CHAPTER 1

Understanding the Relationship between Math and Art

> *After a certain high level of technical skill is achieved, science and art tend to coalesce in esthetics, plasticity, and form.* —ALBERT EINSTEIN (1879–1955), SCIENTIST AND ARTIST

Math and fine art (or simply art) are two of the most interesting subjects to explore. However, their relationship may seem paradoxical. Unlike the bond of math and science, the association between math and art is not so obvious. The average person might have difficulty seeing any direct connection between math and art because of their distinct content and methodologies. Instead, the strict logic used in math and the emotional expressions portrayed in art might make the two subjects seem like polar opposites. Yet some experts think otherwise.

So, what is the truth? What are the differences and similarities between math and art? Are they really linked in any sense? If yes, how do they affect and help each other? This chapter provides answers to these questions. In order to lay the foundation for later introducing a new method of learning art, this chapter also focuses on the following topics:

- **Ways that math can be used to help us learn about art**
- **Existing drawing methods and their connections with math**
- **Reasons that separating math from art is a serious mistake**

Understanding How Math and Art Differ

Art and math are indeed very different in at least the following two aspects:

- In terms of content, math is an abstraction of patterns, whereas art is a visual product that provides aesthetic value.
- In terms of methodology, we must process everything in math sequentially by strictly following logic and proven rules. In art, the way we create and learn is subjective and empirical.

Differences in Content

Math and art exhibit distinct differences in their content. Math is the study of numbers, properties, and the relationships of quantities. Either a research paper or a student exercise (later referred to as "math work") tries to show that some quantity patterns exist in a particular problem. Although math is abstract, it comes from and applies to almost everything: science, technology, economics, and even art.

> *Mathematics and Culture I*, edited by Michele Emmer (Springer-Verlag, 2004), details some of these interconnections between math and other disciplines.

Art is a visual product that displays graphics appealing to an audience and the artist. An artwork may originate from real scenes, but it is most influenced by the artist's emotions and senses.

A math work involves a sequence of logic operations that delivers results by using theories, symbols, and established procedures. Math concepts may be fully abstract and imaginary, as they are not necessarily associated with any real objects in the physical world. Such abstractness allows math to be used as a language to describe and solve problems in science and engineering. Math integrates a strong sense of correct or incorrect. Its content is objective—that is, a math work has the same meaning across all cultures at all times. The purpose of math work is either to develop a new theory for future use or to solve a problem based on existing theories. In math circles, a new math research article is accepted only if it presents new results and is proven to be correct by experts. Otherwise, showing the article to the public would be almost meaningless, even if the work were done by a famous mathematician.

In contrast, a piece of artwork, either two-dimensional (2D) or three-dimensional (3D), presents visual structures that express an artist's thoughts or represent objects the artist observed or imagined. By using shapes, colors, textures, and patterns, artists intend to communicate with viewers in a graphic language that is easy to understand. Artwork is subjective. Generally speaking, the content carries no sense of right or wrong, provided that the artwork does not go against its cultural standard. Any work completed by an artist can be exhibited to the public as long as the artist is satisfied with the product. Visitors in an art gallery can be quickly drawn into imaginary worlds presented by artists. These viewers have emotional reactions to a painting or object based on their individual experiences that they associate with the contents of the artwork. Artwork will never provide everyone with the same experience because its meaning is rooted deeply in each viewer's culture and personal history.

Differences in Methodologies

The methodologies used in math and art are quite different too. In math, every step has to be processed sequentially by following logic and proven theories without emotion. Although mathematicians may use their wildest imaginations when solving a problem or coming up with a new concept, all of their thoughts must rely on reasoning and proven results. After a problem is solved or a new theory is conceived, the outcome then becomes a stepping-stone for future work, and the best approach to achieve such results may be standardized for others' use. This procedural mechanism makes math easy to learn.

For a math problem, even when there are multiple ways to solve it, the correct answer will be the same no matter which method you use. The key to learning math is to understand its theories and related processes. A math work is about definitions, principles, and logic operations. After the process is understood, anyone can duplicate the entire work perfectly without errors—but without understanding, you could easily get stuck at any step. That is why most of us can master basic math knowledge and operations in school.

Figure 1.1 shows a simple algebra problem—solving a quadratic equation—that can be worked out in two ways. The first method turns the equation into a special form so that the square root can be removed in order to solve the problem. The process requires careful manipulation of the numbers. A mistake at any point would result in a wrong answer. However, mathematicians have already articulated these tedious steps and come up with a standard formula for solving any quadratic equation. The second method involves applying that formula directly, resulting in fewer steps and therefore less hassle.

Solve $x^2 + x - 1 = 0$

Method #1:

$$x^2 + x + \frac{1}{4} - \frac{5}{4} = 0$$

$$(x + \frac{1}{2})^2 = \frac{5}{4}$$

$$x + \frac{1}{2} = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$$

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

Method #2 (using formula):

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

Figure 1.1

An example of a math problem solved in two ways

In contrast, the method we use to study and create art is subjective and experimental. The process of creating art is affected by the artist's opinions on the topics and that individual's skills in using particular art tools; thus, artwork is hard to reproduce exactly. Although students are taught theories and principles in art class, in order to turn the theories into real skills, students have to spend most of their time doing hands-on exercises to apply those theories and to master the actual skills of using pens, brushes, and other media. Because of human physical limitations, no one can draw a perfect line or circle, let alone a complicated object, even if the drawing principles are understood. Figure 1.2 shows an observational drawing exercise. Although it is quite simple, drawing the object perfectly is extremely hard, even for experienced artists. In creating a piece of art, artists rely more on emotion, incidental decisions, and skill than theories to lay out graphic components.

Figure 1.2

An example of a basic drawing exercise

Unless duplicating an existing object, artists have a lot of freedom to manipulate contents by all available means. Because of artists' diverse approaches and the lack of standardized processes, it is tough for art students to acquire the methods by which other artists create masterpieces. It is almost impossible, even for the artists themselves, to reproduce their existing work exactly. Meanwhile, the lack of restrictions in creating content allows everyone to produce "artwork" without requiring any formal approval. Learning how to create art does take substantial practice, as it is a hands-on activity. Artists with various levels of skills can sit together to draw the same object and repeat such basic exercises again and again. You will never witness such phenomena when learning math, where you must master lower-level knowledge before learning the higher level. A math professional may never need to practice a basic math operation after it has been mastered.

The strange relationship between creativity and art skills contributes to the poor results of art education in our society. A good art skill does not guarantee a good job in creating artwork. To create an artwork, imagination is more important than skills. You may find some artworks in exhibitions are not different from a work done by a novice. For this reason, many students studying art emphasize expression and imagination more than basic hands-on skills. As a result, even though most adults have spent some time in art classes in elementary and secondary school, they still haven't truly mastered the basic skills.

Understanding How Math and Art Are Similar

Some experts claim that math and art are similar and profoundly connected despite their noticeable differences. For example, the Greek philosopher and scientist Aristotle

(384 BC–322 BC) said, "The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful."

Russian painter Wassily Kandinsky (1866–1944) claimed, "The final abstract expression of every art is number."

Many scholars today draw similar conclusions. In a February 2005 article in *Math Horizons* magazine, Carla Farsi et al. also pointed out that both math and art are creative and share beauty. Although art is traditionally considered a creative activity, the imagination evident in math can be easily proven by the massive production of research work published every year. Such abstract theories, not necessarily linked with real objects or phenomena, cannot be completed without innovative minds. A typical procedure used to develop a new math theory is first to predict a possible result and then provide a logical proof. Obtaining such theoretical outcomes normally needs both professional training and inventive thinking.

Mathematicians strongly agree that math procedures and results exhibited by clean formulas and nice patterns such as repetition, symmetry, uniformity, parallelism, and orthogonality are a matter of "beauty." Even math topics that appear chaotic may well have very structured and neat features. For instance, consider the mathematical constant pi (π), which indicates the ratio of any circle's circumference to its diameter. Usually represented as 3.14, π is actually an irrational number, which means that its decimal digits never end and never repeat in any pattern and that the value seems to be unpredictable. However, mathematicians have already proved that π can be expressed in various "beautiful" formulas, including the following *Madhava-Leibniz series*, which can be found in many math textbooks:

$$\pi = 4\,(1/1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \ldots)$$

Such phenomenon can also be seen in *probability theory*, which tells us various systematic and predicable outcomes about random events. The Gaussian distribution, nicknamed the *bell curve* based on its shape, is one good example of the certainty of uncertain numbers. The beauty of math is also used as a tool to detect possible mistakes by mathematicians because correct answers and processes tend to be neat and clean. Certainly, mathematicians enjoy such "abstract beauty," just as artists love the visual pleasure in their artwork.

GAUSSIAN DISTRIBUTION OR BELL CURVE

Gaussian distribution is a math model that describes random variables. German great mathematician Carl Friedrich Gauss discovered this distribution in 1809. Many things in nature, such as human ages and heights, are distributed in a bell curve; Gaussian distribution is considered the most important distribution model in statistics.

