1

Introduction

With all thy getting get understanding.

—Proverbs 4:7 (KJV)

The central focus of *multiscale structural mechanics* is to apply fundamental principles of mechanics to develop models that predict the behavior of structures, particularly those made of advanced materials exhibiting general anisotropy and heterogeneity. In these cases, considering multiple length scales (and, in some cases, temporal scales) becomes crucial for accurate modeling. To keep the terminology simple, we use the phrase *composite material* to describe any material exhibiting anisotropy and heterogeneity. In other words, any material that is not isotropic and homogeneous is called a composite material in this book. Similarly, any structure not made of a single isotropic homogeneous material is referred to as a composite structure.

This chapter serves as an introduction to the basic concepts of multiscale structural mechanics. It introduces topics such as the continuum hypothesis, isotropic versus anisotropic materials, homogeneous versus heterogeneous materials, materials versus structures. Common engineering structural models, an overview of advanced composite materials, and the challenges of modeling composite materials and structures are also covered. Finally, this chapter introduces multiscale modeling and mechanics of structure genome (MSG).

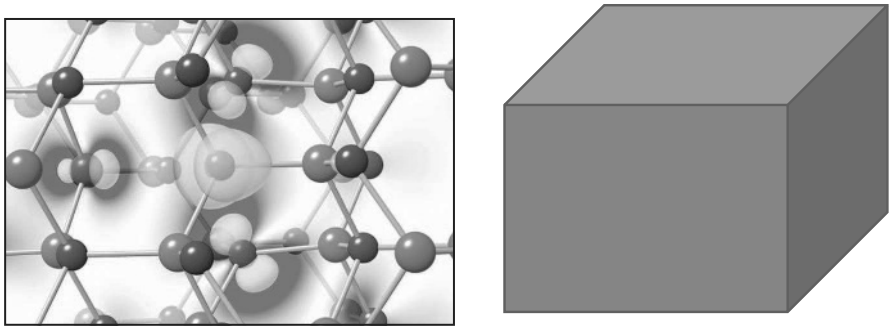
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1.1 Continuum Hypothesis

A *composite material*, often referred to simply as a composite, is a combination of two or more distinct, physically or chemically different constituent materials, also known as phases. To understand a composite material, we must first define a material. In general, a *material* can be understood as the substance from which an object is made.

The characterization of a material, including its composition and behavior, varies across different length scales depending on the application. At the smallest scale measurable by modern scientific instruments, it is well-established that all materials possess specific atomic structures (see the left figure of Figure 1.1). This microscopic description is crucial

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(a) (b)

Figure 1.1 Atomic structure and corresponding continuum for a typical material: (a) atomic structure of a material and (b) the corresponding hypothetical continuum.

in fields such as atomic physics, chemistry, medical sciences, nanotechnology, and materials science and engineering.

However, this book's focus lies on the macroscopic behavior of systems or their components, such as airplane wings and fuselages, wind turbine blades, and car bodies. For the design and analysis of these structures, we describe materials using physical properties widely used in engineering, such as density, stiffness, and strength. To bridge the gap between atomic structures and these physical properties, we employ the continuum hypothesis, which treats materials as continuous, ignoring their discrete atomic nature.

A *continuum*, also known as continuous media, is a theoretical construct (see the right figure of Figure 1.1) that provides a unified framework for describing various types of matter, including solids, liquids, and gases [1–4]. It is an abstract body of matter defined based on the continuum hypothesis, a concept derived from mathematics. The real number system can be considered as a mathematical continuum because a distinct real number can always be found between any two different real numbers. Furthermore, there are infinitely many real numbers between any two distinct numbers. Similarly, we can represent time as a one-dimensional (1D) continuum using a real number t, while the physical space we inhabit can be represented as a three-dimensional (3D) continuum with coordinates described by three real numbers (x_1, x_2, x_3).

This concept of a mathematical continuum can be extended to describe a material body, whether it occupies a finite or infinite space. We assume that there is a continuous distribution of matter in the space it occupies, meaning that matter fills the space completely, without gaps or voids. Holes in a solid are excluded from the continuum. Additionally, we assume that all mathematical functions used to describe the properties of matter (e.g., density, velocity, energy) are continuously defined throughout the body, except possibly at a finite number of surfaces that separate continuous regions. This classical definition of a *material continuum*, often simply referred to as *continuum*, establishes a one-to-one correspondence between the material and the real number system. According to this definition, the space occupied by matter can be divided indefinitely into smaller and smaller volumes (see the right figure of Figure 1.1). This assumption enables the analysis of material behavior using differential calculus.

