac{9!}{5!}$ b. $\frac{10!}{4!}$ c. $\frac{7!}{3!}$ d. $\frac{6!}{0!}$
3. **WE6d** Evaluate the following factorials.
   a. $\frac{n!}{(n-5!)}$ b. $\frac{(n+3)!}{(n+1)!}$ c. $\frac{(n-3)!}{n!}$ d. $\frac{(n-2)!}{(n+2)!}$

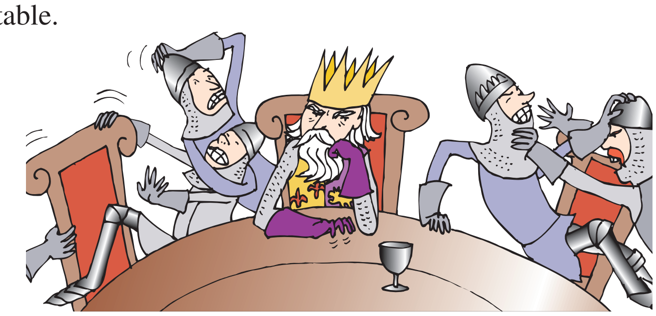
Technology active

4. **WE6a, b** Evaluate the following factorials.
   a. $4!$ b. $5!$ c. $6!$ d. $10!$
   e. $14!$ f. $9!$ g. $7!$ h. $3!$
5. **WE7a** Calculate each of the following by expressing it in expanded form.
   a. $^8P_2$ b. $^7P_5$ c. $^8P_7$
6. **WE7b** Write each of the following as a quotient of factorials and hence evaluate.
   a. $^9P_6$ b. $^5P_2$ c. $^{18}P_5$
7. Use your calculator to determine the value of:
   a. $^{20}P_6$ b. $^{800}P_2$ c. $^{18}P_5$
8. **WE8** A soccer club will appoint a president and a vice-president. Eight people have volunteered for either of the two positions.
   a. In how many different ways can these positions be filled?
   b. For three years the same 8 people volunteered for these positions. What is the probability one of them is president for both years?
9. There are 26 players in an online game. How many different results for 1st, 2nd, 3rd and 4th can occur?
10. A rowing crew consists of 4 rowers who sit in a definite order. How many different crews are possible if 5 people try out for selection?

11. The school musical needs a producer, director, musical director and script coach. Nine people have volunteered for any of these positions. In how many different ways can the positions be filled? (*Note:* One person cannot take on more than 1 position.)



12. There are 14 swimmers in a race. In how many different ways can the 1st, 2nd and 3rd positions be filled?
13. **WE9** a. By recalling the appropriate formula, give an expression for the number of different arrangements if, from a group of 15 people, 4 are to be seated at a round table.
    b. Evaluate this expression.
    c. Each table receives a lucky door prize, a lucky seat prize and a best-dressed prize. Determine the probability of the same person at one table winning all 3 prizes.
14. A round table seats 6 people. From a group of 8 people, give an expression for, and hence calculate, the number of ways 6 people can be seated at the table.
15. At a dinner party for 10 people all the guests were seated at a circular table. How many different arrangements were possible?
16. At one stage in the court of Camelot, King Arthur and 12 knights would sit at the round table. If each person could sit anywhere determine how many different arrangements were possible.



17. **MC** Which one of the following permutations cannot be calculated?
    A. ${}^{1000}P_{100}$ B. ${}^{1}P_{0}$ C. ${}^{8}P_{8}$ D. ${}^{4}P_{8}$
18. **MC** The result of 100! is greater than 94!.
    Which of the following gives the best comparison between these two numbers?
    A. 100! is 6 more than 94!
    B. 100! is 6 times bigger than 94!
    C. 100! is about 10 000 more than 94!
    D. 100! is ${}^{100}P_6$ times bigger than 94!
19. In how many ways can the letters of the word TODAY be arranged if they are used once only and taken:
    a. 3 at a time? b. 4 at a time? c. 5 at a time?
    Show your answers in the form ${}^{n}P_{r}$ and then evaluate.

1.4 Permutations with restrictions

1.4.1 Like objects

A 5-letter word is to be made from 3 As and 2 Bs. How many different permutations or arrangements can be made?

If the 5 letters were all different, it would be easy to calculate the number of arrangements. It would be $5! = 120$. Perhaps you can see that when letters are repeated, the number of different arrangements will be less than 120. To analyse the situation let us imagine that we can distinguish one A from another. We will write A_1, A_2, A_3, B_1 and B_2 to represent the 5 letters.

As we list some of the possible arrangements we notice that some are actually the same, as shown in the table.

$A_1A_2B_1A_3B_2$ $A_1A_3B_1A_2B_2$ $A_2A_1B_1A_3B_2$ $A_2A_3B_1A_1B_2$ $A_3A_1B_1A_2B_2$ $A_3A_2B_1A_1B_2$	$A_1A_2B_2A_3B_1$ $A_1A_3B_2A_2B_1$ $A_2A_1B_2A_3B_1$ $A_2A_3B_2A_1B_1$ $A_3A_1B_2A_2B_1$ $A_3A_2B_2A_1B_1$	Each of these 12 arrangements is the same — AABAB — if $A_1 = A_2 = A_3$ and $B_1 = B_2$.
$B_2A_1A_2B_1A_3$ $B_2A_1A_3B_1A_2$ $B_2A_2A_1B_1A_3$ $B_2A_2A_3B_1A_1$ $B_2A_3A_1B_1A_2$ $B_2A_3A_2B_1A_1$	$B_1A_1A_2B_2A_3$ $B_1A_1A_3B_2A_2$ $B_1A_2A_1B_2A_3$ $B_1A_2A_3B_2A_1$ $B_1A_3A_1B_2A_2$ $B_1A_3A_2B_2A_1$	Each of these 12 arrangements is the same — BAABA — if $A_1 = A_2 = A_3$ and $B_1 = B_2$.

The number of repetitions is 3! for the As and 2! for the Bs. Thus, the number of different arrangements in choosing 5 letters from 3 As and 2 Bs is $\dfrac{5!}{3! \times 2!}$.

The number of different ways of arranging *n* objects made up of groups of indistinguishable objects, n_1 in the first group, n_2 in the second group and so on, is:

$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}.$$

Note: If there are elements of the group which are not duplicated, then they can be considered as a group of 1. It is not usual to divide by 1!; it is more common to show only those groups which have duplications.

WORKED EXAMPLE 10

Determine how many different permutations of 7 counters can be made from 4 black and 3 white counters.

THINK	WRITE
1. Write down the total number of counters.	There are 7 counters in all; therefore, $n = 7$.
2. Write down the number of times any of the coloured counters are repeated.	There are 3 white counters; therefore, $n_1 = 3$. There are 4 black counters; therefore, $n_2 = 4$.
3. Write down the rule for arranging groups of like things.	$\dfrac{n!}{n_1!\,n_2!\,n_3!\ldots n_r!}.$
4. Substitute the values of n, n_1 and n_2 into the rule.	$= \dfrac{7!}{3! \times 4!}$
5. Expand each of the factorials.	$= \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1}$
6. Simplify the fraction.	$= \dfrac{7 \times 6 \times 5}{6}$
7. Evaluate.	$= 35$

8. Answer the question.	Thirty-five different arrangements can be made from 7 counters, of which 3 are white and 4 are black.

1.4.2 Restrictions

Sometimes restrictions are introduced so that a smaller number of objects from the original group need to be considered. This results in limiting the numbers of possible permutations.

WORKED EXAMPLE 11

A rowing crew of 4 rowers is to be selected, in order from the first seat to the fourth seat, from 8 candidates. Determine how many different arrangements are possible if:

a. **there are no restrictions**
b. **Jason or Kris must row in the first seat**
c. **Jason must be in the crew, but he can row anywhere in the boat**
d. **Jason is not in the crew.**

THINK / **WRITE**

a. 1. Write down the permutation formula. *Note:* 4 rowers are to be selected from 8 and the order is important.

a. $^{n}\text{P}_{r} = \dfrac{n!}{(n-r)!}$

2. Substitute the given values of n and r into the permutation formula.

$^{8}\text{P}_{4} = \dfrac{8!}{(8-4)!}$

$= \dfrac{8!}{4!}$

3. Expand the factorials or use a calculator to evaluate 8! and 4!.

$= \dfrac{8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$

$= 8 \times 7 \times 6 \times 5$

4. Evaluate.

$= 1680$

5. Answer the question.

There are 1680 ways of arranging 4 rowers from a group of 8.

b. 1. Apply the multiplication principle since two events will follow each other; that is, Jason will fill the first seat and the remaining 3 seats will be filled in $7 \times 6 \times 5$ ways or Kris will fill the first seat and the remaining 3 seats will be filled in $7 \times 6 \times 5$ ways.

J	7	6	5

or

K	7	6	5

b. No. of arrangements
= no. of ways of filling the first seat × no. of ways of filling the remaining 3 seats.
$= 2 \times {}^{n}\text{P}_{r}$

2. Substitute the values of n and r into the formula and evaluate.

$= 2 \times {}^7P_3$
$= 2 \times 210$
$= 420$

3. Answer the question.

There are 420 ways of arranging the 4 rowers if Jason or Kris must row in the first seat.

c. 1. Apply the addition principle, since Jason must be in either the first, second, third or fourth seat. The remaining 3 seats will be filled in $7 \times 6 \times 5$ ways each time.

J	7	6	5	+	7	J	6	5
7	6	J	5	+	7	6	5	J

c. No. of arrangements
= no. of arrangements with Jason in seat 1
+ No. of arrangements with Jason in seat 2
+ No. of arrangements with Jason in seat 3
+ No. of arrangements with Jason in seat 4.

2. Substitute the values of n and r into the formula.

No. of arrangements
$= 1 \times {}^7P_3 + 1 \times {}^7P_3 + 1 \times {}^7P_3 + 1 \times {}^7P_3$
$= 4 \times {}^7P_3$
$= 4 \times 210$

3. Evaluate.

$= 840$

4. Answer the question.

There are 840 ways of arranging the 4 rowers if Jason must be in the crew of 4.

d. As Jason is not in the crew, there are only 7 candidates. Four rowers are to be chosen from 7 and order is important.

d. $${}^7P_4 = \frac{7!}{(7-4)!} = \frac{7!}{3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 7 \times 6 \times 5 \times 4 = 840$$

There are 840 ways of arranging the crew when Jason is not included.