Math in Art

Math can be found in art in several areas. Artists may intentionally or unintentionally use special math knowledge in their artwork. In addition, math is about hidden patterns in almost everything, including visual and aesthetic features. Math embedded in art is a huge topic, but here I'll briefly show you some obvious examples.

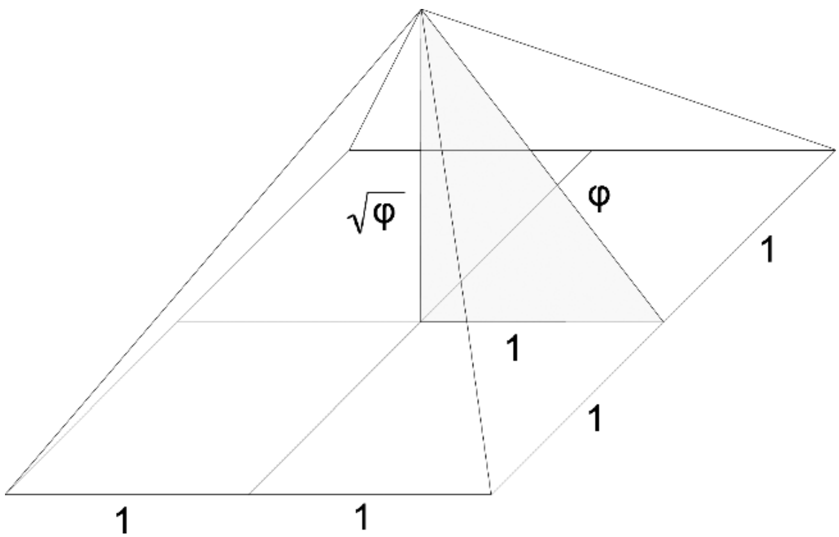
Math Numbers Used in Art

There are quite a few well-known irrational numbers, or *math constants*, that have unusual meanings: φ (≈1.61803), π (≈3.14159), e (≈2.71828), and so on. These numbers exist in calculations and formulas across a wide range of mathematics fields.

According to math history books, the ancient Egyptians and Greeks discovered the *golden ratio* (φ) and found that it generated pleasing results. This interesting number was used in building the Great Pyramids and other architectural wonders. Figure 1.3 shows the golden ratio embedded in the Egyptian Great Pyramid.

> **GOLDEN RATIO**
>
> Assume you have two numbers, a and b. They form a golden ratio if "a+b is to a as a is to b," i.e., (a+b)/a = a/b. This relationship holds only when a/b = φ (≈1.61803).

Figure 1.3
Golden ratio in the Great Pyramid



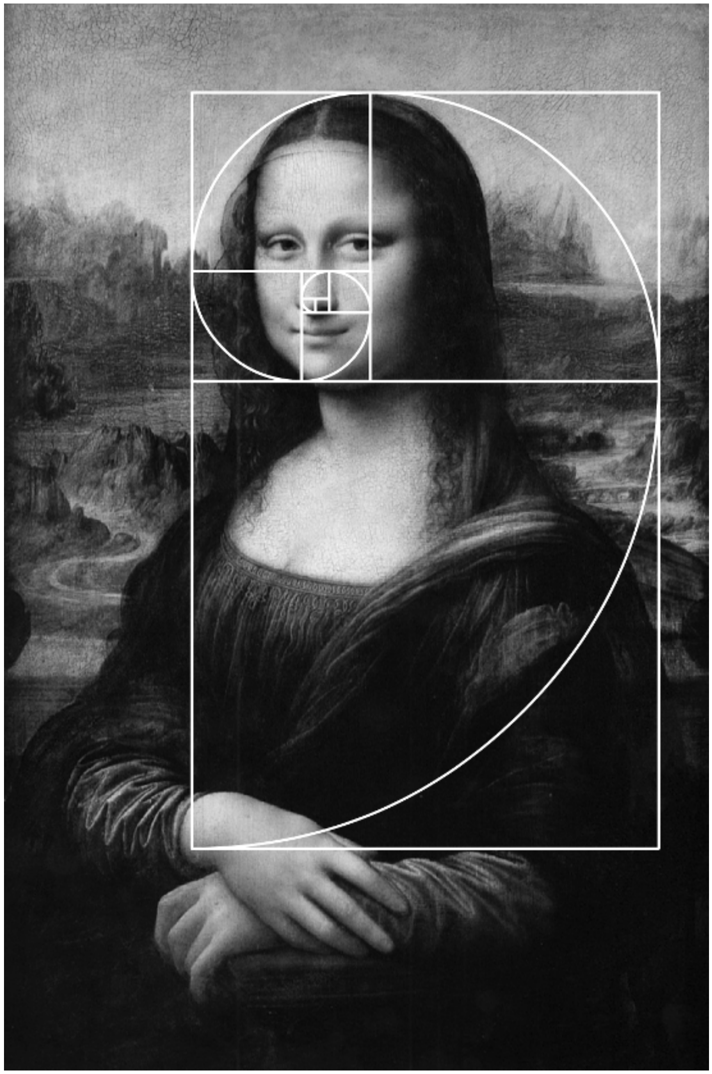
SOURCE: REPRODUCED BY THE AUTHOR FROM *SQUARING THE CIRCLE: GEOMETRY IN ART AND ARCHITECTURE* BY PAUL CALTER (WILEY, 2008).

The *Fibonacci sequence* demonstrates another excellent example of beauty in numbers. The sequence starts from 0 and 1. All other numbers are the sum of the previous two numbers: 0, 1, 1, 2, 3, 5, 8, 13…. The sequence has been proven to be intimately related to the golden ratio. The structures of trees, flowers, and even snail shells can be represented by Fibonacci numbers, which abstract a widely existing pattern in nature in general.

Many books, including *Lindenmayer Systems, Fractals, and Plants* by Przemyslaw Prusinkiewicz et al. (Springer, 1989), further explore Fibonacci numbers in the nature.

Some researchers have indicated that the Fibonacci sequence and associated Fibonacci spiral are hidden in the famous painting *Mona Lisa* by Leonardo da Vinci (1452–1511), shown in Figure 1.4. However, not everyone agrees with these findings because da Vinci did not describe such a configuration in his writings. Whether the suggestion is true or not, it is fairly certain that the golden ratio was widely used for composition in many masterpieces because the ratio provides visual harmony. For example, we can find such a "coincidence" in Michelangelo's painting *The creation of Adam*, in which the two fingertips partition the whole painting roughly at 1:1.618 in both horizontal and vertical directions, as indicated in Figure 1.5.

Figure 1.4
Fibonacci structures found in *Mona Lisa*

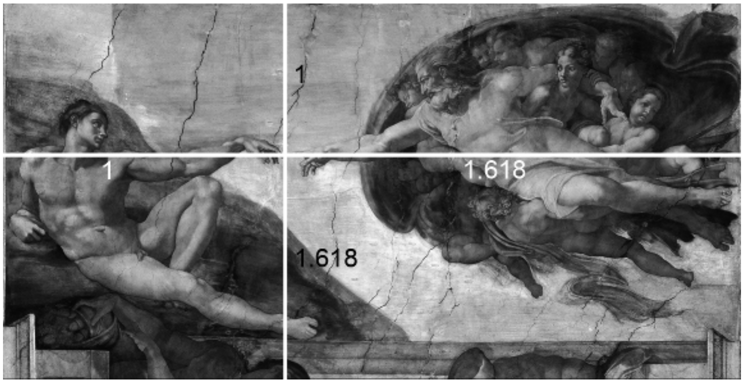


SOURCE: THIS IMAGE IS AVAILABLE AT `http://en.wikipedia.org/wiki/File:Mona_Lisa.jpg`.

Geometry Used in Art

Geometry is a branch of math that originated from measuring distances in ancient Egyptian land surveys. Over the past thousands of years, geometry has been used almost everywhere, from the sciences to art to our everyday lives. The fundamental concepts such as points, lines, curves, and polygons used in elementary geometry are automatically connected to visual art. Graphic patterns and structures contained in geometric shapes are of artistic value.

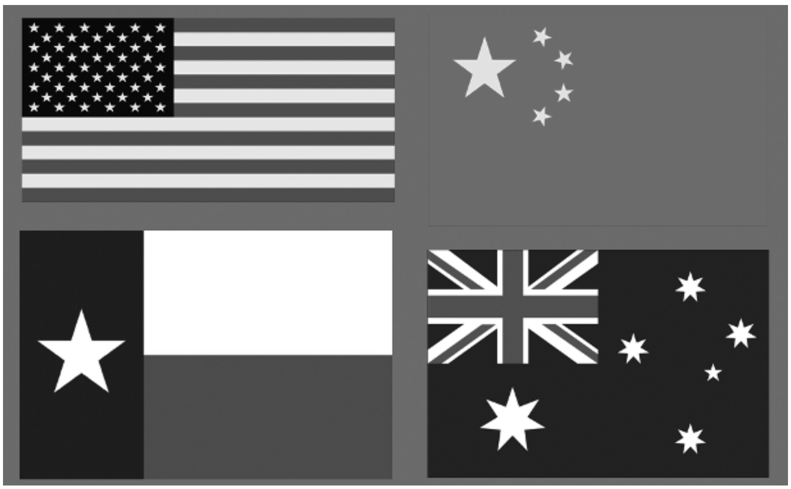
Figure 1.5
The golden ratio used for composition



SOURCE: THIS IMAGE IS AVAILABLE AT http://upload.wikimedia.org/wikipedia/commons/0/0c/Creation_of_Adam_Michelangelo.jpg.