The concept of density is defined for a material continuum. Let P be a point within a material body, located inside a small volume element $\mathrm{d}V$. If the mass of the small volume element is denoted as $\mathrm{d}m$, we can define the density as

$$\rho = \lim_{\mathrm{d}V \to 0} \frac{\mathrm{d}m}{\mathrm{d}V} \tag{1.1}$$

This definition implies that as the volume element $\mathrm{d}V$ shrinks to zero at point P, the ratio of its mass to its volume converges to a limit. This limit is the density of the material continuum at point P. For a material continuum at any time, such a quantity is defined for all points inside the continuum:

$$\rho = \rho(x_1, x_2, x_3, t) \tag{1.2}$$

The density might vary with position and time. For instance, the density of air at the top of a mountain differs from that at the foot.

The continuum hypothesis asserts that a material continuum remains continuous when subjected to external forces. This means that two material particles initially adjacent will remain adjacent throughout deformation. While the continuum can break, as seen in solid fractures, the fracture surfaces are treated as newly created external surfaces.

The material continuum is a hypothetical concept because the atomic and molecular composition of matter is a well-established fact. At sufficiently small scales, such as the atomic scale, all materials are discontinuous. For example, a helicopter rotor blade made of aluminum is essentially a vast collection of aluminum atoms organized in a particular pattern. At an even smaller scale, an atom consists of electrons, positively charged protons, and uncharged neutrons. However, in a material continuum, we idealize the real discrete atomic/molecular structure of matter as a hypothetical continuous medium. This idealization is commonly justified in design and analysis of engineering structures.

In engineering, we primarily focus on a macroscopic view of material bodies, such as airplanes, spacecraft, bridges, and automobiles. The dimensions of these bodies far exceed the *mean free path*, which represents the average distance a molecule travels between successive collisions with other molecules (approximately 10^{-9} m for solids and 10^{-8} m for liquids). Designing and analyzing common engineering bodies with the atomic or molecular details is neither practical nor necessary. Instead, *it is commonly accepted that the behavior at length scales of μm (10^{-6} m) and larger can be sufficiently described using the continuum hypothesis*. In this book, we use the "continuum scale" to refer to length scales of μm and larger.

Nevertheless, this does not mean that the microscopic effects of the material's atomic reality cannot be considered. These effects can be incorporated through material properties, typically obtained from experiments by testing specimens with dimensions much larger than the mean free path. Recently, computer simulations, such as the molecular dynamics (MD) simulation, are increasingly being used to compute material properties. Both experiments and MD simulations account for the discrete atomic composition of matter to derive properties defined for the idealized material continuum.

In the definition of density in Eq. (1.1), for a real material, $\mathrm{d}V$ cannot continuously shrink to zero. Instead, $\mathrm{d}V$ is bounded below by a volume element $\mathrm{d}V_0$ with sides at a length scale much larger than the mean free path. By dividing the total mass of atoms or molecules

contained within that volume element by its volume, we define the density at point P of the continuum as the limit of that ratio:

$$\rho = \lim_{\mathrm{d}V \to \mathrm{d}V_0} \frac{\mathrm{d}m}{\mathrm{d}V} \tag{1.3}$$

A continuum model effectively analyzes the macroscopic behavior of engineering systems, with numerous engineering theories—such as elasticity, plasticity, and fluid mechanics—based on the continuum concept providing accurate predictions consistent with experimental observations across a broad range of working conditions. However, a continuum model cannot accurately capture material behavior at length scales comparable to or smaller than the mean free path.

1.2 Isotropic vs. Anisotropic Materials

Many material properties, such as the coefficients of thermal expansion (CTEs) and stiffness, depend on direction. A material is considered *isotropic* if its properties remain the same in all directions at the same point. Conversely, a material is *anisotropic* if its properties differ in different directions at the same point. A well-known example of an anisotropic material found in nature is wood. Cutting wood along the grain direction is typically much easier than cutting it along the growth ring direction. Nearly all materials possess some level of anisotropy. However, the anisotropy of certain materials, such as most metals, is negligible for practical purposes, allowing them to be approximated as isotropic materials.

1.3 Homogeneous vs. Heterogeneous Materials

A material is *homogeneous* if its properties remain the same at all points throughout the material's domain. Conversely, if the properties vary within the domain, the material is *heterogeneous*.