WORKED EXAMPLE 12

a. **Calculate the number of permutations of the letters in the word COUNTER.**
b. **In how many of these do the letters C and N appear side by side?**
c. **In how many permutations do the letters C and N appear apart?**
d. **Determine the probability of the letters C and N appearing side by side.**

THINK

a. 1. Count the number of letters in the given word.

2. Determine the number of ways the 7 letters may be arranged.

WRITE

a. There are 7 letters in the word COUNTER.

The 7 letters may be arranged $7! = 5040$ ways.

3. Answer the question.

There are 5040 permutations of letters in the word COUNTER.

b. 1. Imagine the C and N are 'tied' together and are therefore considered as 1 unit. Determine the number of ways C and N may be arranged: CN and NC.

b. Let C and N represent 1 unit.
They may be arranged $2! = 2$ ways.

2. Determine the number of ways 6 things can be arranged.
Note: There are now 6 letters: the 'CN' unit along with O, U, T, E and R.

Six things may be arranged $6! = 720$ ways.

3. Determine the number of permutations in which the letters C and N appear together.

$$\begin{aligned}\text{The number of permutations} &= 2 \times 6! \\ &= 2 \times 720 \\ &= 1440\end{aligned}$$

4. Answer the question.

There are 1440 permutations in which the letters C and N appear together.

c. 1. Determine the total number of arrangements of the 7 letters.

c. $$\begin{aligned}\text{Total number of arrangements} &= 7! \\ &= 5040\end{aligned}$$

2. Write down the number of arrangements in which the letters C and N appear together, as obtained in **a**.

Arrangements with C and N together = 1440

3. Determine the difference between the values obtained in steps 1 and 2.
Note: The number of arrangements in which C and N are apart is the total number of arrangements less the number of times they are together.

$$\begin{aligned}\text{The number of arrangements} &= 5040 - 1440 \\ &= 3600\end{aligned}$$

4. Answer the question.

The letters C and N appear apart 3600 times.

d. 1. Recall the probability formula.

d. $\text{P}(A) = \dfrac{n(\xi)}{n(A)}$

2. State the number of elements in the set ξ, that is $n(\xi)$.

From part **a**, there are 5040 permutations of letters in the word COUNTER.
$n(\xi) = 5040$

3. Determine the number of elements in the set A, that is the number of arrangements in which the letters C and N appear side by side.

From part **b**, there are 1440 permutations in which the letters C and N appear side by side.
$n(A) = 1440$

4. Calculate the answer.

$$\begin{aligned}\text{P}(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{1440}{5040} \\ &= \frac{2}{7}\end{aligned}$$

The probability of the letters C and N appearing side by side is $\frac{2}{7}$.

WORKED EXAMPLE 13

Consider the two words 'PARALLEL' and 'LINES'.

a. How many arrangements of the letters of the word LINES have the vowels grouped together?

b. How many arrangements of the letters of the word LINES have the vowels separated?

c. How many arrangements of the letters of the word PARALLEL are possible?

d. Determine the probability that in a randomly chosen arrangement of the word PARALLEL, the letters A are together.

THINK	WRITE
a. 1. Group the required letters together.	**a.** There are two vowels in the word LINES. Treat these letters, I and E, as one unit.
2. Arrange the unit of letters together with the remaining letters.	Now there are four groups to arrange: (IE), L, N, S. These arrange in 4! ways.
3. Use the multiplication principle to allow for any internal rearrangements.	The unit (IE) can internally rearrange in 2! ways. Hence, the total number of arrangements is: $4!\times 2! = 24\times 2$ $= 48$
b. 1. State the method of approach to the problem.	**b.** The number of arrangements with the vowels separated is equal to the total number of arrangements minus the number of arrangements with the vowels together.
2. State the total number of arrangements.	The five letters of the word LINES can be arranged in $5! = 120$ ways.
3. Calculate the answer.	From part **a**, there are 48 arrangements with the two vowels together. Therefore, there are $120 - 48 = 72$ arrangements in which the two vowels are separated.
c. 1. Count the letters, stating any identical letters.	The word PARALLEL contains 8 letters of which there are 2 As and 3 Ls.
2. Recall the rule $\dfrac{n!}{n_1!\,n_2!\ldots}$ and state the number of distinct arrangements.	There are $\dfrac{8!}{2!\times 3!}$ arrangements of the word PARALLEL.
3. Calculate the answer.	$\dfrac{8!}{2!\times 3!} = \dfrac{8\times 7\times 6\times 5\times \cancel{4}^{2}\times \cancel{3!}}{\cancel{2}\times \cancel{3!}}$ $= 3360$ There are 3360 arrangements.
d. 1. State the number of elements in the sample space.	**d.** There are 3360 total arrangements of the word PARALLEL, so $n(\xi) = 3360$ or $\dfrac{8!}{2!\times 3!}$.
2. Group the required letters together.	For the letters A to be together, treat these two letters as one unit. This creates seven groups: (AA), P, R, L, L, E, L, of which three are identical Ls.

3. Calculate the number of elements in the event.	The seven groups arrange in $\frac{7!}{3!}$ ways. As the unit (AA) contains two identical letters, there are no distinct internal rearrangements of this unit that need to be taken into account. Hence, $\frac{7!}{3!}$ is the number of elements in the event.
4. Calculate the required probability. *Note:* It helps to use factorial notation in the calculations.	The probability that the As are together $= \frac{\text{number of arrangements with the As together}}{\text{total number of arrangements}}$ $= \frac{7!}{3!} \div \frac{8!}{2! \times 3!}$ $= \frac{7!}{3!} \times \frac{2! \times 3!}{8 \times 7!}$ $= \frac{2}{8}$ $= \frac{1}{4}$

studyon

Units 1 & 2 › Area 1 › Sequence 1 › Concept 4

Permutations Summary screen and practice questions

Exercise 1.4 Permutations with restrictions

Technology active

1. Recall the appropriate formula and calculate the number of different arrangements can be made using the 6 letters of the word NEWTON, assuming:
   a. the first N is distinct from the second N
   b. there is no distinction between the 2 Ns.
2. How many different permutations can be made using the 11 letters of the word ABRACADABRA?
3. **WE10** Determine how many different arrangements of 5 counters can be made using 3 red and 2 blue counters.
4. Determine how many different arrangements of 9 counters can be made using 4 black, 3 red and 2 blue counters.
5. A collection of 12 books is to be arranged on a shelf. The books consist of 3 copies of *Great Expectations*, 5 copies of *Catcher in the Rye* and 4 copies of *Huntin’, Fishin’ and Shootin’*. How many different arrangements of these books are possible?

6. A shelf holding 24 cans of dog food is to be stacked using 9 cans of Yummy and 15 cans of Ruff for Dogs. Determine how many different ways the shelf can be stocked.
7. **WE11** A cricket team of 11 players is to be selected, in batting order, from 15. Determine how many different arrangements are possible if:
   a. there are no restrictions
   b. Arjun must be in the team at number 1
   c. Arjun must be in the team but he can be anywhere from 1 to 11
   d. Arjun is not in the team.
8. The Student Council needs to fill the positions of president, secretary and treasurer from 6 candidates. Each candidate can fill only one of the positions. Determine how many ways can this be done if:
   a. there are no restrictions
   b. Tan must be secretary
   c. Tan must have one of the 3 positions
   d. Tan is not in any of the positions.
9. The starting 5 in a basketball team is to be picked, in order, from the 10 players in the squad. Determine how many ways can this be done if:
   a. there are no restrictions
   b. Jamahl needs to be player number 5
   c. Jamahl and Anfernee must be in the first 5 players (starting 5)
   d. Jamahl is not in the team.
10. **WE12** a. Calculate the number of permutations of the letters in the word MATHS.
   b. In how many of these do the letters M and A appear together?
   c. In how many permutations do the letters M and A appear apart?
   d. Determine the probability of the letters M and A appearing apart.
11. A rowing team of 4 rowers is to be selected in order from 8 rowers.
   a. In how many different ways can this be done?
   b. In how many of these ways do 2 rowers, Jane and Lee, sit together in the boat?
   c. In how many ways can the crew be formed without using Jane or Lee?
   d. In how many ways can the crew be formed if it does not contain Jane?

12. A decathlon has 12 runners.
   a. In how many ways can 1st, 2nd and 3rd be filled?
   b. In how many ways can 1st, 2nd and 3rd be filled if Najim finishes first?
13. **WE13** Consider the words SIMULTANEOUS and EQUATIONS.
   a. How many arrangements of the letters of the word EQUATIONS have the letters Q and U grouped together.
   b. How many arrangements of the letters of the word EQUATIONS have the letters Q and U separated?
   c. How many arrangements of the letters of the word SIMULTANEOUS are possible?
   d. Determine the probability that in a randomly chosen arrangement of the word SIMULTANEOUS, both the letters U are together.
14. **MC** If the answer is 10, which of the following options best matches this answer?
   A. The number of ways 1st and 2nd can occur in a race with 5 entrants
   B. The number of distinct arrangements of the letters in NANNA
   C. The number of permutations of the letters in POCKET where P and O are together
   D. ${}^{10}P_2 \div {}^{4}P_2$

15. MC If the answer is 480, which of the following options best matches this answer?
    A. The number of ways 1st and 2nd can occur in a race with 5 entrants
    B. The number of distinct arrangements of the letters in NANNA
    C. The number of permutations of the letters in POCKET where P and O are apart
    D. ${}^{10}P_2 \div {}^{4}P_2$
16. The clue in a crossword puzzle says that a particular answer is an anagram of STOREY. An anagram is another word that can be obtained by rearranging the letters of the given word.
    a. Determine the number of possible arrangements of the letters of STOREY.
    b. The other words in the crossword puzzle indicate that the correct answer is O__T__.How many arrangements are now possible? Can you see the word?

1.5 Combinations

1.5.1 When order does not matter

A group of things chosen from a larger group where order is not important is called a combination. In previous sections we performed calculations of the number of ways a task could be done where order is important — permutations or arrangements. We now examine situations where *order does not matter*.