Some special shapes, such as circles, squares, and triangles, are so precious that they serve as fundamental elements in art. For instance, a star is one of the most beautiful geometric shapes in graphic design, and it is used in many national and regional flags (see Figure 1.6).

Figure 1.6
Stars used in flag design



As the ancient Greek tragedian Euripides (480 BC–406 BC) said, “Mighty is geometry; joined with art, resistless.”

Artists have used geometry in various ways to create their artwork. You can easily find geometric patterns used explicitly in abstract artwork created by many great artists such as Wassily Kandinsky (1866–1944) and Pablo Picasso (1881–1973).

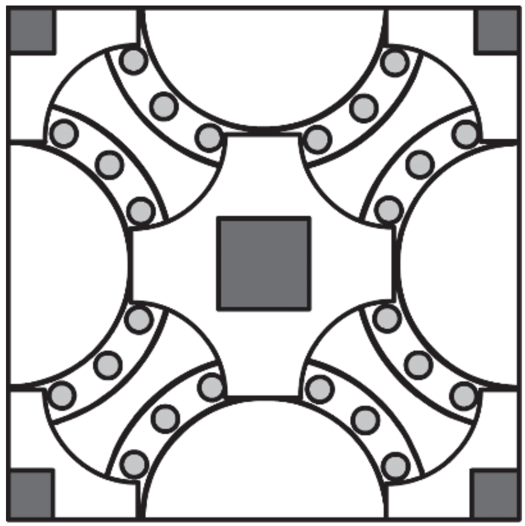
Dutch graphic artist M.C. Escher (1898–1972) is also well known for his elegant use of geometry in art. He used geometric elements and various styles of symmetries to create unique images that are fascinating to viewers. Some graphic configurations in his artwork are technically impossible in the physical world. Besides using well-designed graphic figures, he also applied math transformations to generate smooth transactions between different shapes.

MATH TRANSFORMATION

Loosely speaking, a math transformation is a math function that turns one image into a new look. Some popular and simple examples are translation, rotation, reflection, and scaling.

Although Escher was not a mathematician, he conducted his own math research for creating artwork, according to Doris Schattschneider in the June/July 2004 issue of the magazine *Notices of the American Mathematical Society*. Using math transforms is an efficient way to create interesting images. Typically, we can start with a simple pattern and then duplicate it with different rotations and scaling to obtain a new pattern. Repeating this process a couple of times will immediately generate a complicated geometric pattern. Figure 1.7 shows one of such examples.

Figure 1.7
A geometric pattern



Math Processes Used in Art

In observational drawing, artists usually simplify regular objects into spheres, cylinders, cones, cubes, or combinations of these basic forms. When sketching these geometric structures on paper, they need a careful consideration of the spatial relationship among the objects. These activities are actually forms of math. Figure 1.8 is a sample of an observational drawing in which the artist tried to make sure that every object has its correct location, orientation, and connection with the rest of the scene by performing a 3D analysis. Hidden lines and section curves are drawn to describe spatial features of objects. Parallel lines are sketched to confirm the perspective in the drawing.

Drawing realistically also requires a math operation for projecting an object from 3D space onto paper. This process, known as *perspective transform*, is the core of perspective theory. Officially established in the Renaissance, this theory became a fundamental principle that guides artists in drawing what they observe. You'll learn more details about this theory later in this chapter. For now, Figure 1.9 shows one famous example of the application of the theory: the oil painting *Last Supper* by da Vinci. The ceiling of the room is formed from angled lines that clearly create the illusion of a 3D perspective, as shown by the white lines.

Figure 1.8
3D structures in an observational drawing

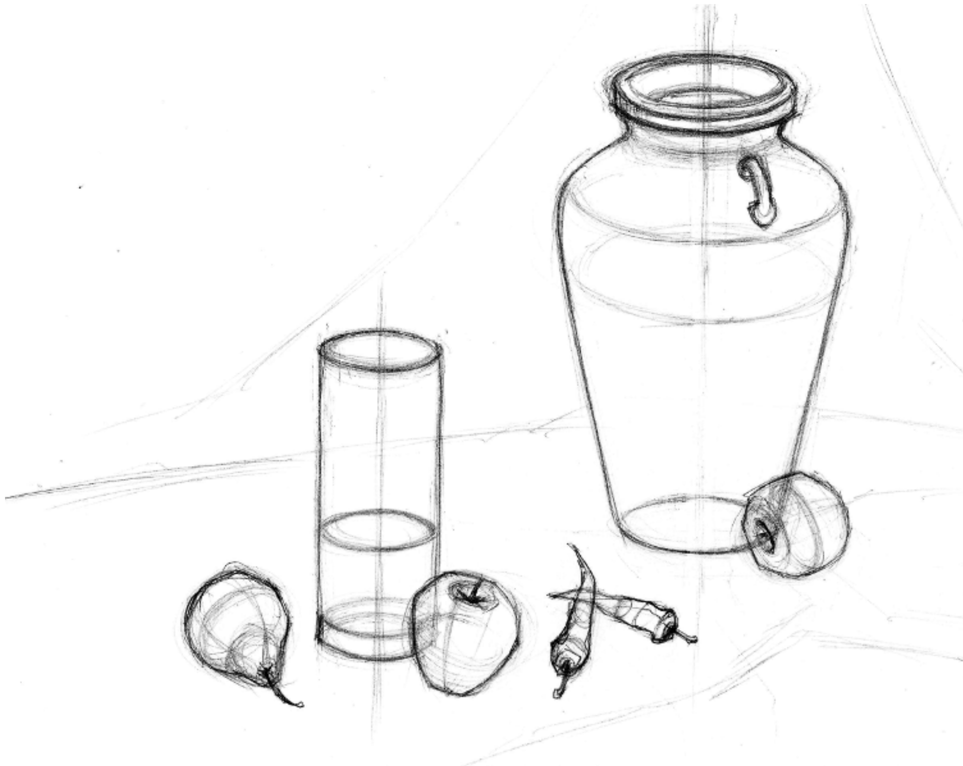
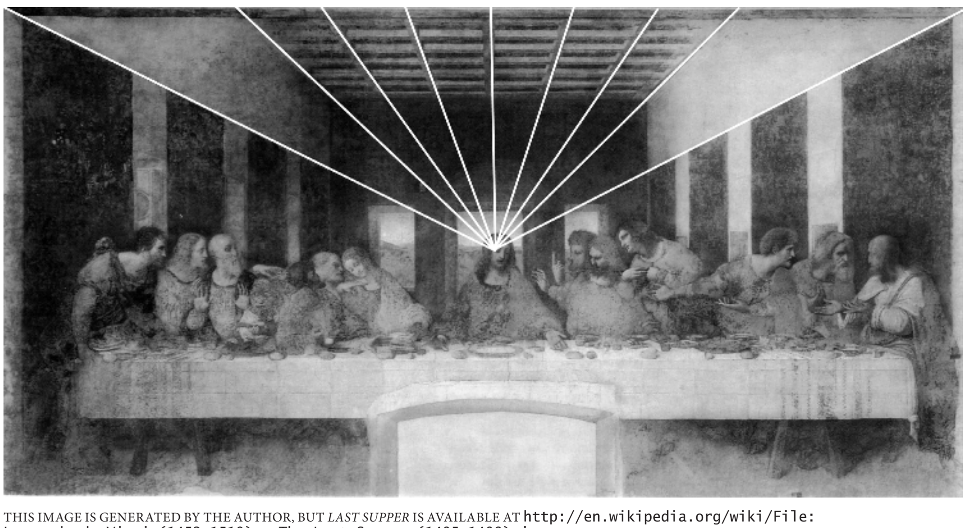


Figure 1.9
Perspective view in the *Last Supper*



This image is generated by the author, but *Last Supper* is available at http://en.wikipedia.org/wiki/File:Leonardo_da_Vinci_(1452-1519)_-_The_Last_Supper_(1495-1498).jpg.

Art in Math

Although generally most mathematicians are not artists, they often get inspiration from art in their research. To solve problems involved in multiple variables and unknowns, imagining and thinking visually are useful ways for a mathematician to "see" undiscovered patterns. At the same time, modern computers provide a powerful tool to visualize complicated data artistically.

Visual Assistance for Math Work

A math problem is usually very abstract, but it doesn't have to stay that way. Using visual assistance is common in math research. The proof of the Pythagorean theorem and the use of Bézier curves in computer graphics (CG) are good examples.

One of the most famous math discoveries, the Pythagorean theorem, was named for the Greek mathematician Pythagoras (570 BC–495 BC). The theorem unveils a "magical" pattern existing in all right triangles: the square of the length of the hypotenuse (the side opposite the right angle) is always equal to the sum of the squares of the lengths of the other two sides.