It is essential to note that material heterogeneity is meaningful only at the continuum scale, as all materials are inherently heterogeneous at the atomic scale (see Figure 1.1). In other words, heterogeneous materials are continuous bodies with properties that vary with position. These materials could be composed of distinguishable constituents, each with constant properties (e.g., fiber-reinforced polymers composed of two different constituents: fiber and resin). Alternatively, they may be composed of one or more constituents with properties that vary continuously with location (e.g., functionally graded materials). Furthermore, a material may also be considered as anisotropic when it consists of a single anisotropic phase that is oriented differently at various locations (i.e., polycrystals).

1.4 Materials vs. Structures

Although the above definition of materials using the continuum hypothesis offers a rational connection to the atomic or molecular structure of the matter, it may not be practical for engineers working on systems much larger than atomic scales. In engineering,

a *material* is more usefully defined as the substance used to construct an object. If the object is a structure, we are referring to structural materials.

Structures and (structural) materials are load-bearing solid bodies. These terms are often used interchangeably, mainly because the distinction between a structure and a material is not always explicitly defined. This overlap often begins in the first solid mechanics course, commonly referred to as *mechanics of materials*. However, that course primarily addresses deformation and stress analyses of simple structures, such as rods, shafts, columns, beams, and pressure vessels.

With the increasing use of composites in industry and advancements in fabricating materials with complex microstructures through additive manufacturing (AM) or 3D printing, the distinction between structures and materials has become even less clear.

One possible distinction lies in the presence of boundaries: a material, as described by the continuum hypothesis, has no explicit boundaries and is often considered as a point in a structure. In contrast, a structure has clearly defined boundaries and interacts with its environment through boundary conditions, such as imposed displacements, applied loads, temperature variations, or moisture exposure. For example, we describe a panel structure using dimensions like thickness, length, and width, but such geometric descriptions are unnecessary when referring to a material like aluminum.

Another distinction is how we characterize materials and structures. Materials are characterized by intrinsic properties such as Young's modulus, Poisson's ratio, CTE, yield strength, and ultimate strength. These properties are intrinsic because they remain unchanged regardless of the structure made from the material. For instance, a linear elastic material with a Young's modulus of 73 GPa retains this value whether used to construct a shaft, a pressure vessel, or any other structure, provided it remains in the elastic region. Even in nonlinear behavior, material properties depend on the current stress or strain state but remain intrinsic because they do not change with the structure itself. Due to these intrinsic properties, the material will exhibit the same behavior when subjected to the same stress or strain state, even if used in two entirely different structures. In contrast, the behavior of a structure is described using parameters, such as displacements, strains, stresses, natural frequencies, and buckling loads. These parameters depend on the structure's size, geometry, and boundary conditions.

In some cases, novel material properties can be achieved by engineering innovative microstructures. The emerging field of metamaterials focuses on designing microstructures to produce properties not naturally found in materials. For example, Frenzel et al. [5] discovered a 3D mechanical *metamaterial* that exhibits twist under compression (see Figure 1.2). However, it is crucial to distinguish between structural behavior and material behavior. Strictly speaking, Frenzel et al. discovered a structural behavior rather than a material behavior. This behavior disappears when many unit cells are added along all three dimensions. It can be explained as the extension–twist coupling behavior of the Euler–Bernoulli beam model or the Kirchhoff–Love plate model. Nevertheless, Frenzel et al. used the complex micropolar continuum mechanics model to interpret this as a 3D material behavior.

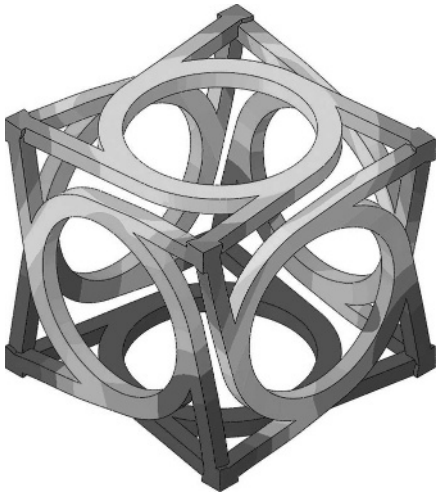


Figure 1.2 A 3D metamaterial with a twist.