Suppose 5 people have nominated for a committee consisting of 3 members. It does not matter in what order the candidates are placed on the committee, it matters only whether they are there or not. If order was important we know there would be ${}^{5}P_3$, or 60, ways in which this could be done. Here are the possibilities:

ABC	ACB	BAC	BCA	CAB	CBA
ABD	ADB	BAD	BDA	DAB	DBA
ABE	AEB	BAE	BEA	EAB	ABA
ACE	AEC	CAE	CEA	EAC	ECA
ACD	ADC	CAD	CDA	DAC	DCA
ADE	AED	DAE	DEA	EAD	EDA
BCD	BDC	CBD	CDB	DBC	DCB
BCE	BEC	CBE	CEB	EBC	ECB
BDE	BED	DBE	DEB	EBD	EDB
CDE	CED	DCE	DEC	ECD	EDC

The 60 arrangements are different only if we take order into account; that is, ABC is different from CAB and so on. You will notice in this table that there are 10 distinct committees corresponding to the 10 distinct rows. Each row merely repeats, in a different order, the committee in the first column. This result (10 distinct committees) can be arrived at logically:

1. There are ${}^{5}P_3$ ways of choosing or selecting 3 from 5 in order.
2. Each choice of 3 is repeated 3! times.
3. The number of distinct selections or combinations is ${}^{5}P_3 \div 3! = 10$.

This leads to the general rule of selecting r objects from n objects.

The number of ways of choosing or selecting r objects from n distinct objects, where order is not important, is given by ${}^{n}C_r$:

$$ {}^{n}C_r = \frac{{}^{n}P_r}{r!} $$

C is used to represent combinations.

WORKED EXAMPLE 14

Write these combinations as statements involving permutations, then calculate them.

a. $^{7}C_2$ **b.** $^{20}C_3$

THINK	WRITE
a. 1. Recall the rule for $^{n}C_r$.	**a.** $^{n}C_r = \dfrac{^{n}P_r}{r!}$
2. Substitute the given values of n and r into the combination formula.	$^{7}C_2 = \dfrac{^{7}P_2}{2!}$
3. Simplify the fraction.	$= \dfrac{\left(\frac{7!}{5!}\right)}{2!}$ $= \dfrac{7!}{5!} \div 2!$ $= \dfrac{7!}{5!} \times \dfrac{1}{2!}$ $= \dfrac{7 \times 6 \times 5!}{5! \times 2 \times 1}$
4. Evaluate.	$= \dfrac{7 \times 6}{2 \times 1}$ $= \dfrac{42}{2}$ $= 21$
b. 1. Write down the rule for $^{n}C_r$.	**b.** $^{n}C_r = \dfrac{^{n}P_r}{r!}$
2. Substitute the values of n and r into the formula.	$^{20}C_3 = \dfrac{^{20}P_3}{3!}$
3. Simplify the fraction.	$= \dfrac{\left(\frac{20!}{17!}\right)}{3!}$ $= \dfrac{20!}{17!} \div 3!$ $= \dfrac{20!}{17!} \times \dfrac{1}{3!}$ $= \dfrac{20 \times 19 \times 18 \times 17!}{17! \times 3 \times 2 \times 1}$
4. Evaluate.	$= \dfrac{20 \times 19 \times 18}{3 \times 2 \times 1}$ $= \dfrac{6840}{6}$ $= 1140$

WORKED EXAMPLE 15

Apply the concept of $^{n}C_{r}$ to calculate the number ways a basketball team of 5 players be selected from a squad of 9 if the order in which they are selected does not matter.

THINK	WRITE
1. Recall the rule for $^{n}C_{r}$. *Note:* Since order does not matter, use the $^{n}C_{r}$ rule.	$^{n}C_{r} = \dfrac{^{n}P_{r}}{r!}$
2. Substitute the values of n and r into the formula.	$^{9}C_{5} = \dfrac{^{9}P_{5}}{5!}$
3. Simplify the fraction.	$= \dfrac{\left(\frac{9!}{4!}\right)}{5!}$ $= \dfrac{9!}{4!} \div 5!$ $= \dfrac{9!}{4!} \times \dfrac{1}{5!}$ $= \dfrac{9!}{4!\,5!}$ $= \dfrac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!}$
4. Evaluate.	$= \dfrac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$ $= \dfrac{3024}{24}$ $= 126$

The formula we use to determine the number of ways of selecting r objects from n distinct objects, where order is not important, is useful but needs to be simplified.

$$\begin{aligned} ^{n}C_{r} &= \frac{^{n}P_{r}}{r!} \\ &= \frac{\frac{n!}{(n-r)!}}{r!} \\ &= \frac{n!}{r!\,(n-r)\,!} \end{aligned}$$

$$^{n}C_r = \frac{n!}{r!\,(n-r)!} = \binom{n}{r}$$

$0 \le r \le n$ where r and n are non-negative integers.

The formula for $^{n}C_r$ is exactly that for the binomial coefficients used in the binomial theorem, which is explored with reference to Pascal's triangle in section 1.7.

on Resources

Interactivity: Counting techniques (int-6293)

WORKED EXAMPLE 16

Determine the value of the following.

a. $^{12}C_5$ **b.** $\binom{10}{2}$

THINK	WRITE
a. 1. Recall the formula for $^{n}C_r$.	**a.** $^{n}C_r = \frac{n!}{(n-r)!\,r!}$
2. Substitute the given values of n and r into the combination formula.	$^{12}C_5 = \frac{12!}{(12-5)!\,5!}$ $= \frac{12!}{7!\,5!}$
3. Simplify the fraction.	$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7!}{7! \times 5 \times 4 \times 3 \times 2 \times 1}$ $= \frac{12 \times 11 \times \cancel{10} \times \cancel{9}^3 \times \cancel{8}^2}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 12 \times 11 \times 3 \times 2$ $= 792$
b. 1. Recall the rule for $\binom{n}{r}$.	**b.** $\binom{n}{r} = {}^{n}C_r$ $= \frac{n!}{(n-r)!\,r!}$
2. Substitute the given values of n and r into the combination formula.	$\binom{10}{2} = \frac{10!}{(10-2)!\,2!}$ $= \frac{10!}{8!\,2!}$
3. Simplify the fraction.	$= \frac{10 \times 9 \times 8!}{8! \times 2 \times 1}$ $= \frac{10 \times 9}{2 \times 1}$

4. Evaluate. $= \frac{90}{2}$

$= 45$

1.5.2 Probability calculations

The combination formula is always used in selection problems. Most calculators have a ${}^nC_r$ key to assist with the evaluation when the figures become large.

Both the multiplication and addition principles apply and are used in the same way as for permutations.

The calculation of probabilities from the rule $P(A) = \frac{n(A)}{n(\xi)}$ requires that the same counting technique is used for the numerator and denominator. We have seen for permutations that it can assist calculation to express numerator and denominator in terms of factorials and then simplify. Similarly for combinations, express the numerator and denominator in terms of the appropriate combinatoric coefficients and then carry out the calculations.

WORKED EXAMPLE 17

A committee of 5 students is to be chosen from 7 boys and 4 girls. Use ${}^nC_r$ and the multiplication and addition principles to answer the following.

a. Calculate how many committees can be formed.

b. Calculate how many of the committees contain exactly 2 boys and 3 girls.

c. Calculate how many committees have at least 3 girls.

d. Determine the probability of the oldest and youngest students both being on the committee in part a.

THINK

a. 1. As there is no restriction, choose the committee from the total number of students.

2. Use the formula ${}^nC_r = \frac{n!}{r! \times (n-r)!}$ to calculate the answer.

WRITE

a. There are 11 students in total from whom 5 students are to be chosen. This can be done in ${}^{11}C_5$ ways.

$$\begin{aligned}{}^{11}C_5 &= \frac{11!}{5! \times (11-5)!} \\ &= \frac{11!}{5! \times 6!} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6!}{5! \times 6!} \\ &= \frac{11 \times \cancel{10}^2 \times \cancel{9 \times 8}^3 \times 7}{\cancel{5} \times \cancel{4 \times 3 \times 2} \times 1} \\ &= 462\end{aligned}$$

There are 462 possible committees.

b. 1. Select the committee to satisfy the given restriction.

b. The 2 boys can be chosen from the 7 boys available in ${}^7C_2$ ways.The 3 girls can be chosen from the 4 girls available in ${}^4C_3$ ways.

2. Use the multiplication principle to form the total number of committees.
Note: The upper numbers on the combinatoric coefficients sum to the total available, $7 + 4 = 11$, while the lower numbers sum to the number that must be on the committee, $2 + 3 = 5$.

The total number of committees which contain two boys and three girls is ${}^7C_2 \times {}^4C_3$.

3. Calculate the answer.

$$\begin{aligned} {}^7C_2 \times {}^4C_3 &= \frac{7!}{2! \times 5!} \times 4 \\ &= \frac{7 \times 6}{2!} \times 4 \\ &= 21 \times 4 \\ &= 84 \end{aligned}$$

There are 84 committees possible with the given restriction.

c. 1. List the possible committees which satisfy the given restriction.

c. As there are 4 girls available, at least 3 girls means either 3 or 4 girls.The committees of 5 students which satisfy this restriction have either 3 girls and 2 boys, or they have 4 girls and 1 boy.

2. Write the number of committees in terms of combinatoric coefficients.

3 girls and 2 boys are chosen in ${}^4C_3 \times {}^7C_2$ ways.
4 girls and 1 boy are chosen in ${}^4C_4 \times {}^7C_1$ ways.

3. Use the addition principle to state the total number of committees.

The number of committees with at least three girls is ${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1$.

4. Calculate the answer.

$$\begin{aligned} {}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 &= 84 + 1 \times 7 \\ &= 91 \end{aligned}$$

There are 91 committees with at least 3 girls.

d. 1. State the number in the sample space.

d. The total number of committees of 5 students is ${}^{11}C_5 = 462$ from part **a**.

2. Form the number of ways the given event can occur.

Each committee must have 5 students. If the oldest and youngest students are placed on the committee, then 3 more students need to be selected from the remaining 9 students to form the committee of 5. This can be done in ${}^9C_3$ ways.

3. State the probability in terms of combinatoric coefficients.

Let A be the event the oldest and the youngest students are on the committee.

$$\begin{aligned} P(A) &= \frac{n(A)}{n(\xi)} \\ &= \frac{{}^9C_3}{{}^{11}C_5} \end{aligned}$$

4. Calculate the answer.

$$P(A) = \frac{9!}{3! \times 6!} \div \frac{11!}{5! \times 6!}$$
$$= \frac{9!}{3! \times 6!} \times \frac{5! \times 6!}{11!}$$
$$= \frac{1}{3!} \times \frac{5!}{11 \times 10}$$
$$= \frac{5 \times 4}{110}$$
$$= \frac{2}{11}$$

The probability of the committee containing the youngest and the oldest students is $\frac{2}{11}$.

WORKED EXAMPLE 18

Evaluate the following using your calculator and comment on your results.