After its discovery, mathematicians found several neat ways to prove the theorem. One of them, shown in Figure 1.10, is a visual approach. By tiling four copies of a right triangle together to form a larger square, that square's area is the sum of the areas of those four triangles and the small square in the middle. This simple fact results in the theorem:

$$c^2 = 4 \times 0.5ab + (b - a)2$$

$$c^2 = a^2 + b^2$$

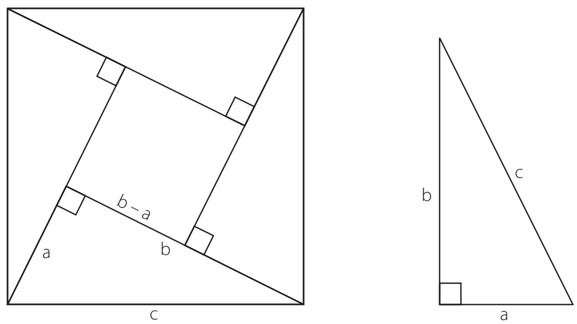


Figure 1.10

Graphic proof of the Pythagorean theorem

Bézier curves are a perfect example of the mixture of math and art. Named after French engineer and mathematician Pierre Bézier (1910–1999) for his popularizing of this method, Bézier curves have a beautiful graphical structure and can be generated efficiently in computers using following so-called subdivision method. The left image in Figure 1.11 demonstrates this method for a simple case of a cubic Bézier curve determined by four given points: P_0, P_1, P_2, and P_3. The curve is defined as the following math equation:

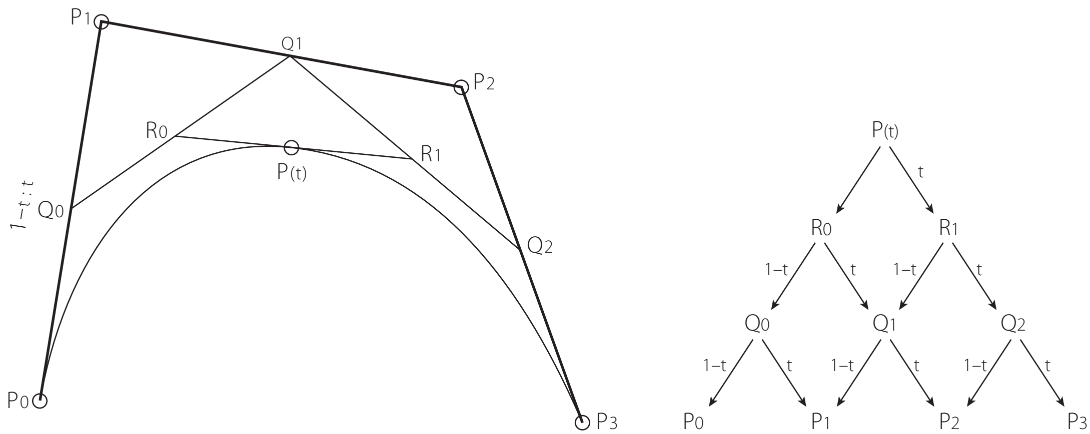
$$P(t) = (1 - t)^3P_0 + 3(1 - t)^2tP_1 + 3(1 - t)^2tP_2 + t^3P_3, \text{ for } 0 \le t \le 1.$$

P(t) represents the position of the curve at any particular t – an internal variable that describes the curve. For example, t=0 stands for the starting point, t=1 is the end point, and t=0.5 can be consider the middle point.

The previous math expression may appear scary to many artists, but its construction process is intuitive and stunningly simple. The pyramidal diagram on the right side of Figure 1.11 shows how the value P(t) is obtained step-by-step. For example, point $Q_0 = (1 - t)P_0 + tP_0$ is a weighted average of points P_0 and P_1, and therefore Q_0 is on the line segment P_0P_1. The rest of the vertices in the diagram can be found exactly the same way.

Because of their simplicity and elegance, Bézier curves serve as the industry standard to create curves in today's graphics software programs, such as Adobe Creative Suite (CS) and Autodesk Maya.

Figure 1.11
Bézier curve construction (left) and vertex dependency (right)



Visualization of Math

In computer technology, color is represented numerically by red, green, and blue components. This allows us to turn an array of numbers into a full screen of colorful dots through visualization techniques. An application of modern CG, *computer visualization* is extremely useful in the sciences and engineering, in which massive numerical information needs to be explored. When numbers are associated with colors and shapes, they can be displayed or even animated onscreen for people to recognize hidden patterns and features. Map coloring and graphical presentations of weather data are some simple examples that are familiar to everyone. This process, although it does not involve emotion and can be reproduced exactly, does output some interesting digital artwork.

In math, we often work with various quantities such as tangents, derivatives, convergence rates, and so forth. This kind of numerical information can also be visualized, and the results are often far more astonishing than you can imagine. One well-known example is fractal images. The Mandelbrot set is one of the earliest such abstract pieces of "artwork" created by the famous mathematician Benoit Mandelbrot (1924–2010), the father of fractal geometry (see Figure 1.12). Today, fractals are viewed and enjoyed by the public as digital artwork rather than their original math formulas.

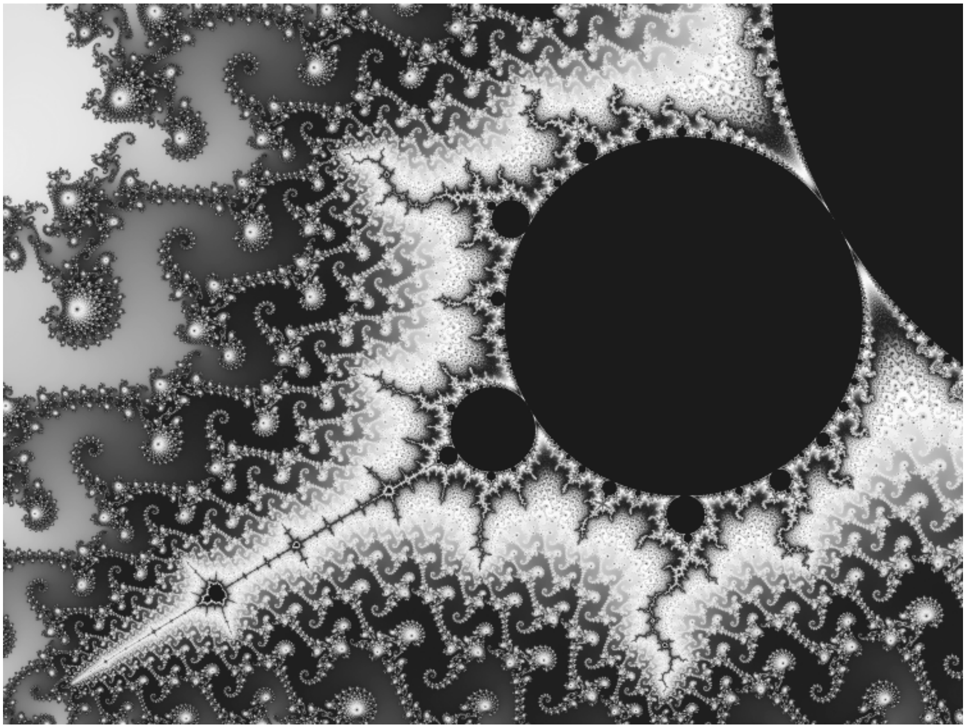


Figure 1.12
The Mandelbrot set

THIS IMAGE IS AVAILABLE AT http://commons.wikimedia.org/wiki/File:Mandel_zoom_08_satellite_antenna.jpg.

WHAT IS FRACTAL?

A fractal is a rough geometric shape that has a self-similarity; that is, the shape can be split into pieces, and each piece looks similar to the whole shape. In nature, trees, clouds, and mountains are perfect examples of fractals.

Using Math to Create Digital Art

The numerical representation of colors, shapes, and spaces makes math a powerful tool for generating artwork via computers. In the past several decades, the use of CG has expanded from military simulations to daily use in cartoons, movies, video games, and photo processing. With graphics software, artists are able to create an entire virtual environment and then simulate all physical properties and interactions between lights and objects.

This approach opened the door for artists to create digital artwork on the computer. In 1995, Pixar released the first computer-animated feature film, *Toy Story*, which inspired many to learn CG, and it had a huge impact in the improvement of animation production. Today, CG scientists are working day and night to invent new technologies and develop advanced algorithms for building faster and better systems. Their innovations, together with continuous improvements in computer hardware, have made graphics software the industry-standard tools for CG artists to create artwork efficiently. For instance, concept artists use Adobe Photoshop to draw and paint directly on computers, while animators create 3D models and animations via various software programs such as Autodesk Maya and 3ds Max and Pixologic's ZBrush. Motion editors manipulate video clips onscreen with Adobe After Effects.

As a result, film studios are using stunning computer-generated special effects. *Avatar*, a science-fiction film directed by James Cameron in 2009, lifted the visual-effects standard to another level with its 3D effects and its large-scale virtual world. The good news is that this revolution is just starting. Cinematic screens are becoming more like computer screens, on which moving pictures are not necessarily images of real objects but are created from "faked" digital characters and artificial props. The wonderful part of this revolution is that we are no longer restricted by any physical laws or scientific developments. Motion pictures are an arena for imagination—just like painting. You should expect more computer-generated, eye-popping digital effects in the future so that viewers can experience things that never actually happened in our universe.