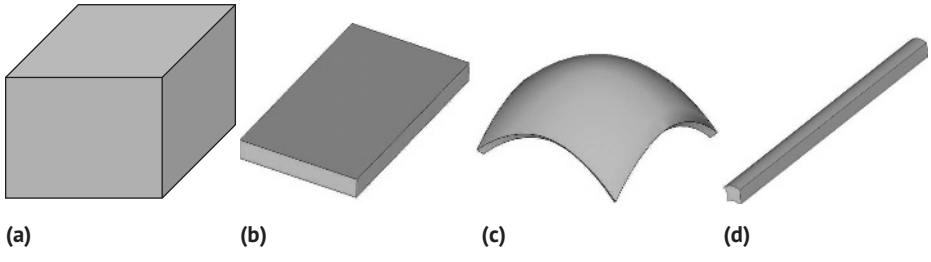


Figure 1.3 Typical structural components: (a) 3D structure, (b) plate, (c) shell, and (d) beam.

1.5 3D Structures, Plates, Shells, and Beams

Structures can be categorized based on their external geometry. If all three dimensions of a structure are of similar size, it is referred to as a 3D structure or solid (Figure 1.3a). If one dimension is much *smaller* than the other two, the structure is called a plate (Figure 1.3b) or a shell (Figure 1.3c), depending on whether the undeformed configuration is flat or curved. The two larger dimensions are referred to as in-plane directions, while the smaller dimension is called the thickness, which can be either constant or variable. For plates and shells, a reference surface can be defined using two in-plane coordinates, classifying them as two-dimensional (2D) structures.

If one dimension of the structure is much *larger* than the other two, the structure is known as a beam (Figure 1.3d). The large dimension is referred to as the span of the beam, and a reference line (or axis) can be defined along the span. The reference line can be as general as a spatial curve, as seen in initially curved and twisted beams. Consequently, beams are also referred to as 1D structures. The two smaller dimensions constitute the cross-section, which may be constant or variable along the reference line. If the wall thickness of a beam is much smaller than its cross-sectional dimension, it is referred to as a thin-walled beam. In such cases, the structure's three dimensions differ significantly in magnitude.

Collectively, plates, shells, beams, and thin-walled beams are referred to as *dimensionally reducible structures* [6]. This term emphasizes that one or more small dimensions can be eliminated to create simplified structural models with reduced dimensions while still adequately predicting their behavior.

As far as mathematical modeling is concerned, this classification is based solely on the external geometry of the structure, while its internal construction can be arbitrary. For instance, a porous material may be modeled as a 3D structure (see Figure 1.4), a flat panel with grid stiffeners as a plate (see Figure 1.5), and a high-aspect-ratio wing, even without clearly defined cross-sections, as a beam. In this sense, all engineering structural systems, regardless of their complexity, can be considered as combination of structural components categorized as 3D structures, plates, shells, and beams (see Figure 1.3), possibly with complex internal constructions.

The classification of structures based on external geometric characteristics does not restrict the materials from which the structures are made. A structure categorized as

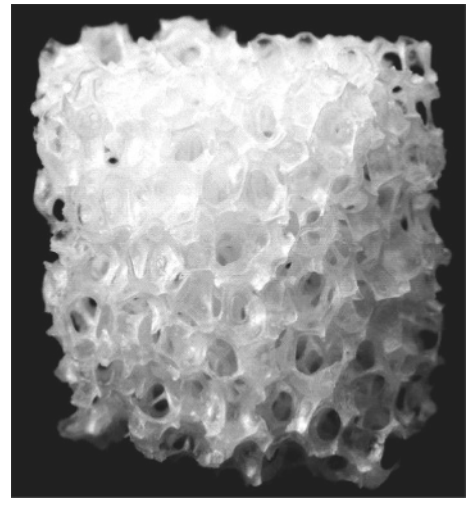


Figure 1.4 A sample porous material.

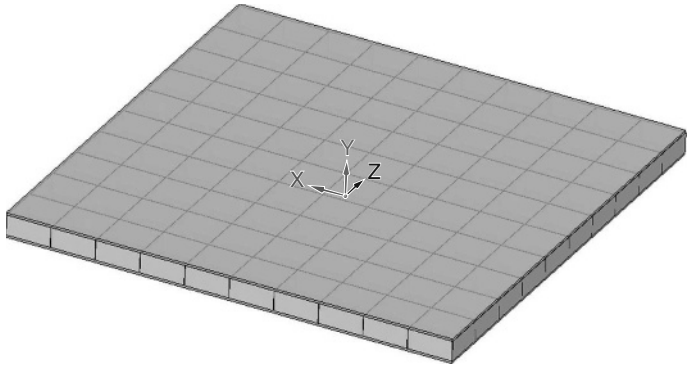


Figure 1.5 A sample heterogeneous panel with grid stiffeners.

a beam, plate, shell, or 3D structure can be made from isotropic (e.g., aluminum) or anisotropic materials (e.g., carbon fiber-reinforced composites). If a structure is made from a single material, it is referred to as a *homogeneous structure*. Conversely, if it is composed of multiple materials, it is termed a *heterogeneous structure*.