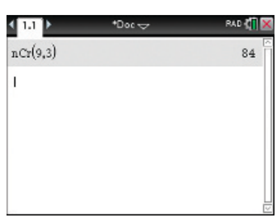
a. $^{9}C_{3}$ **b.** $^{9}C_{6}$ **c.** $^{15}C_{5}$ **d.** $^{15}C_{10}$ **e.** $^{12}C_{7}$ **f.** $^{12}C_{5}$

THINK	WRITE
a–f. Use your calculator to evaluate the listed combinations.	**a.** $^{9}C_{3} = 84$ **b.** $^{9}C_{6} = 84$ **c.** $^{15}C_{5} = 3003$ **d.** $^{15}C_{10} = 3003$ **e.** $^{12}C_{7} = 792$ **f.** $^{12}C_{5} = 792$
Comment on your results.	So $^{9}C_{3} = {}^{9}C_{6}$, $^{15}C_{5} = {}^{15}C_{10}$ and $^{12}C_{7} = {}^{12}C_{5}$. Its appear that when, for example, $^{12}C_{p} = {}^{12}C_{q}$ $p + q = 12$.

TI | THINK

a.1. On a Calculator page, press MENU, then select:
5: Probability
3: Combinations.
Complete the entry line as:
nCr(9, 3)
then press ENTER.

WRITE



2. The answer appears on the screen.

$^{9}C_{3} = 84$

CASIO | THINK

a.1. On a Run-Matrix screen, press OPTN then F6. Select PROB by pressing F3, then select nCr by pressing F3. Complete the entry line as:
9C3
then press EXE.

WRITE

2. The answer appears on the screen.

$^{9}C_{3} = 84$

For each of the preceding examples, it can be seen that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$. This may be derived algebraically:

$$\begin{aligned}
{}^n\mathrm{C}_{n-r} &= \frac{{}^n\mathrm{P}_{n-r}}{(n-r)!} \\
&= \frac{\left(\frac{n!}{[n-(n-r)]!}\right)}{(n-r)} \\
&= \frac{\left(\frac{n!}{r!}\right)}{(n-r)!} \\
&= \frac{n!}{r!} \times \frac{1}{(n-r)!} \\
&= \frac{n!}{r!\,(n-r)!} \\
&= \frac{n!}{(n-r)!\,r!} \\
&= \frac{{}^n\mathrm{P}_r}{r!} \\
&= {}^n\mathrm{C}_r
\end{aligned}$$

Drawing on our understanding of combinations, we have:
- ${}^nC_r = {}^nC_{r-1}$, as choosing r objects must leave behind $(n-r)$ objects and vice versa
- ${}^nC_0 = 1 = {}^nC_n$, as there is only one way to choose none or all of the n objects
- ${}^nC_1 = n$, as there are n ways of choosing 1 object from a group of n objects.

on Resources

Digital document: SkillSHEET Listing possibilities (doc-26826)
Digital document: SpreadSHEET Combinations (doc-26827)

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 5

Combinations Summary screen and practice questions

Exercise 1.5 Combinations

Technology free

1. **WE14** Write each of the following as statements in terms of permutations.

a. ${}^8\mathrm{C}_3$ **b.** ${}^{19}\mathrm{C}_2$ **c.** ${}^1\mathrm{C}_1$ **d.** ${}^5\mathrm{C}_0$

2. Write each of the following using the notation ${}^n\mathrm{C}_r$.

a. $\frac{{}^8\mathrm{P}_2}{2!}$ **b.** $\frac{{}^9\mathrm{P}_3}{3!}$ **c.** $\frac{{}^8\mathrm{P}_0}{0!}$ **d.** $\frac{{}^{10}\mathrm{P}_4}{4!}$

3. **WE15** Apply the concept of ${}^n\mathrm{C}_r$ to calculate the number of ways three types of ice-cream can be chosen in any order from a supermarket freezer if the freezer contains:

a. 3 types **b.** 6 types **c.** 10 types **d.** 12 types.

4. A mixed netball team must have 3 women and 4 men in the side. If the squad has 6 women and 5 men wanting to play, determine how many different teams are possible.

5. A *quinella* is a bet made on a horse race which pays a win if the punter selects the first 2 horses in any order. Determine how many different quinellas are possible in a race that has:
   a. 8 horses b. 16 horses.
6. **MC** At a party there are 40 guests and they decide to have a toast. Each guest ‘clinks’ glasses with every other guest. How many clinks are there in all?
   A. 39 B. 40 C. 40! D. 780
7. **MC** On a bookshelf there are 15 books — 7 geography books and 8 law books. Abena selects 5 books from the shelf — 2 geography books and 3 law books. How many different ways can she make this selection?
   A. ${}^{15}C_2 \times {}^{15}C_3$ B. ${}^{15}C_7 \times {}^{15}C_8$ C. ${}^{7}C_2 \times {}^{8}C_3$ D. ${}^{7}C_2 + {}^{8}C_3$

Technology active

8. A cricket team of 11 players is to be chosen from a squad of 15 players. Determine how many ways can this be done.
9. A basketball team of 5 players is to be chosen from a squad of 10 players. Determine how many ways can this be done.
10. **WE16** Determine the value of the following:
    a. ${}^{12}C_4$ b. ${}^{11}C_1$ c. ${}^{12}C_{12}$
    d. $\binom{21}{15}$ e. $\binom{100}{1}$ f. $\binom{17}{14}$
11. From a pack of 52 cards, a hand of 5 cards is dealt.
    a. How many different hands are there?
    b. How many of these hands contain only red cards?
    c. How many of these hands contain only black cards?
    d. How many of these hands contain at least one red and at least one black card?
12. **WE17** A committee of 5 students is to be chosen from 6 boys and 8 girls. Use ${}^{n}C_r$ and the multiplication principle to answer the following.
    a. Calculate how many committees can be formed.
    b. Calculate how many of the committees contain exactly 2 boys and 3 girls.
    c. Calculate how many committees have at least 4 boys.
    d. Determine the probability of neither the oldest nor the youngest student being on the committee.
13. A rugby union squad has 12 forwards and 10 backs in training. A team consists of 8 forwards and 7 backs. Determine how many different teams can be chosen from the squad.
14. A music collection contains 32 albums. Determine how many ways 5 albums can be chosen from the collection.

Questions 15, 16 and 17 refer to the following information. The Maryborough Tennis Championships involve 16 players. The organisers plan to use 3 courts and assume that each match will last on average 2 hours and that no more than 4 matches will be played on any court per day.

15. In a ‘round robin’ each player plays every other player once.
    a. If the organisers use a round robin format, determine how many games will be played in all.
    b. For how many days would the tournament last?
16. The organisers split the 16 players into two pools of 8 players each. After a ‘round robin’ within each pool, a final is played between the winners of each pool.
    a. Determine how many matches are played in the tournament.
    b. How long does the tournament last?

17. A 'knock out' format is one in which the loser of every match drops out and the winners proceed to the next round until there is only one winner left.
 a. If the game starts with 16 players, determine how many matches are needed before a winner is obtained.
 b. How long would the tournament last?

18. Lotto is a gambling game played by choosing 6 numbers from 45. Gamblers try to match their choice with those numbers chosen at the official draw. No number can be drawn more than once and the order in which the numbers are selected does not matter.
 a. Calculate how many different selections of 6 numbers can be made from 45.
 b. Suppose the first numbers drawn at the official draw are 42, 3 and 18. How many selections of 6 numbers will contain these 3 numbers?

Note: This question ignores supplementary numbers. Lotto is discussed further in the next section.

19. a. **WE18** Calculate the value of:
 i. ${}^{12}C_3$ and ${}^{12}C_9$ **ii.** ${}^{15}C_8$ and ${}^{15}C_7$ **iii.** ${}^{10}C_1$ and ${}^{10}C_9$ **iv.** ${}^{8}C_3$ and ${}^{8}C_5$
 b. What do you notice? Give your answer as a general statement such as 'The value of ${}^{n}C_r$ is ...'.

1.6 Applications of permutations and combinations

1.6.1 Permutations and combinations in the real world

Counting techniques, particularly those involving permutations and combinations, can be applied in gambling, logistics and various forms of market research. In this section we investigate when to use permutations and when to use combinations as well as examining problems associated with these techniques.

Permutations are used to count when order is important. Some examples are:

- the number of ways the positions of president, secretary and treasurer can be filled
- the number of ways a team can be chosen from a squad *in distinctly different positions*
- the number of ways the first three positions of a race can be filled.

Combinations are used to count when order is not important. Some examples are:

- the number of ways a committee can be chosen
- the number of ways a team can be chosen from a squad
- the number of ways a hand of 5 cards can be dealt from a deck.

These relatively simple applications of permutations and combinations are explored in the worked examples that follow. However, it is important to be mindful that the modern world relies on combinatorial algorithms. These algorithms are important for any system that benefits from finding the fastest ways to operate. Examples include communication networks, molecular biology, enhancing security and protecting privacy in internet information transfer, data base queries and data mining, computer chip design, simulations and scheduling.

WORKED EXAMPLE 19

a. Ten points are marked on a page and no three of these points are in a straight line. Determine how many triangles can be drawn joining these points.

b. Determine how many different 3-digit numbers can be made using the digits 1, 3, 5, 7 and 9 without repetition.

THINK	WRITE
a. 1. *Note:* A triangle is made by choosing 3 points. It does not matter in what order the points are chosen, so ^nC_r is used. Recall the rule for ^nC_r.	**a** $^nC_r = \dfrac{n!}{(n-r)!\,r!}$
2. Substitute the given values of n and r into the combination formula.	$^{10}C_3 = \dfrac{10!}{(10-3)!\,3!}$ $= \dfrac{10!}{7!\,3!}$
3. Simplify the fraction.	$= \dfrac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1}$ $= \dfrac{10 \times \cancel{9}^3 \times \cancel{8}^4}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 10 \times 3 \times 4$ $= 120$
5. Answer the question.	120 triangles may be drawn by joining 3 points.
6. Verify the answer obtained by using the combination function on a calculator.	
b. 1. *Note:* Order is important here. Recall the rule for ^nP_r.	**b** $^nP_r = \dfrac{n}{(n-r)!}$
2. Substitute the given values of n and r into the permutation formula.	$^5P_3 = \dfrac{5!}{(5-3)}$ $= \dfrac{5!}{2!}$
3. Evaluate.	$= \dfrac{5 \times 4 \times 3 \times 2}{2!}$ $= 5 \times 4 \times 3$ $= 60$
4. Answer the question.	Sixty 3-digit numbers can be made without repetition from a group of 5 numbers.
5. Verify the answer obtained by using the permutation function on a calculator.	