While we enjoy these beautiful cinematic art products, most people don't realize how essential math is to this revolution. The technologies behind CG software are direct applications of mathematics theories. 3D objects are created with polygons and polynomial functions in an xyz-coordinate system. These concepts come from *geometry*, *algebra*, and *topology*—three major branches of math. Light-modeling is nothing more than math formulas that simulate the distribution of light energy. By changing the parameters of the models, we are able to replicate different materials such as stone and glass to make objects look real. Animating objects is a process of solving differential equations that describe the dynamic systems of the scene. The supporting math theories have been well studied in traditional physics.

Keep in mind, however, that, although math provides the radical and theoretical contributions for all these magical results, it is computer scientists who develop the data structures and algorithms and who implement graphics software on top of the mathematics theories. Artists are the last ones in this assembly line to finalize the imagery products.

Although software programs have brought us new media and tools for art creation, artists still may choose to mimic traditional art styles in digital space. *Non-photorealistic rendering (NPR)*, a branch of CG, is dedicated to this special type of computer art. In order to create conventional artwork numerically, researchers study the mathematics principles involved in the production process, such as pen movements and physical behaviors, media properties, or even an entire art-creation process from sketching to rendering. Although these technologies are not mature at this moment, they enable us to have fun in creating digital watercolor or oil paintings very conveniently.

Figure 1.13 is an oil painting generated by an NPR system called IMPaSto developed by William Baxter et al. at the University of North Carolina at Chapel Hill in 2004. It seems that most of the traditional styles will be simulated fully by computers sooner or later, and NPR has a huge potential to create many new art styles in the future. You can visit the home page of the International Symposium on Non-Photorealistic Animation and Rendering at `www.npar.org` to check out the latest technical developments in this field.

Figure 1.13

A painting using IMPaSto

SOURCE: ©2004 WILLIAM V. BAXTER III

Using Math in Traditional Drawing

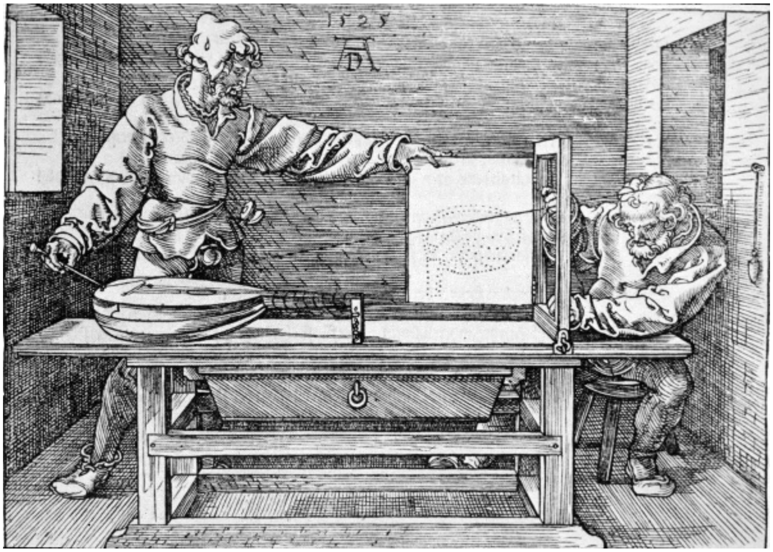
Art is a visual communication language through which artists express their emotions. In order to do that, several art skills are required: the artist must be able to draw, master color theory, use media techniques, and be capable of artistic expression. Among these skills, drawing ability is the most essential, and it serves as the foundation of art. Just as words and grammar are critical to writing, drawing is the first skill that needs to be mastered before other techniques can be acquired.

Technically, drawing skill, sometimes referred to as *draftsmanship*, is the ability to handle lines, shapes, and gray values on 2D surfaces so that the resulting picture resembles what the artist desired or observed. In the past 200 years, especially after the invention of photography, artists were no longer constrained to realistic styles in creating their work. Nonetheless, it is still a most desirable skill to be able to draw realistically. That is why observational drawing (that is, drawing what you see) remains a core class in the curriculum of today's professional art schools.

Perspective Theory

Artists have sought out techniques to master observational drawing over thousands of years. They wanted to draw objects realistically, but an invisible magic force complicated everything: all parallel lines in the real world were no longer parallel when viewed through human vision—circles became ovals, and the size of an object varied in a strange pattern corresponding to the distance between the object and the viewer. Unable to solve these puzzles, artists couldn't efficiently create artwork that precisely resembled what they observed. They even invented devices to investigate the hidden rules governing the view (see Figure 1.14).

Figure 1.14
An ancient drawing device



THIS IMAGE IS AVAILABLE AT http://en.wikipedia.org/wiki/File:Dürer_-_Man_Drawing_a_Lute.jpg.

Without solid principles, mastering drawing was a slow process, relying solely on practice. This frustration ended in the Renaissance when art and math were integrated more closely than ever before. According to Paul Calter's book *Squaring the Circle: Geometry in Art and Architecture* (Wiley, 2008), perspective theory, the theory of how 3D objects appear on 2D surfaces, was invented by the artists Filippo Brunelleschi (1377–1446) and Leon Alberti (1404–1472). The theory was further developed by several artists including da Vinci.

It shouldn't surprise you that both Brunelleschi and da Vinci were also mathematicians, because the core foundation of perspective drawing is based on similar triangles—a simple elementary geometry problem solved a long time ago. It is interesting to note that many artists prior to the Renaissance were not able to figure out such simple geometry problems in their creative process, probably because they lacked the basic math knowledge.

Perspective theory is a big milestone in art history. "Perspective is the rein and rudder of painting," da Vinci said. The theory solves the fundamental problem of observational drawing and tells us how to draw 3D objects realistically on paper—that is, it shows us the way to draw what we actually *see*.

Sketching an object in front of us is the equivalent of projecting the object onto an imaginary 2D plane called the *picture plane*, as shown on the left side of Figure 1.15. An alternative way of understanding this process is to consider the plane as a glass window between the object and our eyes. The scene we see through the window can be viewed as an image painted on the glass—that image is what we want to draw. A side view of this scenario of mapping an object to the picture plane is shown on the right side of Figure 1.15.

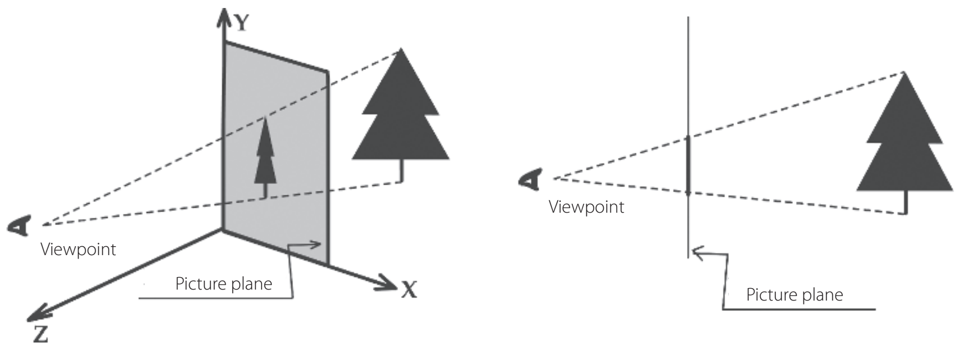


Figure 1.15

Picture plane front view (left) and side view (right)

The main idea behind perspective theory is not complicated at all. It consists of several rules based on the following key concepts:

- Horizon line or eye level
- Central vanishing point (CVP)
- Vanishing point left (VPL)
- Vanishing point right (VPR)
- Viewpoint
- Picture plane

Under a perspective projection, all straight lines in 3D space become straight lines in the picture plane. Parallel lines in a horizontal plane are no longer parallel. Instead, their extensions merge at some vanishing point on the horizon line. There are three commonly used perspective-drawing styles, depending on the number of vanishing points that are used, or equivalently, the way that an object is oriented with respect to the viewpoint. These perspective styles are as follows:

- One-point perspective
- Two-point perspective
- Three-point perspective

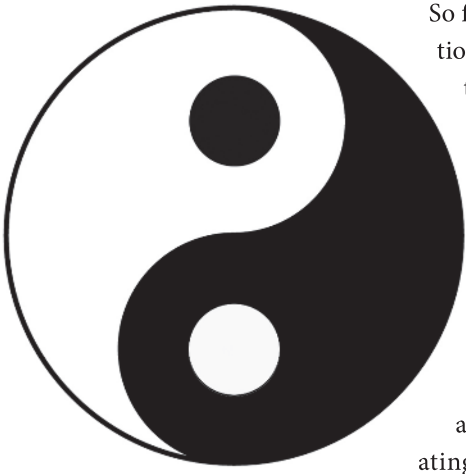
In-depth coverage of these styles is beyond the scope of this book. If you wish to learn further details, study the popular literature readily available.