Furthermore, the classification of structures in this book is independent of the loading conditions they may experience. While engineers often associate beam structures with bending loads, the definition of a beam structure presented here encompasses a broader range of loading scenarios. A beam structure, as classified in this book, can be subjected to forces and moments in all three directions, including an axial force, a torque, two transverse shear forces, and two bending moments.

1.6 Structures, Models, and Assumptions

The field of applied mechanics known as *structural mechanics* focuses on predicting displacements, strains, stresses, failure, and other phenomena for various types of structures using mathematical models. Structures are classified as 3D structures, plates, shells, and beams based on their geometric characteristics, which guide the construction of different models for structural analysis.

Beam models, also called 1D structural models, are developed for slender structures. In these models, field variables are functions of a single coordinate that represents the reference line along the beam. Conversely, plate and shell models, referred to as 2D structural models, are designed for panel structures, where field variables depend on two in-plane coordinates that define the reference surface of the plate or shell.

It is important to note that the terms 1D, 2D, and 3D refer to the number of coordinates required to describe the analysis domain, not the dimensionality of the structural behavior. For example, a 1D beam model can capture displacements and rotations in all three directions.

In structural mechanics, each structural model consists of three sets of equations: kinematics, kinetics, and energetics (constitutive relations). The constitutive relations, which describe the material behavior, are the only equations that change based on the material used in the structure. Predicting these constitutive relations is known as *constitutive modeling*. Typically, a specific form is assumed for the constitutive relation, with parameters either measured through physical testing or calculated using micromechanics.

For homogeneous structures, 3D material properties such as Young's modulus E, Poisson's ratio ν, and shear modulus G can be directly used in structural analysis with a 3D solid model. For isotropic, homogeneous thin panels or slender structures, geometric characteristics are combined with 3D material properties to compute structural stiffness parameters for plate, shell, and beam models. For example, extension stiffness EA, bending stiffness EI, and torsion stiffness GJ are used for beam models. Traditionally, these stiffness parameters are determined based on specific assumptions, such as the Euler–Bernoulli assumptions and Saint Venant assumptions for beams.

When dealing with homogeneous structures made of anisotropic materials or heterogeneous structures, constitutive modeling becomes more complex. For example, in composite laminates, refined assumptions such as higher-order assumptions, zigzag assumptions, and

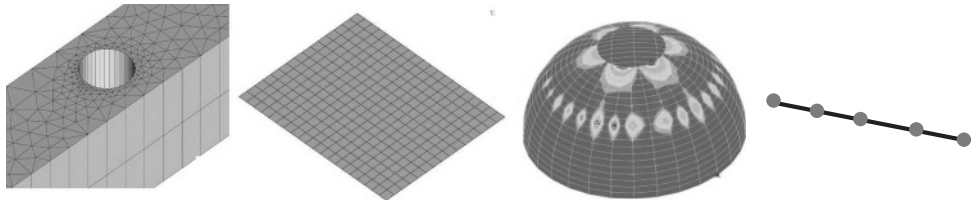


Figure 1.6 Typical structural elements: (a) solid element, (b) plate element, (c) shell element, and (d) beam element.

layerwise assumptions have been introduced to capture the kinematics along the smaller dimensions of the structure (e.g., the thickness of a plate).

Structural analyses are often performed using finite element analysis (FEA) tools such as Abaqus, Ansys, and Nastran. These analyses employ 3D solid elements, 2D plate or shell elements, or 1D beam elements (see Figure 1.6), each corresponding to a specific structural model comprising three sets of equations: kinematics, kinetics, and constitutive relations.

In engineering, different beam models are used, including the Euler–Bernoulli model, the Timoshenko model, and the Vlasov model:

1. Euler–Bernoulli Model: Also referred to as the classical beam model, it neglects transverse shear effects.
2. Timoshenko Model: Known as the first-order shear deformation theory (FSDT) for composite beams, it accounts for transverse shear effects.
3. Vlasov model: Includes nonuniform torsion effects.