WORKED EXAMPLE 20

Jade and Kelly are 2 of the 10 members of a basketball squad. Calculate how many ways can a team of 5 be chosen if:

a. both Jade and Kelly are in the 5

b. neither Jade nor Kelly is in the 5

c. Jade is in the 5 but Kelly is not.

THINK	WRITE
a. 1. *Note:* Order is not important, so ^nC_r is used. Recall the rule for ^nC_r.	**a** $^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade and Kelly are included then there are 3 positions to be filled from the remaining 8 players. Substitute the given values of n and r into the combination formula.	$^8C_3 = \frac{8!}{(8-3)!\,3!}$ $= \frac{8!}{5!\,3!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5}{5! \times 3 \times 2 \times 1}$ $= \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 8 \times 7$ $= 56$
5. Answer the question.	If Jade and Kelly are included, then there are 56 ways to fill the remaining 3 positions.
b. 1. *Note:* Order is not important, so ^nC_r is used. Recall the rule for ^nC_r.	**b** $^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade and Kelly are not included then there are 5 positions to be filled from 8 players. Substitute the given values of n and r into the combination formula.	$^8C_5 = \frac{8!}{(8-5)!\,5!}$ $= \frac{8!}{3!\,5!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!}$ $= \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1}$
4. Evaluate.	$= 8 \times 7$ $= 56$
5. Answer the question.	If Jade and Kelly are not included, then there are 56 ways to fill the 5 positions.
c. 1. *Note:* Order is not important, so ^nC_r is used. Recall the rule for ^nC_r.	**c** $^nC_r = \frac{n!}{(n-r)!\,r!}$
2. *Note:* If Jade is included and Kelly is not then there are 4 positions to be filled from 8 players. Substitute the given values of n and r into the combination formula.	$^8C_4 = \frac{8!}{(8-4)!\,4!}$ $= \frac{8!}{4!\,4!}$
3. Simplify the fraction.	$= \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!}$ $= \frac{\cancel{8} \times 7 \times \cancel{6}^2 \times 5}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$

4. Evaluate.
$= 7 \times 2 \times 5$
$= 70$

5. Answer the question.
If Jade is included and Kelly is not, then there are 70 ways to fill the 4 positions.

6. Verify each of the answers obtained by using the combination function on a calculator.

1.6.2 Lotto systems

An interesting application of combinations as a technique of counting is a game that Australians spend many millions of dollars on each week — lotteries. There are many varieties of lottery games in Australia. To play Saturday Gold Lotto in Queensland, a player selects 6 numbers from 45 numbers. The official draw chooses 6 numbers and 2 supplementary numbers. Depending on how the player’s choice of 6 numbers matches the official draw, prizes are awarded in different divisions.

Division 1: 6 winning numbers
Division 2: 5 winning numbers and one of the supplementary numbers
Division 3: 5 winning numbers
Division 4: 4 winning numbers
Division 5: 3 winning numbers and one of the supplementary numbers
If the official draw was:

Winning numbers						Supplementaries	
13	42	6	8	20	12	2	34

A player who chose:

8 34 13 12 20 45

would win a Division 4 prize and a player who chose:

8 34 13 12 22 45

would win a Division 5 prize.

A player may have 7 lucky numbers 4, 7, 12, 21, 30, 38 and 45, and may wish to include all possible combinations of these 7 numbers in a 6 numbers lotto entry.

This can be done as follows:

4	7	12	21	30	38
4	7	12	21	30	45
4	7	12	21	38	45
4	7	12	30	38	45
4	7	21	30	38	45
4	12	21	30	38	45
7	12	21	30	38	45

The player does not have to fill out 7 separate entries to enter all combinations of these 7 numbers 6 at a time but rather can complete a ‘System 7’ entry by marking 7 numbers on the entry form.

A System 9 consists of all entries of 6 numbers from the chosen 9 numbers.

WORKED EXAMPLE 21

Use the information on lottery systems given above.
A player uses a System 8 entry with the numbers 4, 7, 9, 12, 22, 29, 32 and 36.
The official draw for this game was 4, 8, 12, 15, 22, 36 with supplementaries 20 and 29.

a. **How many single entries are equivalent to a System 8?**
b. **List 3 of the player's entries that would have won Division 4.**
c. **Determine how many of the player's entries would have won Division 4.**

THINK	WRITE
a 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**a** ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$
2. *Note:* A System 8 consists of all entries consisting of 6 numbers chosen from 8. Substitute the given values of n and r into the combination formula.	${}^8C_6 = \dfrac{8!}{(8-6)!\,6!}$ $= \dfrac{8!}{2!\,6!}$
3. Simplify the fraction.	$= \dfrac{8 \times 7 \times 6!}{2 \times 7 \times 6!}$ $= \dfrac{\not{8}^4 \times 7}{\not{2} \times 1}$
4. Evaluate.	$= 4 \times 7$ $= 28$
5. Answer the question.	A System 8 is equivalent to 28 single entries.
6. Verify each of the answers obtained by using the combination function on a calculator.	
b *Note:* Division 4 requires 4 winning numbers. The player's winning numbers are 4, 12, 22 and 36. Any of the other 4 numbers can fill the remaining 2 places. List 3 of the player's entries that would have won Division 4.	**b** Some of the possibilities are: 4 12 22 36 7 9 4 12 22 36 7 29 4 12 22 36 7 32
c 1. *Note:* Order is not important, so ${}^nC_r$ is used. Recall the rule for ${}^nC_r$.	**c** ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$
2. *Note:* To win Division 4 the numbers 4, 12, 22 and 36 must be included in the entry. The other 2 spaces can be filled with any of the other 4 numbers in any order. Substitute the given values of n and r into the combination formula.	${}^4C_2 = \dfrac{4!}{(4-2)!\,2!}$ $= \dfrac{4!}{2!\,2!}$
3. Simplify the fraction.	$= \dfrac{4 \times 3 \times 1!}{\not{2} \times 1 \times 2!}$ $= \dfrac{\not{4}^2 \times 3}{\not{2} \times 1}$
4. Evaluate.	$= 2 \times 3$ $= 6$

5. Answer the question. — Six of the player's entries would have won Division 4.

6. Verify each of the answers obtained by using the combination function on a calculator.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concept 6

Applications of permutations and combinations Summary screen and practice questions

Exercise 1.6 Applications of permutations and combinations

Technology active

1. WE19 Determine how many ways there are:
   a. to draw a line segment between 2 points on a page with 10 points on it
   b. to make a 4-digit number using the digits 2, 4, 6, 8 and 1 without repetition
   c. to choose a committee of 4 people from 10 people
   d. for a party of 15 people to shake hands with one another.
2. Determine how many ways there are:
   a. for 10 horses to fill 1st, 2nd and 3rd positions
   b. to choose a team of 3 cyclists from a squad of 5
   c. to choose 1st, 2nd and 3rd speakers for a debating team from 6 candidates
   d. for 20 students to seat themselves in a row of 20 desks.
3. The French flag is known as a tricolour flag because it is composed of the 3 bands of colour. Determine how many different tricolour flags can be made from the colours red, white, blue and green if each colour can be used only once in one of the 3 bands and order is important.

4. In a taste test a market research company has asked people to taste 4 samples of coffee and try to identify each as one of four brands. Subjects are told that no 2 samples are the same brand. Determine how many different ways the samples can be matched to the brands.
5. WE20 A volleyball team of 6 players is to be chosen from a squad of 10 players. Calculate how many ways can this be done if:
   a. there are no restrictions
   b. Stephanie is to be in the team
   c. Stephanie is not in the team
   d. two players, Stephanie and Alison, are not both in the team together.

6. A cross-country team of 4 runners is to be chosen from a squad of 9 runners. Determine how many ways this can be done if:
   a. there are no restrictions
   b. Cecily is to be one of the 4
   c. Cecily and Michael are in the team
   d. either Cecily or Michael but not both are in the team.

7. MC A netball team consists of 7 different positions: goal defence, goal keeper, wing defence, centre, wing attack, goal attack and goal shooter. The number of ways a squad of 10 players can be allocated to these positions is:

   A. 10! B. 7! C. $\frac{10!}{7!}$ D. ${}^{10}P_7$

8. WE21 Use the information on lotteries given on page 41.
   A player uses a System 8 entry with the numbers 9, 12, 14, 17, 27, 34, 37 and 41. The official draw for this game was 9, 13, 17, 20, 27, 41 with supplementaries 25 and 34.
   a. How many single entries are equivalent to a System 8?
   b. List 3 of the player's entries that would have won Division 4.
   c. Determine how many of the player's entries would have won Division 4.
9. Use the information on lotteries given on page 41.
   A player uses a System 9 entry with the numbers 7, 10, 12, 15, 25, 32, 35, 37 and 41. The official draw for this game was 7, 11, 15, 18, 25, 39 with supplementaries 23 and 32.
   a. To how many single entries is a System 9 equivalent?
   b. List 3 of the player's entries that would have won Division 5.
   c. How many of the player's entries would have won Division 5?
10. In the gambling game roulette, if a gambler puts \$1 on the winning number he will win \$35. Suppose a gambler wishes to place five \$1 bets on 5 different numbers in one spin of the roulette wheel. If there are 36 numbers in all, determine how many ways the five bets can be placed.
11. A soccer team of 11 players is to be chosen from a squad of 17. If one of the squad is selected as goalkeeper and any of the remaining players can be selected in any of the positions, determine how many ways can this be done if:
    a. there are no other restrictions
    b. Karl is to be chosen
    c. Karl and Andrew refuse to play in the same team
    d. Karl and Andrew are either both in or both out.
12. MC A secret chemical formula requires the mixing of 3 chemicals. A researcher does not remember the 3 chemicals but has a shortlist of 10 from which to choose. Each time she mixes 3 chemicals and tests the result she takes 15 minutes.
    How long does the researcher need, to be absolutely sure of getting the right combination?
    A. 120 hours B. 7.5 hours
    C. 15 hours D. 30 hours

Questions **13** and **14** refer to the following information: Keno is a popular game in clubs and pubs around Australia. In each round a machine randomly generates 20 winning numbers from 1 to 80. In one entry a player can select up to 15 numbers.

13. Suppose a player selects an entry of 6 numbers.
  a. Determine how many ways an entry of 6 numbers can contain 6 winning numbers.
  Suppose an entry of 6 numbers has exactly 3 winning numbers in it.
  b. In how many ways can the 3 winning numbers be chosen?
  c. In how many ways can the 3 losing numbers be chosen?
  d. How many entries of 6 numbers contain 3 winning numbers and 3 losing numbers?