Analytic Geometry

In 1637, French mathematician René Descartes (1596–1650) invented *analytic geometry*, also called coordinate geometry, which integrated geometry with algebra. Analytic geometry greatly increases our abilities to handle geometry problems. By using a coordinate system, spatial properties are turned into algebraic expressions so that we can use well-developed algebraic operations and theories to work through geometry challenges.

This intelligent treatment helps art too. Modern CG is probably the biggest beneficiary because its 3D modeling ability would be impossible without coordinate systems. The century-old *grid method* is a direct application of using coordinate systems for artistic purposes. Besides, analytic geometry also allows us to understand and further investigate the perspective theory with algebraic tools.

Figure 1.16
Yin and yang



So far, I have provided you with much evidence that shows the connection between math and art. This is just a small part of the whole picture that requires much more time to discover completely. Art vs. math is similar to *yin vs. yang*, an ancient Chinese philosophical concept describing a pair of things that are polar opposites yet interconnected. Figure 1.16 illustrates the interdependencies and autonomous aspects of their coexistence.

Though perspective theory gives us a solid principle to convert 3D objects into 2D data, is it all you need to master? If not, what else should you learn? The next section presents the skills you need to acquire in order to create art. From there, you will be able to see what theories are missing and what role math plays in creating art.

Using Math to Help You Draw

Mathematics is the queen of the sciences…

—Carl Friedrich Gauss (1777–1855), mathematician and scientist

Math penetrates into every corner of science, engineering, and technology. By using the language of math, we can explore hidden patterns and behaviors within a real problem concerned with quantities. In turn, the results usually make us not only understand the relationship of the quantities in this particular problem better but also solve the issues with guide of the math analysis. That is why we can barely find a scientific research article that does not use math. Mathematics is a powerful tool in pushing new technologies forward in almost all fields.

In traditional drawing, the situation seems to be quite different, even though we do see the influence of math in perspective theory, which is a core aspect of today's art training. However, this theory was developed about 600 years ago, and we haven't seen any noticeable new progress since then to make the theory more efficient or easier to use. Over the past several decades, computer technology has changed almost everything in our lives by providing new solutions or improving efficiency, but traditional drawing seems to lag behind as an exception to such advances. Does that mean perspective theory is perfect and powerful enough to solve all drawing issues? If the answer is yes, then why do we still have big problems in learning how to create art efficiently? Is it because math or science is really useless in art beyond perspective theory?

This section answers these questions by presenting a new look at what we should learn in art and how math can help us. Ideally, this new analysis will give you a much clearer picture of why math should be integrated with art in order to make learning how to draw easier and faster.

Three Levels of Drawing Ability

In order to learn drawing efficiently, you first need to know what skills you have to master and what kind of approaches are practical. Therefore, this section introduces some concepts that will help you understand the drawing process and enable you to easily see the problems that can arise in learning how to draw.

We often hear the term *drawing ability* when people are talking about art training. What *is* it? What components does it contain? Is it mainly the ability of controlling your hand movements or the ability to managing graphic elements with corresponding theories? Are they acquirable? How should we obtain them? If a person uses tools to help them draw, does that person still have real drawing skill?

To make these questions even clearer, we can take an analogous look at *math ability*. How do we learn math in school? Are we taught to make our brains run faster when processing numbers, or do we learn methods so that numerical processing can be performed more intelligently and efficiently?

If a student uses a calculator to successfully complete a complicated calculation, does that dramatically improve the student's math ability? No. Obviously, students learn *methods* to gain the ability to complete math operations. This is what math instructors believe math ability is supposed to be based on. Math teachers never focus on pushing a student's neurosystem to run faster or on teaching students to do math based on *feelings* or *senses* instead of principles.

Being able to use calculators is considered an insignificant part of math abilities, although calculators are widely used in American schools. Some educators, including me, suggest that calculators should not be used by kids in elementary school because these devices cannot directly improve calculating ability. Instead, overdependence on calculators weakens a person's ability to do basic math operations such as multiplication.

Similar to math ability, there are different levels of drawing skills:

Hand-Coordination Ability This is the ability to control the movement of the dominant hand without any assistance.

Theory-Guided Drawing Ability This represents the skillfulness with which a person can draw using techniques guided by some drawing theories, but not using any extra tools such as rulers or compasses.

Tool-Assisted Drawing Ability As the name indicates, this reflects how well a person can draw when using drafting tools or devices such as rulers or even projectors.

Hand-Coordination Ability

Hand-coordination ability (HCA) represents how well your hand “listens” to your brain. It should not surprise you that HCA overlaps with the concept of eye-hand coordination in neuroscience.

A simple way to test your HCA is to draw some basic figures such as circles or straight lines directly on paper *without* any assistance. It is a well-known fact that none of us can draw them *perfectly*, although we fully understand what these graphics are supposed to resemble. This test undoubtedly tells us that everyone’s HCA is limited, no matter how hard we try. This limitation can be much more noticeable when we draw large-sized figures.

You can improve HCA through practice. An obvious proof is the difference in coordination abilities between our two hands. After being trained for many years, the dominant hand of an adult can follow directions much better than the nondominant hand, even though both hands are controlled by the same brain.

Nevertheless, we can hardly push HCA beyond the bottleneck caused by physical limitations. Therefore, enhancing HCA should *not* be the major goal for art classes, because the HCA of an adult has *already* been established over many years. If improving HCA were to be a major focus, learning progress would be slow, and drawing skills would not improve significantly. We are better off teaching students to draw “smartly” instead of pushing their HCAs.

Theory-Guided Drawing Ability

Theory-guided drawing ability (TGDA) takes advantage of theories and corresponding techniques needed to draw without using any physical tools. For instance, when drawing a very large circle, we may start with a square and gradually cut its corners based on certain calculations to generate a good-looking rounded curve. Although every step in this long procedure is done without tools, the circle created can still be much better shaped than the result of drawing it freehand without a strategy—that is, using merely HCA. In this example, it is the technique that instantly increases our circle-drawing ability, although HCA is not changed at all per se. The theories used to guide our drawing may focus on various aspects, such as how to lay out structures, how to draw shapes, how to control orientations and proportions, and so forth. Using knowledge and related techniques is the key that differentiates TGDA from HCA.

Tool-Assisted Drawing Ability

Drawing with tools is like doing calculations with calculators, although drawing tools are more and more high-tech these days. If we simply judge the drawing results only, *tool-assisted drawing ability (TADA)* is appealing because it enables us to generate good artwork with little or no training. Unfortunately, TADA is not considered a real drawing skill, mainly because tools are not part of the human body. If you cannot draw without tools, you would not be called an “artist” by today’s standards.

What Makes Us Draw Better?

Now that we have classified drawing ability at three levels, let's look at what we really need in order to master how to draw. Surely, TADA is not the option. Although both HCA and TGDA enable us to draw, are they equally approachable and learnable? I haven't seen any drawing books that separate these two abilities for the purpose of learning to create art, and many people consider them the same or at least inseparable. This is a major reason why incorrect drawing methods are often taught at schools.

Because the hand is the instrument doing the real work of drawing, it gives people the wrong impression that HCA and TGDA are basically the same. The truth is, these two abilities are quite different, even though related. TGDA indeed does depend on HCA, but it is drawing theory and technique that turn a person into an artist. Many artists cannot draw circles and lines much more accurately than a nonartist. Real drawing skill comes from an artist's knowledge, especially techniques that enable them to maneuver pencils on paper fluidly.

You may wonder how I know that HCA is not the driving force in an artist's drawing. In the fall of 2010, I conducted an interesting experiment that separated HCA from drawing theories and techniques. In a drawing room of the Art Institute of California, San Diego, I performed quick sketches with my *left hand*—that is, my nondominant one, for the first time in my life. You would expect my left-hand coordination ability to be terrible, because I had *never* used that hand for drawing before. Therefore, drawing with my left hand would be mainly governed by my knowledge of art and its techniques, and my poor left HCA should be of little use beyond holding a pencil and moving it slowly on paper.

However, the drawing result turned out to be a big surprise. Both the students at the scene and I were truly shocked at the quality of the 5-minute sketches I did with my left hand. One of these sketches appears on the left side of Figure 1.17. To see the influence of my right-hand HCA, a similar 5-minute quick sketch completed with my right hand appears on the right side of Figure 1.17.

In the two drawings, pronounced differences exist in the line quality and amount of work (mainly rendering) accomplished within the same time frame. Otherwise, the difference in the accuracy of structures is really unnoticeable! This test simply demonstrates that my drawing knowledge and techniques are the main force I use to manipulate lines and shapes. HCA has relatively little impact on the drawing.

When I took art classes at various educational institutions in the past, I observed that few art theories are taught besides perspective theory and human anatomy. It seems that the major goal in art classes is to improve HCA, which can be acquired with practice and a little theoretical support. Although some drawing techniques are taught, they are generally more experience-driven rather than developed systematically via principles. Such tactics sometimes go against scientific principles, although art instructors may not be aware of this.