For plate and shell structures, commonly used models include the Kirchhoff–Love model and the Reissner–Mindlin model:

1. Kirchhoff–Love Model: Also called the classical plate and shell model, or the classical lamination theory (CLT) for composite laminated plates and shells, it neglects the transverse shear effects.
2. Reissner–Mindlin Model: Also known as FSDT for composite plates and shells, it accounts for the transverse shear effects.

For 3D structures, the commonly used model is the Cauchy continuum model, also referred to as the classical continuum model.

In some cases, specific models and their associated assumptions are used to name a structure in technical literature and communications. For example, terms like "Euler–Bernoulli beam" or "Timoshenko beam" imply that the structure must conform to certain assumptions to be analyzed using these beam models. However, a beam structure is simply a solid with a much larger axial dimension compared to its other two dimensions. Such a structure can be analyzed using various models, including the 3D Cauchy continuum model, the Euler–Bernoulli model, the Timoshenko model, or higher-order models, depending on the desired behavior to be captured.

To avoid unnecessary confusion, it is essential to differentiate between the structure, the model, and the associated assumptions. While assumptions like those according to Euler–Bernoulli or Timoshenko are helpful, they are not strictly necessary. For instance, one can develop an Euler–Bernoulli beam model without adhering to Euler–Bernoulli assumptions.

Next, we will focus on structures made of linear elastic materials to introduce the fundamentals of each model. While equations will be presented to describe these models in this chapter, you only need to understand the general concepts behind them. Detailed theoretical discussions will be covered in later chapters.

1.6.1 Cauchy Continuum Model

The kinematics of the Cauchy continuum model involves three displacements (u_1, u_2, u_3) and six strains (ε_{11}, ε_{22}, ε_{33}, ε_{23}, ε_{13}, ε_{12}), all functions of the three coordinates x_1, x_2, and x_3, which describes the 3D body. This dependence on three coordinates is why the Cauchy model is classified as a 3D model. The strain–displacement relations are expressed as

$$\begin{aligned} \varepsilon_{11} &= \frac{\partial u_1}{\partial x_1}, \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ 2\varepsilon_{23} &= \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \qquad 2\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \qquad 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{aligned} \tag{1.4}$$

The kinetics of the Cauchy continuum model uses six stresses (σ_{11}, σ_{22}, σ_{33}, σ_{23}, σ_{13}, σ_{12}), governed by the following three equilibrium equations:

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 &= 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 &= 0 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 &= 0 \end{aligned} \tag{1.5}$$

Here, f_1, f_2, and f_3 represent distributed body forces per unit volume in the three directions.

The constitutive relations for the linear elastic behavior according to the Cauchy continuum model are described using the Hooke's law as:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \tag{1.6}$$

The 6×6 matrix C_{ij} is the stiffness matrix, with its inverse being the compliance matrix. Both matrices are symmetric. Anisotropic material properties are typically given in the material coordinate system. However, kinematics and kinetics are often expressed in the global coordinate system. A proper transformation is needed to transfer the constitutive relations from the material coordinate system to the global coordinate system.

For isotropic materials, the constitutive relations can be written using the compliance matrix in terms of Young's modulus E and Poisson's ratio ν:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{1.7}$$

which can be inverted to obtain the same expression as Eq. (1.6) with

$$C_{11} = C_{22} = C_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$C_{12} = C_{13} = C_{23} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$C_{44} = C_{55} = C_{66} = \frac{E}{2(1+\nu)}$$

and all other terms in the stiffness matrix of Eq. (1.6) are zero.

Equations (1.4), (1.5), and (1.6) form a system of 15 equations underpinning the Cauchy continuum model to be solved along with boundary conditions for 15 unknowns: three displacements, six strains, and six stresses. Kinematics and kinetics remain the same no matter whether the structure is made of an isotropic or anisotropic material; only the constitutive relations vary.

1.6.2 Kirchhoff–Love Model

The Kirchhoff plate model, also referred to as the classical plate model, was originally developed by Kirchhoff for thin plates made of isotropic materials. It is based on a set of a priori assumptions, including that the transverse normal material line (described by the coordinate x_3) along the thickness direction is rigid, remains straight, and remains

perpendicular to the reference surface (described by coordinates x_1 and x_2) during deformation. Furthermore, the model assumes that the structure experiences a plane stress state during deformation.