14. Suppose a player selects an entry of 20 numbers.
  a. Determine how many ways an entry of 20 numbers can contain 20 winning numbers.
  b. Suppose an entry of 20 numbers has exactly 14 winning numbers in it.
    i. In how many ways can the 14 winning numbers be chosen?
    ii. In how many ways can the 6 losing numbers be chosen?
    iii. How many entries of 20 numbers contain 14 winning numbers and 6 losing numbers?
    iv. How many entries of 20 numbers contain no winning numbers?

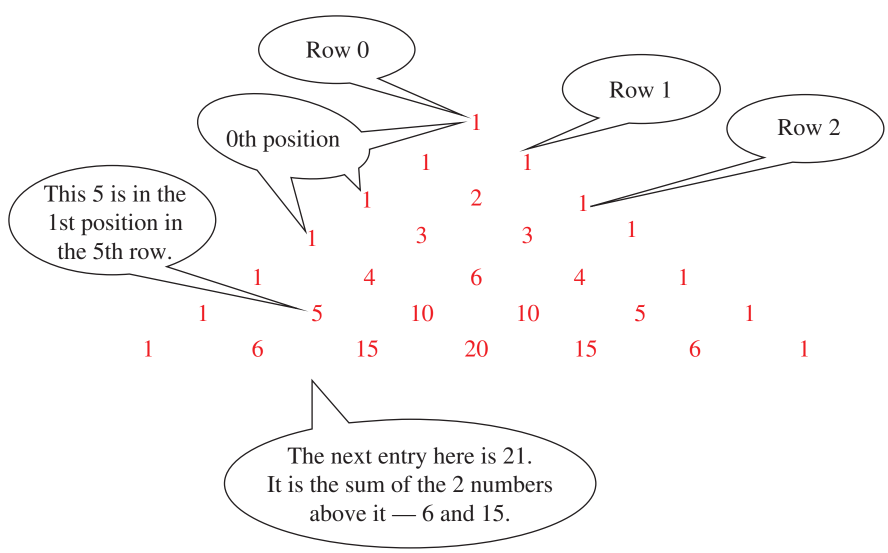
1.7 Pascal's triangle and the pigeon-hole principle

1.7.1 Pascal's triangle

Combinations are useful in other areas of mathematics, such as probability and binomial expansions. If we analyse the ${}^nC_r$ values closely, we notice that they produce the elements of any row in **Pascal's triangle** or each of the coefficients of a particular binomial expansion.

The triangle shown was named after the French mathematician Blaise Pascal. He was honoured for his application of the triangle to his studies in the area of probability.

Each new row in Pascal's triangle is obtained by first placing a 1 at the beginning and end of the row and then adding adjacent entries from the previous row.



Each element in Pascal's triangle can be calculated using combinations. For example, 10 is the 2nd element in the 5th row of Pascal's triangle; that is, ${}^5C_2 = 10$ (assuming 1 is the zeroth (0th) element). Hence, the triangle can be written using $\binom{n}{r}$ or ${}^nC_r$ notation.

$n = 0$:	1	${}^0C_0$
$n = 1$:	1 1	${}^1C_0$ ${}^1C_1$
$n = 2$:	1 2 1	${}^2C_0$ ${}^2C_1$ ${}^2C_2$
$n = 3$:	1 3 3 1	${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$
$n = 4$:	1 4 6 4 1	${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$
$n = 5$:	1 5 10 10 5 1	${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$

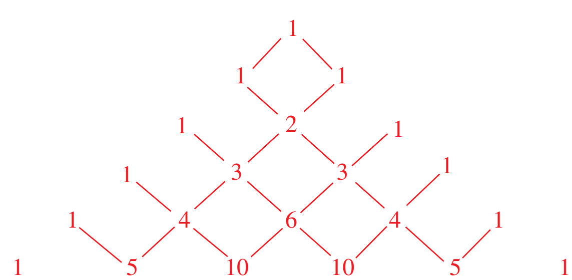
Note that the first and last number in each row is always 1.
Each coefficient is obtained by adding the two coefficients immediately above it.
The binomial expansion can therefore be generalised using combinations.

Pascal's triangle shows that the rth element of the nth row of Pascal's triangle is given by ${}^nC_r$.
It is assumed that the 1 at the beginning of each row is the 0th element.
This gives Pascal's identity:

$$ {}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \text{ for } 0 < r < n $$

The relationship between Pascal's triangle and combinations can be extended to the **binomial theorem**. This theorem gives a rule for expanding an expression such as $(a + b)^n$. Expanding expressions such as this may become quite difficult and time consuming using the usual methods of algebra. Consider the coefficients of the binomial expansion.

$(x + y)^0 = 1$
$(x + y)^1 = x + y$
$(x + y)^2 = x^2 + 2xy + y^2$
$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$



The coefficients in the binomial expansion are equal to the numbers in Pascal's triangle.
These relationships are summarised in the following table.

$(x + y)^0 = 1$	Coefficient: 1	${}^0C_0$
$(x + y)^1 = x + y$	Coefficients: 1 1	${}^1C_0$, ${}^1C_1$,
$(x + y)^2 = x^2 + 2xy + y^2$	Coefficients: 1 2 1	${}^2C_0$, ${}^2C_1$, ${}^2C_2$
$(y + x)^3 = y^3 + 3x^2y + 3xy^2 + x^3$	Coefficients: 1 3 3 1	${}^3C_0$, ${}^3C_1$, ${}^3C_2$, ${}^3C_3$
$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$	Coefficients: 1 4 6 4 1	${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$, ${}^4C_4$
$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$	Coefficients: 1 5 10 10 5 1	${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$, ${}^5C_5$

The binomial expansion can therefore be generated using combinations:

$$
\begin{aligned}
(x+y)^n &= {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n \\
&= x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + y^n
\end{aligned}
$$

on Resources

Interactivity: Pascal’s triangle and binomial coefficients (int-2554)
Digital document: SpreadSHEET Pascal’s triangle (doc-26828)

WORKED EXAMPLE 22

Refer to Pascal’s triangle above and answer the following questions.
a. What number is in the 4th position in the 6th row?
b. Complete the 7th row in Pascal’s triangle.
c. The numbers 7 and 21 occur side by side in the 7th row. What element in the 8th row occurs below and in between these numbers?

THINK	WRITE
a 1. Locate the 6th row and the 4th position. *Note:* Remember the 0th row is 1 and the first row is 1 1. In the 6th row the 1 on the left is in the 0th position.	**a** 6th row ⇒ 1 6 15 20 **15** 6 1
2. Answer the question.	The number in the 4th position in the 6th row is 15.
b 1. Write down the elements of the 6th row.	**b.** 6th row ⇒ 1 6 15 20 15 6 1
2. Obtain the 7th row. a. Place the number 1 at the beginning of the row. b. Add the first 2 adjacent numbers from the 6th row (1 and 6). c. Place this value next to the 1 on the new row and align the value so that it is in the middle of the 2 numbers (directly above) which created it. d. Repeat this process with the next 2 adjacent numbers from the 6th row (6 and 15). e. Once the sums of all adjacent pairs from the sixth row have been added, place a 1 at the end of the row.	7th row ⇒ 1 7 21 35 35 21 7 1
3. Answer the question.	The 7th row is 1 7 21 35 35 21 7 1.
c 1. Add the numbers 7 and 21 in order to obtain the element in the 8th row which occurs below and in between these numbers.	**c** 7 21 28

2. Answer the question. — The element in the 8th row which occurs below and in between 7 and 21 is 28.

WORKED EXAMPLE 23

Use combinations to calculate the number in the 5th position in the 9th row of Pascal's triangle.

THINK	WRITE
1. Write down the combination rule.	${}^{n}C_{r}$
2. Substitute the values for n and r into the rule. *Note:* The row is represented by $n = 9$. The position is represented by $r = 5$.	${}^{9}C_{5} = 126$
3. Evaluate using a calculator.	
4. Answer the question.	The value of the number in the 5th position in the 9th row is 126.

WORKED EXAMPLE 24

Use the binomial theorem to expand $(a + 2)^4$.

THINK	WRITE
1. Recall the rule for the binomial theorem.	$(x+y)^n = x^n + {}^{n}C_{1}x^{n-1}y^1 + \ldots {}^{n}C_{r}x^{n-r}y^r + \ldots y^n$
2. Substitute the values for a, b and n into the rule: $x = a$, $y = 2$ and $n = 4$.	$(a+2)^4 = a^4 + {}^{4}C_{1}a^3 2^1 + {}^{4}C_{2}a^2 2^2 + {}^{4}C_{3}a^1 2^3 + 2^4$
3. Simplify.	$= a^4 + 4 \times a^3 \times 2 + 6 \times a^2 \times 4 + 4 \times a \times 8 + 16$ $= a^4 + 8a^3 + 24a^2 + 32a + 16$

WORKED EXAMPLE 25

What is the 4th term in the expansion of $(x + y)^7$?

THINK	WRITE
1. Recall that the rule for the 4th term can be obtained from the binomial theorem: $(x+y)^n = x^n + {}^{n}C_{1}x^{n-1}y^1 + \ldots {}^{n}C_{r}x^{n-r}y^r + \ldots y^n$ Write down the rule for the r th term.	r th term $= {}^{n}C_{r}a^{n-r}b^r$
2. Substitute the values for x, y, n and r into the rule: $a = x$, $b = y$, $n = 7$ and $r = 4$.	${}^{n}C_{r}x^{n-r}y^r = {}^{7}C_{4}x^{7-4}y^4$
3. Simplify. *Note:* The 0th term corresponds to the first element of the expansion.	$= 35x^3y^4$
4. Answer the question.	The 4th term is equal to $35x^3y^4$.

1.7.2 Pigeon-hole principle

Henri Poincaré, a famous mathematician, once described mathematics as 'the art of giving the same name to different things'. Consider three phenomena, which on the surface appear different — population growth, the value of investments and radioactive decay. Each can be described by one mathematical concept: exponential change. The mathematician gives three seemingly different things the same name.

The **pigeon-hole principle** is a good example of how mathematics gives the same name to different things.

The pigeon-hole principle:
If there are $(n + 1)$ pigeons to be placed in n pigeon-holes, then there is at least one pigeon-hole with at least two pigeons in it.

Notes:
- Note the precise use of language in this statement, in particular the importance of the phrase 'at least'.
- Some may view the pigeon-hole principle as an obvious statement, but used cleverly it is a powerful problem-solving tool.