Figure 1.17
Left-hand quick sketch (left) and right-hand quick sketch (right)

So, what are the missing pieces in today's artist training? What kind of theory can make us draw better and more efficiently? What *are* the knowledge and techniques that we should master? In my experience, drawing is a process of extracting structural information from the scene, followed by arranging geometric elements correctly on paper. Therefore, we should know how objects are structured in general and learn methods to obtain graphic elements representing the objects we want to draw. When we put pencil to paper, we must have knowledge of shapes and their spatial behaviors in 2D spaces. Additionally, we should master corresponding techniques for manipulating shapes efficiently.

The list of "must knows" is pretty short, but how well have we done so far in teaching students to master these knowledge? In the next section, you will review some of the existing methods used in today's training. I present this topic for two reasons: besides examining the usefulness of these methods in improving your drawing ability, you should know the real pros and cons of each in case you're using any of the methods discussed.

Examining Existing Drawing Methods

There are a number of popular drawing methods; some are math-driven, while some others use very little (if any) math at all. It will be helpful for you to become familiar with these methods in order to more readily learn the new method I introduce in Chapter 3, "Drawing with the ABC Method."

Perspective Drawing and Its Variations

In the previous section, I briefly mentioned the perspective theory as an application of math as well as the basic concepts involved. The theory is essential in today's art training, although some artists may not fully understand its supporting geometry theories.

To create a perspective drawing precisely, various drafting tools are typically required. Rulers, triangles, and templates are good helpers to make every line or curve perfect. Thus, executing the standard perspective drawing can be tedious and time-consuming. The method is not suitable for the outdoors, and it is often performed by professionals for commercial applications such as architecture and interior design. Figure 1.18 is a simple perspective drawing based on two vanishing points.

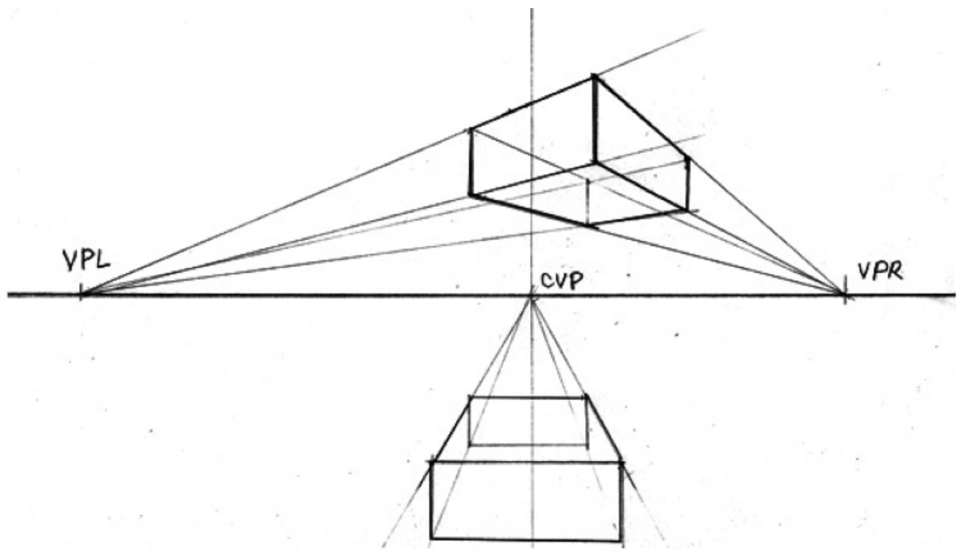


Figure 1.18
A simple perspective drawing

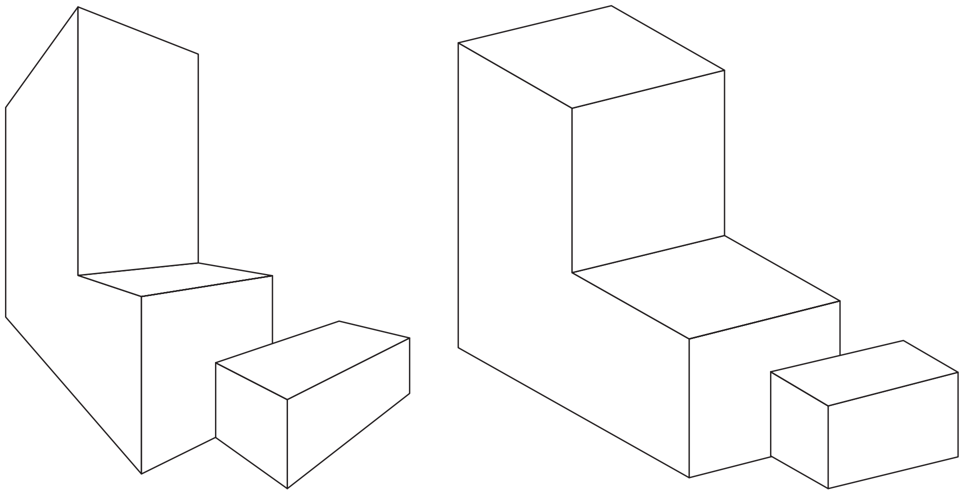
There are several variations of perspective drawing. The most popular one is applying perspective theory casually in order to make it more practical to use. In many situations, artists draw freehand without using tools—all points and lines are not required to be perfectly positioned and oriented. I call it *casual perspective drawing* in this book in order to differentiate it from the other versions.

In the casual perspective drawing method, perspective theory is used only as a general guide. By setting up the horizon line and vanishing points as a reference system on paper, artists can arrange shapes with respect to these references with reasonable accuracy. Most artists use this approach today.

The perspective projection is not the only way to map 3D objects onto 2D planes, however. In engineering and some video games, we often want to illustrate 3D objects without distortions. In this case, all parallel lines in a 3D space remain parallel on paper.

A simple way to do this is using *orthogonal transform*, which is theoretically equivalent to setting the viewpoint an infinite distance away from the scene in a standard perspective drawing. For this reason, we can consider this method a variation of perspective drawing. The two images in Figure 1.19 illustrate a comparison of simple cubes in perspective and orthogonal drawings. The latter is much easier for entry-level artists to draw without worrying about tedious rules.

Figure 1.19
Cubes in perspective view (left) and cubes in orthogonal view (right)



The orthogonal transform is widely used in engineering drawings for illustrating man-made objects such as car engines or pipelines. Because of the large number of components in various machines, you might think that this type of drawing would be difficult for engineers who lack art training. Amazingly, however, engineers usually don't have trouble accomplishing this complicated task. The secret is quite simple. An engineer-designed object can be deconstructed into simple, standard forms with predesigned locations and orientations. The orthogonal transform then turns 3D forms into shapes on the picture plane. All an engineer needs to do is to draw basic shapes such as ovals and rectangles in a step-by-step procedure, pretty much like solving math equations. This process is a great example that proves the power of math in drawing.

Although perspective theory is derived from math knowledge, it does not work well in all circumstances. Its application scope is limited to 3D only. If we want to copy an existing piece of 2D artwork, perspective theory is totally useless in the sense that it does not tell us how shapes in the source image should be mapped onto paper. Technically, there are no vanishing points, horizon line, or even a 3D system inside a given image upon which perspective theory arranges objects.

Using only a small number of references is another major technical weakness of perspective theory. Even in standard 3D cases, vanishing points may often occur outside the paper, and therefore this causes big operational hassles for perspective drawing. The theory handles all shapes corresponding to the few references, but is incapable of telling us the spatial relationship directly *between* shapes.

Unfortunately, managing spatial relations between shapes is the most difficult part for beginners, according to my observations. In general, it is fairly easy to draw simple shapes such as ovals or rectangles to approximate objects or components. But most of us have trouble putting these shapes at the right places with the correct orientation. Perspective theory does a very poor job in providing assistance because of its technical limitations.

The Grid Method

Whereas perspective theory fails to work for 2D images, the legendary *grid method* seems to be a good solution to fix the problem. Some art scholars consider this method an amazing way to teach people to draw 2D pictures accurately and without difficulty. By putting vertical and horizontal lines (grids) on both an original image and a paper copy, the grid system breaks the image into smaller pieces or cells that contain very simple graphics (see Figure 1.20).

This smart tactic dramatically reduces the complexity of the drawing and enables an average person to reproduce images easily. The number of grid lines depends on the size of the image to be copied, the visual complexity of the image, and the skill level of the person who is doing the drawing. The denser the grid lines, the easier the duplicating task will be. The grid system in both the original image and the paper copy must have the same number of grid lines, although the size of cells in the two systems could be different. This allows a user to enlarge or shrink the image.