Love later extended these assumptions to curved panels, resulting in the classical shell model [7]. Since both models share the same assumptions and constitutive relations, they are collectively referred to as the Kirchhoff–Love model. For simplicity, we illustrate the plate model below.

The kinematics of the Kirchhoff–Love model involves three displacements ($\bar{u}_1, \bar{u}_2, \bar{u}_3$), as well as six strain variables, also known as the classical strain measures. These include the in-plane strains ($\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$) and the curvatures ($\kappa_{11}, \kappa_{22}, \kappa_{12}$). Specifically for a plate, the strain–displacement relations are given by

$$\begin{aligned} \epsilon_{11} &= \frac{\partial \bar{u}_1}{\partial x_1}, \qquad \epsilon_{22} = \frac{\partial \bar{u}_2}{\partial x_2}, \qquad 2\epsilon_{12} = \frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1} \\ \kappa_{11} &= -\frac{\partial^2 \bar{u}_3}{\partial x_1^2}, \qquad \kappa_{22} = -\frac{\partial^2 \bar{u}_3}{\partial x_2^2}, \qquad \kappa_{12} = -\frac{\partial^2 \bar{u}_3}{\partial x_1 \partial x_2} \end{aligned} \tag{1.8}$$

The kinetics of the Kirchhoff–Love model involves six stress resultants: in-plane force resultants (N_{11}, N_{22}, N_{12}) and moment resultants (M_{11}, M_{22}, M_{12}). These are governed by the following equilibrium equations:

$$\begin{aligned} &\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = 0 \\ &\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = 0 \\ &\frac{\partial^2 M_{11}}{\partial x_1^2} + \frac{\partial^2 M_{22}}{\partial x_2^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial q_2}{\partial x_1} - \frac{\partial q_1}{\partial x_2} + p_3 = 0 \end{aligned} \tag{1.9}$$

Here, p_1, p_2, and p_3 represent equivalent forces per unit area in the three directions, and q_1 and q_2 are equivalent moments per unit area in the two in-plane directions, distributed over the reference surface.

The constitutive relations for the Kirchhoff–Love model are expressed as:

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{61} & B_{62} & B_{66} \\ B_{11} & B_{21} & B_{61} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{62} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix} \tag{1.10}$$

This 6×6 matrix is termed the stiffness matrix for the Kirchhoff–Love model, and its inverse is the compliance matrix. The submatrices A, B, and D are well-known in CLT with A as the in-plane stiffness, D as the bending stiffness, and B as the coupling stiffness between in-plane and bending behavior. Although B is symmetric in CLT, it is not symmetric for general anisotropic heterogeneous panels. For special composite panels, such as symmetric laminates, B vanishes, decoupling bending behavior from in-plane behavior.

If the plate is made of an isotropic material and the origin of x_3 is at the center of the thickness, the constitutive relations simplify to

$$\begin{aligned}
\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} &= \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{Bmatrix} \\
\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} &= \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \end{Bmatrix}
\end{aligned} \tag{1.11}$$

Here, h is the thickness of the plate and the in-plane behavior and bending behaviors are decoupled and can be solved separately.

Equations (1.8), (1.9), and (1.10) form a system of 15 equations underpinning the Kirchhoff–Love model to be solved along with boundary conditions for 15 unknowns: three displacements, six strain variables, and six stress resultants. Kinematics and kinetics remain the same for homogeneous or heterogeneous plates made of isotropic or anisotropic materials. The only difference is that the stiffness matrix in Eq. (1.10) could be fully populated for anisotropic homogeneous or heterogeneous panels.

Although the Kirchhoff–Love model was developed based on a priori assumptions, these assumptions are not strictly necessary. Advanced modeling techniques, such as the mechanics of structure genome (MSG), can derive this model, as will be presented later in Chapter 5. Thus, in this book, the Kirchhoff–Love model refers to the system of 15 field variables governed by the 15 equations in Eqs. (1.8), (1.9), and (1.10), regardless of the derivation method.

1.6.3 Reissner–Mindlin Model

When the thickness of the panel is not sufficiently small compared to its in-plane dimensions, or when the deformation cannot be adequately captured using only the classical strain measures, a more refined model is needed. The Reissner–Mindlin model, named after the independent contributions of Reissner and Mindlin, addresses these limitations and extends the capabilities of the Kirchhoff–Love model.