WORKED EXAMPLE 26

In a group of 13 people show that there are at least 2 whose birthday falls in the same month.

THINK	WRITE
1. Think of each person as a pigeon and each month as a pigeon-hole.	There are 12 months and 13 people.
2. If there are 13 pigeons to be placed in 12 holes at least one hole must contain at least two pigeons.	Using the pigeon-hole principle: 13 people to be assigned to 12 months. At least one month must contain at least two people. That is, at least two people have birthdays falling in the same month.

Generalised pigeon-hole principle:
If there are $(nk + 1)$ pigeons to be placed in n pigeon-holes, then there is at least one pigeon-hole with at least $(k + 1)$ pigeons in it.

WORKED EXAMPLE 27

In a group of 37 people show that there are at least 4 whose birthdays lie in the same month.

THINK	WRITE
1. Think of each person as a pigeon and each month as a pigeon-hole.	There are 12 months and 37 people.
2. Recall the generalised pigeon-hole principle.	Using the generalised pigeon-hole principle: 37 people to be assigned to 12 months.

3. $(nk + 1)$ pigeons to be allocated to n holes; $n = 12 \rightarrow k = 3$	The value of n is 12 and k is 3. So at least one month has at least $(k + 1)$ or 4 people in it. That is, at least 4 people have birthdays falling in the same month.

WORKED EXAMPLE 28

On resuming school after the Christmas vacation, many of the 22 teachers of Eastern High School exchanged handshakes. Mr Yisit, the Social Science teacher, said, 'Isn't that unusual — with all the handshaking, no two people shook hands the same number of times'.

Not wanting to spoil the fun, the Mathematics teacher, Mrs Pigeon, said respectfully, 'I am afraid you must have counted incorrectly. What you say is not possible'

How can Mrs Pigeon make this statement?

THINK	WRITE
1. Think of the possible number of handshakes by a person as a pigeon-hole.	For each person there are 22 possible numbers of handshakes; that is, 0 to 21.
2. If two or more people have 0 handshakes, the problem is solved. Consider the cases where there is 1 person with 0 handshakes or 0 persons with 0 handshakes.	1 person with 0 handshakes: If there is 1 person with 0 handshakes, there can be no person with 21 handshakes. Thus, there are 21 people to be assigned to 20 pigeon-holes. Therefore, there must be at least one pigeon-hole with at least two people in it. 0 people with 0 handshakes: If there is no person with 0 handshakes, there are 22 people to be assigned to 21 pigeon-holes. Therefore, there must be at least one pigeon-hole with at least two people in it (at least two people have made the same number of handshakes).
3. Conclude using a sentence.	Thus, there are at least two people who have made the same number of handshakes.

studyon

Units 1 & 2 > Area 1 > Sequence 1 > Concepts 7 & 8

Pascal's triangle Summary screen and practice questions
The pigeon-hole principle Summary screen and practice questions

Exercise 1.7 Pascal's triangle and the pigeon-hole principle

Technology free

1. Write the first 8 rows in Pascal's triangle.
2. **WE22** Refer to Pascal's triangle on page 45 and answer the following questions.
   a. What number is in the 4th position in the 8th row?
   b. Complete the 9th row in Pascal's triangle.
   c. If 9 and 36 occur side by side in the 9th row, what element in the 10th row occurs below and in between these numbers?

Technology active

3. **WE23** Use combinations to:
   a. calculate the number in the 7th position of the 8th row of Pascal's triangle
   b. calculate the number in the 9th position of the 12th row of Pascal's triangle
   c. generate the 10th row of Pascal's triangle.
4. **WE24** Use the binomial theorem to expand:
   a. $(x + y)^2$ b. $(n + m)^3$ c. $(a + 3)^4$
5. **WE25** a. What is the 4th term in the expansion of $(x + 2)^5$?
   b. What is the 3rd term in the expansion of $(p + q)^8$?
   c. What is the 7th term in the expansion of $(x + 2)^9$?
6. **MC** A row of Pascal's triangle is given below. What number is located at position x?

$$\begin{matrix} & & & & & x \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \end{matrix}$$

   A. 8 B. 28 C. 45 D. 120
7. **MC** $16x^3$ is definitely a term in the binomial expansion of:
   A. $(x + 2)^3$ B. $(x + 4)^3$ C. $(x + 2)^4$ D. $(x + 4)^4$
8. a. In Pascal's triangle, calculate the sum of all elements in the:
   i. 0th row ii. 1st row iii. 2nd row
   iv. 3rd row v. 4th row vi. 5th row
   b. i. What do you notice?
   ii. Complete the statement: 'The sum of the elements in the nth row of Pascal's triangle is ...'
9. Use the result from question 8 to deduce a simple way of calculating:

$${}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$$

10. **WE26** In a cricket team consisting of 11 players, show that there are at least 2 whose phone numbers have the same last digit.

11. **WE27** A squad of 10 netballers is asked to nominate when they can attend training. They can choose Tuesday only, Thursday only or Tuesday and Thursday. Show that there is at least one group of at least 3 players who agree with one of these options.
12. J&L lollies come in five great colours — green, red, brown, yellow and blue. How many J&Ls do I need to select to be sure I have 6 of the same colour?

13. The new model WBM roadster comes in burgundy, blue or yellow with white or black trim. That is, the vehicle can be burgundy with white or burgundy with black and so on. How many vehicles need to be chosen to ensure at least 3 have the same colour combination?
14. Is it possible to show that in a group of 13 people, there are at least 2 whose birthdays fall in February?
15. **WE28** Nineteen netball teams entered the annual state championships. However, it rained frequently and not all games were completed. No team played the same team more than once. Mrs Organisit complained that the carnival was ruined and that no two teams had played the same number of games. Show that she is incorrect in at least part of her statement and that at least two teams played the same number of games.

16. Prove that in any group of 6 people either at least 3 are mutual friends or at least 3 are strangers.

1.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. a. State the inclusion–exclusion principle for 3 sets, R, S and T.
   b. Use the inclusion–exclusion principle to calculate the number of cards in a deck of 52 cards that are either red, a jack or a court card (king, queen or jack).
2. a. State the multiplication principle.
   b. One or two letters are to be chosen from the letters A, B, C, D, E and F. In how many different ways can this be done without replacement, if order is important?
3. a. State the definition of ${}^{n}P_{r}$.
   b. Without a calculator, compute the value of ${}^{10}P_{3}$.
   c. Prove that ${}^{n}P_{r} = \frac{n!}{(n-r)!}$.
4. A free-style snowboard competition has 15 entrants. In how many ways can the first, second and third places be filled? You may wish to use technology to answer this question.
5. a. Suppose 5 people are to be seated. Explain why there are fewer ways of seating 5 people at a circular table compared with seating the group on a straight bench.
   b. How many ways can 5 people be seated at a round table?
6. The main cricket ground in Brisbane is called the Gabba. It is short for Woolloongabba. How many different arrangements of letters can be made from the word WOOLLOONGABBA? You may wish to use technology to answer the question.
7. Apply the concept of ${}^{n}C_{r}$ to calculate the number of ways 12 different ingredients can be chosen from a box of 30 different ingredients. What can you conclude about the ingredients left behind? Do not use algebra to explain this.
8. A committee of 5 men and 5 women is to be chosen from 8 men and 9 women. In how many ways can this be done?
9. A netball team of 7 players is to be chosen from a squad of 11 players. Suppose any squad member can play any position. In how many ways can this be done:
   a. if each player is chosen to play a particular position
   b. If players have no particular position?
   You may choose to use technology to answer this question.

10. A ward in a city hospital has 15 nurses due to work Friday. There are 3 shifts needed to be staffed by 5 nurses on each shift. How many ways can this be done assuming each nurse works only one shift? (The order of the nurses on each shift and the order of each shift are not important.)

11. a. In your own words explain how to construct Pascal's triangle.
    b. Where in Pascal's triangle would you find the number 56, which is equal to ${}^{8}C_{3}$?

12. a. Give an example of the pigeon-hole principle.
    b. Jock has black, blue and beige socks. How many socks does he need to have to be sure that at least one pair are the same colour? Justify your answer.

Complex familiar

13. Assuming that car number plates are sequenced as follows: DLV334 → DLV335 → … DLV999 → DLW000 and so on. Using this sequence, how many number plates are there between DLV334 and DNU211 inclusive?

14. Jane, Laura and Steff are in a netball squad of 13 players. The team only takes 9 players to games. Jane, Laura and Steff's parents share the driving if they can: if 2 or more of the girls are playing, one parent drives them to the game and another parent picks them up. The parents do not share the driving if only one of the girls is playing. For what proportion of games would you predict the girls' parents are able to share the driving?

15. a. Show algebraically that ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r-1}$.
    b. Interpret this result in terms of Pascal's triangle.

16. Consider the expansion $(x + y + z)^5 = x^5 + 5x^4y + 5x^4z + \ldots$
    Use your knowledge of permutations involving identical objects to determine the coefficient of the term x^2yz^2. Recognize that this term could be written as *xxyzz*, *xzyzx* and so on.

Complex unfamiliar

17. a. A school uses identification (ID) cards that consist of two letters from A to D followed by 3 digits chosen from 0 to 9. Each digit may be repeated but letters cannot be repeated. If the school receives about 800 new students each year, after how many years will they run out of unique ID numbers?
    b. For a scene in a movie, five boy–girl couples are needed. If they are to be selected from 10 boys and 12 girls, in how many ways can this be done? (Assume the order of the couples does not matter.)

18. Juan has a bike lock with a 4-digit key but has forgotten the code. He knows that he used either the year he bought the bike, 2016, or the current year, 2019. Also, he jumbled the digits, so 2016 and 2019 are not possible keys. How many different possible keys are there to unlock the bike?

19. a. Consider the first three rows of Pascal's triangle.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

    The sum of the entries of the first three rows of Pascal's triangle is 7 (= 1 + 1 + 1 + 1 + 2 + 1). How many rows of the triangle are needed for the sum to exceed 20 000? Justify your result.
    b. The inclusion–exclusion principle for three sets *R*, *S* and *T* states:
    $n(R \cup S \cup T) = n(R) + n(S) + n(T) - n(R \cap S) - n(R \cap T) - n(S \cap T) + n(R \cap S \cap T)$
    Generalise this principle to four sets, *Q*, *R*, *S* and *T*.