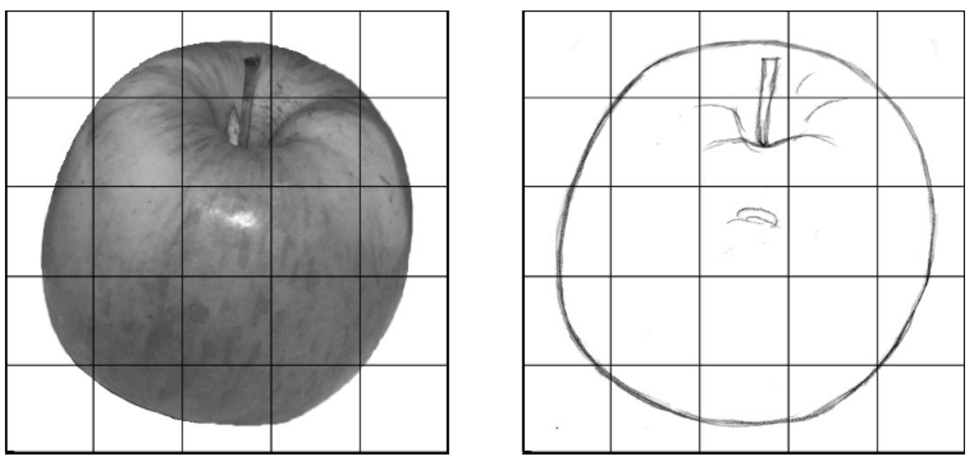


Figure 1.20
The grid method

The grid method is incredibly simple, and so is its supporting theory. A coordination system converts all spatial information into grids and uses the indices of grids to manage locations of graphical structures. This method is a direct application of analytical geometry.

Inexperienced individuals may find the grid method perfect and fall in love with it after one try. Because new users can produce highly accurate artwork, this method can boost confidence in no time. However, the grid method itself doesn't teach us any real drawing skills. Similar to tracing a picture, it does not show you how to organize graphic patterns or orient shapes except by looking at grids. In contrast, using this method excessively will degrade existing drawing skills.

Using this method for art training is similar to using calculators for learning how to do multiplication. Sadly, there are still many books on the market that promote the grid method without explaining the possible consequences it may bring. You should be aware of this crucial issue.

The grid method also suffers from a scope limitation. It is usually not feasible to set up grid lines for 3D objects. Additionally, creating grid systems isn't practical if time is critical. However, artists may use grids to reduce job complexity when they work with large-scale artwork.

I like the attractive design and powerful math principles implicitly embedded in the grid method. In fact, this was one of my major motivations to starting my math-art research—that is, to seek a new drawing approach that is as easy to use as the grid method, but tool-free and universally applicable to all objects in both 2D and 3D.

The Contour-Line Method

Many art books promote the *contour-line method,* which requires an artist to draw lines continuously while their eyes are moving along the boundaries of objects. Usually, the method is carried out *blindly*, that is, without looking at the paper during the entire drawing procedure, in order to let artist better *sense* and *feel* the objects. This unique method aims at recording the eye's movement and therefore achieves "draw what you see."

The actual results, however, are far from what might be expected. Rarely can we draw contour lines accurately—not even close. Many artists I have asked or have observed, including myself, have trouble producing good contour-line artwork in observational drawing. The image shown on the left in Figure 1.21 is an example of a contour-line sketch I did of a simple cup. Compared with my regular sketch of the same cup on the right, the contouring is terrible!

There are several scientific reasons why the contour-line method gives us poor results. When we draw, every single pencil movement produces a small error. If we continue such a task without promptly correcting the error (because our eyes are not focused on the paper), the errors will accumulate quickly to ruin the drawing.

Another piece of evidence is that an artist can hardly make a closed curve, such as the one shown on the left side of Figure 1.21, without looking at the paper. Additionally, the contour-line method mainly trains HCA without teaching any real technique to manage shapes correctly. While drawing, an artist doesn't have a chance to lay out the big picture on paper or take advantage of the graphical structures identified. Working on details without an overall plan is not a good way to solve problems in general.

The contour-line method is designed to train students to use feelings and gain a sense for drawing. In my opinion, feelings and sense are accumulated from experience. They are hardly teachable, learnable, or even describable. Students should learn solid techniques and proven knowledge from their instructors in order to master art skills efficiently.

Figure 1.21

A contour-line drawing (left) and a regular drawing (right)

The Negative-Space Method

When we draw shapes on paper, the background or blank areas are called *negative spaces*. Obviously, the negative spaces are complementary to the shapes we draw, or positive spaces. Technically, if you draw positive spaces correctly, then the negative spaces must be correct too, and vice versa. The main use of negative space is to help artists align shapes on paper. Getting an accurate spatial relationship between two shapes is accomplished by creating a correct negative space.

Figure 1.22 shows a scene with a few simple objects. The left side of the figure shows the objects, or positive spaces, in black on a white background. The right side of the figure shows the corresponding negative spaces, indicated as dark areas. Negative spaces are usually irregularly shaped, and their total number could exceed that of the corresponding positive objects. In real world, the objects have logical connections; for example, handles of clips have same thickness, and a handle is an unbroken piece even if it may be blocked by other objects. By contrast, the negative spaces are usually isolated and lack an obvious association with one another. Therefore, artists rarely use negative spaces as an independent drawing method other than an assistant tool.

Figure 1.22

Positive space (left) and negative space (right)

Stop Separating Math from Art

Although perspective theory is based on math theory, the art community does not emphasize the importance of math. Instead, math and art are not well-integrated, or are even separated in school curriculum. As a general rule, many artists have poor math knowledge, and technical students often cannot draw. There are surely many reasons for this problem. One of them, in my opinion, is the influence of right-brain theory.

Is the Left Brain Useless for Art?

In 1979, Betty Edwards published her famous book, *Drawing on the Right Side of the Brain* (Tarcher/Penguin). In the book, Edwards claimed that drawing is controlled mainly by the right side of the brain, and thus she suggested several unique R-mode methods to learn drawing skills.

With more than 2.5 million copies in 13 languages sold internationally, Edwards' book has certainly reshaped generations of artists over the past three decades. Many art scholars have followed her theory and ignored the importance of the left brain (or *L-mode*). Some even claimed that the left brain is useless or works counter to drawing despite the key role of perspective theory. The influence of her book on society at large is just phenomenal.

As a mathematician, artist, and computer graphics scientist, it is extremely difficult for me to accept Edwards' R-mode theory, especially the ignorance of the left brain for art, because math knowledge has helped me a lot to improve my drawing skill. First, drawing is mainly concerned with managing shapes on paper. *Applying* math knowledge to deal with geometric objects surely is a good choice, if not the best one, for that task. Perspective theory provides strong evidence of math's application to art. How is it possible that the left brain (the "logic" side) is useless for art?

Second, Edwards connected drawing with neuroscience, and thus much of her description of the R-mode is rather mysterious and hard for the average person to grasp. Teaching art by "sense" or "feeling" makes drawing less learnable. As educators, we should teach students solid theories and techniques, not mysterious "senses."

A third reason for my rejection of Edwards' hypothesis is that she suggested several R-mode methods to shut down the L-mode so that the drawing will become better *automatically* as it is handled by the right brain. This assumption is as groundless as the statement that "if you close your eyes (shut down your R-mode), your efforts to learn math will automatically be done in L-mode, and therefore you'll learn math much better." Additionally, the methods she recommended, especially the upside-down method and the contour-line method, do not provide any solid technique to help with shape management as I mentioned earlier. Both methods are aiming to improve HCA.

The most fitting part of Edwards' book is her point that beginners tend to draw from stereotyped images instead of the observed objects. This phenomenon surely is a fatal problem for beginners, and the R-mode methods can fix this problem to a certain degree. For example, the upside-down method tries to prevent an entry-level artist from recognizing any real objects. When drawing a picture, if the picture is placed upside-down, a beginner can hardly recognize the objects in the picture and thus avoid using stereotyped images. However, this treatment also stops students from taking advantage of graphic structures that exist in the scene and slows down the drawing process. Furthermore, for experienced artists who draw based on true observation, the upside-down method does not help at all but rather reduces their efficiency.

It is meaningless to debate whether the left brain is useful to art. The best way to answer this question is to deliver an efficient, logic-based drawing method, as I am doing in this book. I will provide you with a new approach that uses geometry knowledge, CG techniques, and spatial properties of objects to enable you to draw efficiently. You will see how drawing can be improved by some simple math applications. Ideally, this non-conventional method will correct the ill-informed attitude toward math that exists in the art community. Art and math should be merged together just as they were in the time of the Renaissance.

Learning Art with the Full Brain

As pointed out in Edwards' book, logic and reasoning are functions of the left brain, and visual activity is a function of the right brain. In my opinion, the different functionalities of the two hemispheres of the brain perfectly match two of the levels of drawing abilities I defined earlier. HCA is related to spatial sense and thus belongs to the right brain, while TGDA applies to the logical and analytical thinking done in the left brain. Drawing is a process that uses both abilities, although TGDA is much more teachable and learnable than HCA.

Currently, we still don't have enough scientific methods to help us fully control our drawing. Perspective theory is far from a perfect solution. It is time to develop more solid scientific methods so that everyone can easily learn to draw. This book is an attempt to guide you in mastering draftsmanship skills scientifically and efficiently.

Over the past few years, I have hosted numerous workshops on my math-oriented drawing method for different audiences, including art faculties and students at the Art Institute of California, San Diego, and for computer science students at Zhejiang University, China. The result has been very promising. Some tech-minded audiences saw a quick surge in their drawing ability after just a few days!

I think the art community needs to move forward to bring math into art in order to better understand and control spatial properties. The math logic in the left brain definitely helps our art that is naturally connected with the visual acuity of the right brain. Learning art should be a full-brain experience!