20. A plane is covered in points at 1-unit spacing. Each point on the plane is coloured red, blue or white. Show there are three points of the same colour at a maximum distance $2\sqrt{2}$ from each other.

studyon

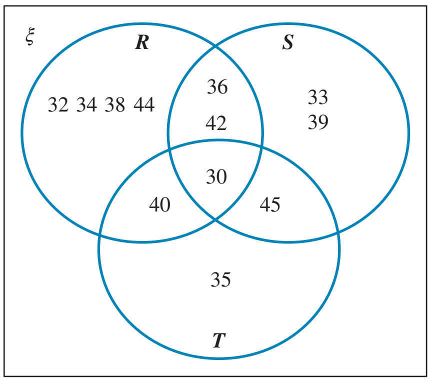
Units 1 & 2 Sit chapter test

Answers

Chapter 1 Permutations and combinations

Exercise 1.2 Counting techniques

1. a.



   b. $n(R \cup S \cup T) = 12$
   c. $n(R \cup S \cup T) = 12$
2. n(red, even or 4) = 36
3. 55 students
4. a. AB BA CA
   AC BC CB
   b. 6 c. $\frac{1}{3}$
5. BG GB YB RB
   BY GY YG RG
   BR GR YR RY
6. ACB BAC CAB
   ABC BCA CBA
7. a. 42 b. 210 c. 840 d. 2520 e. $\frac{1}{7}$
8. a. 24 b. 6 c. 12 d. 24
9. a. 49 b. 252
10. 126
11. a. 200 b. 40 c. 50 d. 290 e. $\frac{1}{40}$
12. a. 13 230
    b. 17 640
    Jake may wear 13 230 outfits with a jacket or 4410 outfits without a jacket. Therefore he has a total of 17 640 outfits to choose from. The assumption made with this problem is that no item of clothing is exactly the same; that is, none of the 7 shirts are exactly the same.
13. C
14. D
15. 100
16. 6
17. 48
18. 256
19. 1080
20. a. 1000
    b. 27
    c.

271	371	471
272	372	472
273	373	473
281	381	481
282	382	482
283	383	483
291	391	491
292	392	492
293	393	493

Exercise 1.3 Factorials and permutations

1. a. $4 \times 3 \times 2 \times 1$
   b. $5 \times 4 \times 3 \times 2 \times 1$
   c. $6 \times 5 \times 4 \times 3 \times 2 \times 1$
   d. $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
2. a. 3024 b. 151 200 c. 840 d. 720
3. a. $n(n-1)(n-2)(n-3)(n-4)$
   b. $(n+3)(n+2)$
   c. $\dfrac{1}{n(n-1)(n-2)}$
   d. $\dfrac{1}{(n+2)(n+1)n(n-1)}$
4. a. 24 b. 120
   c. 720 d. 3 628 800
   e. $8.717\,829\,12 \times 10^{10}$ f. 362 880
   g. 5040 h. 6
5. a. $8 \times 7 = 56$
   b. $7 \times 6 \times 5 \times 4 \times 3 = 2520$
   c. $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 40\,320$
6. a. $\dfrac{9!}{3!} = 60\,480$ b. $\dfrac{5!}{3!} = 20$ c. $\dfrac{18!}{13!} = 1\,028\,160$
7. a. 27 907 200 b. 639 200 c. 1 028 160
8. a. 56 b. $\frac{1}{8}$
9. 358 800
10. 120
11. 3024
12. 2184
13. a. $\frac{^{15}P_4}{4}$ b. 8190 c. $\frac{1}{16}$
14. 3360
15. 362 880
16. 479 001 600
17. D
18. D
19. a. $^5P_3 = 60$ b. $^5P_4 = 120$ c. $^5P_5 = 120$

Exercise 1.4 Permutations with restrictions

1. a. $^6P_6 = 720$ b. $\frac{^6P_6}{2} = 360$
2. 83 160
3. 10
4. 1260

5. 27 720
6. 1 307 504
7. a. 5.45×10^{10} b. 3.63×10^{9} c. 4.00×10^{10} d. 1.45×10^{10}
8. a. 120 b. 20 c. 60 d. 60
9. a. 30 240 b. 3024 c. 6720 d. 15 120
10. a. 120 b. 48 c. 72 d. $\frac{3}{5}$
11. a. 1680 b. 180 c. 360 d. 840
12. a. 1320 b. 110
13. a. 80 640 b. 282 240 c. 119 750 440 d. $\frac{1}{6}$
14. B
15. C
16. a. 720 b. 24, OYSTER

Exercise 1.5 Combinations

1. a. $\frac{^8P_3}{3!}$ b. $\frac{^{19}P_2}{2!}$ c. $\frac{^1P_1}{1!}$ d. $\frac{^5P_0}{0!}$
2. a. $^8C_2$ b. $^9C_3$ c. $^8C_0$ d. $^{10}C_4$
3. a. 1 b. 20 c. 120 d. 220
4. 100
5. a. 28 b. 120
6. D
7. C
8. 1365
9. 252
10. a. 495 b. 11 c. 1 d. 54 264 e. 100 f. 680
11. a. 2 598 960 b. 65 780 c. 65 780 d. 2 467 400
12. a. 2002 b. 840 c. 126 d. $\frac{36}{91}$
13. 59 400
14. 201 376
15. a. 120 b. 10 days
16. a. 57 b. 4 days 6 hours
17. a. 15 b. 1 day 4 hours
18. a. 8 145 060 b. 11 480
19. a. i. 220, 220 ii. 6435, 6435 iii. 10, 10 iv. 56, 56
    b. The value of ^nC_r is the same as $^nC_{n-r}$.

Exercise 1.6 Applications of permutations and combinations

1. a. 45 b. 120 c. 210 d. 105
2. a. 720 b. 10 c. 120 d. 2.4×10^{18}
3. 24
4. 24
5. a. 210 b. 126 c. 84 d. 140
6. a. 126 b. 56 c. 21 d. 70
7. D
8. a. 28
   b. Sample responses include: 9 17 27 41 12 14
   9 17 27 41 12 37 9 17 27 41 12 34
   c. 6
9. a. 84
   b. Sample responses include: 7 15 25 32 10 12
   7 15 25 32 10 35 7 15 25 32 10 37
   c. 10
10. 376 992
11. a. 8008 b. 5005 c. 5005 d. 4004
12. D
13. a. 38 760 b. 1140 c. 34 220 d. 39 010 800
14. a. 1
    b. i. 38 760 ii. 50 063 860 iii. 1 940 475 213 600 iv. 4 191 844 505 805 495

Exercise 1.7 — Pascal's triangle, and the pigeon-hole principle

1.

Row	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1

2. a. 70
   b. 1 9 36 84 126 126 84 36 9 1
   c. 45
3. a. 8
   b. 220
   c.

$^{10}C_0$	$^{10}C_1$	$^{10}C_2$	$^{10}C_3$	$^{10}C_4$	$^{10}C_5$	$^{10}C_6$	$^{10}C_7$	$^{10}C_8$	$^{10}C_9$	$^{10}C_{10}$
1	10	45	120	210	252	210	120	45	10	1

4. a. $x^2 + 2xy + y^2$

b. $n^3 + 3n^2m + 3nm^2 + m^3$

c. $a^4 + 12a^3 + 54a^2 + 108a + 81$

5. a. $80x$ **b.** $56p^5q^3$ **c.** $4608x^2$

6. B

7. D

8. a. i. 1 **ii.** 2 **iii.** 4

iv. 8 **v.** 16 **vi.** 32

b. i. The sum of the elements in each row of Pascal's triangle is a power of 2:

Row	Sum
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$
5	$2^5 = 32$

ii. The sum of the elements in the nth row of Pascal's triangle is 2^n.

9. $2^6 = 64$

10, 11. Sample responses can be found in the worked solutions in the online resources.

12. 26

13. 13

14–16. Sample responses can be found in the worked solutions in the online resources.

1.8 Review: exam practice

1. a. $n(R \cup S \cup T) = n(R) + n(S) + n(T) - n(R \cap S) - n(R \cap T) - n(S \cap T) + n(R \cap S \cap T)$

b. n (red, court card or jack) = 32

2. a. If there are n ways of performing operation A and m ways of performing operation B, then there are $n \times m$ ways of performing A *and* B in the order AB.

b. 64 ways

3. a. ${}^n\mathrm{P}_r$ is the number of ways of choosing r objects from n distinct things, when order is important. ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$

b. ${}^{10}\mathrm{P}_3 = 720$

c.
$$\begin{aligned} {}^n\mathrm{P}_r &= n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \\ &= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

4. ${}^{15}\mathrm{P}_3 = 2730$

5. a. When people are sitting in a circle, we cannot tell the difference between arrangements such as between $\{A, B, C, D, E\}$ and $\{C, D, E, A, B\}$. In a circle, these represent the same arrangement. Therefore, there are fewer ways to arrange people in a circle than in a straight line.

b. 24

6. 3 243 150

7. ${}^{30}\mathrm{C}_{12} = {}^{30}\mathrm{C}_{30-12} = {}^{30}\mathrm{C}_{18} = 86\,493\,225$
Each separate time we choose r things from n distinct things, we also leave $n - r$ objects. (We can interchange taking and leaving.) Hence, ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$

8. 7056

9. a. 1 663 200 **b.** 330

10. 756 756

11. a. In Pascal's triangle, the first two rows (the 0 row and the 1 row) contain only 1s. In subsequent rows, each row begins and ends with a 1, and each other term is the sum of the two terms above it. Row 2 is 1 2 1.

b. Row 8, position 3 (and 5)

12. a. Sample response: In a club with 367 members, there will be at least 2 people with the same birthday (assuming 366 days in a year for leap years).

b. 4 socks. Jock has socks of three different colours. If he selects 4 socks, which is one more than the number of types of socks he has, he would, by the pigeon-hole principle, have at least one matching pair.

13. 50 878

14. 540 out of 715

15. Sample responses can be found in the worked solutions in the online resources.

16. 30

17. a. 15 years **b.** 199 584

18. 46 possible combinations

19. a. 15 rows

b.
$$\begin{aligned} & n(Q \cup R \cup S \cup T) \\ &= n(Q) + n(R) + n(S) + n(T) - n(Q \cap R) - n(Q \cap s) \\ &-n(Q \cap T) - n(R \cap S) - n(R \cap T) - n(S \cap T) \\ &+ n(Q \cap R \cap S) + n(Q \cap R \cap T) + n(Q \cap S \cap T) \\ &+ n(R \cap S \cap T) - n(Q \cap R \cap S \cap T) \end{aligned}$$

20. Sample responses can be found in the worked solutions in the online resources.