The kinematics of the Reissner–Mindlin model includes three displacement variables ($\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$), two rotation variables (θ_1, θ_2), and eight strain variables. These strain variables

include the in-plane strains ($\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$), the curvatures ($\kappa_{11}, \kappa_{22}, \kappa_{12}$), and the transverse shear strains ($\epsilon_{13}, \epsilon_{23}$). For a plate, the strain–displacement relations are given by

$$\begin{aligned}
&\epsilon_{11} = \frac{\partial \bar{u}_1}{\partial x_1}, \qquad \epsilon_{22} = \frac{\partial \bar{u}_2}{\partial x_2}, \qquad 2\epsilon_{12} = \frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1} \\
&\kappa_{11} = \frac{\partial \theta_2}{\partial x_1}, \qquad \kappa_{22} = -\frac{\partial \theta_1}{\partial x_2}, \qquad 2\kappa_{12} = \frac{\partial \theta_2}{\partial x_2} - \frac{\partial \theta_1}{\partial x_1} \\
&2\epsilon_{13} = \frac{\partial \bar{u}_3}{\partial x_1} + \theta_2, \qquad 2\epsilon_{23} = \frac{\partial \bar{u}_3}{\partial x_2} - \theta_1
\end{aligned} \tag{1.12}$$

The kinetics of the Reissner–Mindlin model involves eight stress resultants, including in-plane force resultants (N_{11}, N_{22}, N_{12}), moment resultants (M_{11}, M_{22}, M_{12}), and transverse shear force resultants (N_{13}, N_{23}). These stress resultants are governed by the following five equilibrium equations:

$$\begin{aligned}
&\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = 0 \\
&\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = 0 \\
&\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + p_3 = 0 \\
&\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - q_1 - N_{23} = 0 \\
&\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} + q_2 - N_{13} = 0
\end{aligned} \tag{1.13}$$

The constitutive relations for the Reissner–Mindlin model are expressed as:

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \\ M_{11} \\ M_{22} \\ M_{12} \\ N_{13} \\ N_{23} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & Y_{11} & Y_{12} \\ A_{12} & A_{22} & A_{26} & B_{21} & B_{22} & B_{26} & Y_{21} & Y_{22} \\ A_{16} & A_{26} & A_{66} & B_{61} & B_{62} & B_{66} & Y_{31} & Y_{32} \\ B_{11} & B_{21} & B_{61} & D_{11} & D_{12} & D_{16} & Y_{41} & Y_{42} \\ B_{12} & B_{22} & B_{62} & D_{12} & D_{22} & D_{26} & Y_{51} & Y_{52} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & Y_{61} & Y_{62} \\ Y_{11} & Y_{21} & Y_{31} & Y_{41} & Y_{51} & Y_{61} & G_{11} & G_{12} \\ Y_{12} & Y_{22} & Y_{32} & Y_{42} & Y_{52} & Y_{62} & G_{12} & G_{22} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \\ \kappa_{11} \\ \kappa_{22} \\ 2\kappa_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{Bmatrix} \tag{1.14}$$

This 8×8 matrix is called the stiffness matrix, and its inverse is the compliance matrix for the Reissner–Mindlin model. G_{ij} ($i = 1, 2$; $j = 1, 2$) denotes the transverse shear stiffness, and Y_{ij} ($i = 1, \ldots, 6$; $j = 1, 2$) denotes the coupling stiffness among the classical strain measures and transverse shear strains. The matrices A, B, and D may differ from those in Eq. (1.10) due to possible nonzero Y_{ij} terms.

Equations (1.12), (1.13), and (1.14) form a system of 21 equations underpinning the Reissner–Mindlin model to be solved with appropriate boundary conditions for 21 unknowns: five displacement variables, eight strain variables, and eight stress resultants. Kinematics and kinetics remain unchanged regardless whether the panel is homogeneous or heterogeneous, made of isotropic or anisotropic materials. The only difference lies in the stiffness matrix in Eq. (1.14), which can be fully populated for anisotropic homogeneous or heterogeneous panels.

Although the Reissner–Mindlin model was originally developed using a priori assumptions, these assumptions are not strictly necessary. Advanced modeling techniques, such as MSG, can also derive this model. Thus, the Reissner–Mindlin model described here refers to the system of 21 field variables governed by the 21 equations in Eqs. (1.12), (1.13), and (1.14), independent of the derivation method.

1.6.4 Euler–Bernoulli Model

The Euler–Bernoulli model is used to analyze slender structures where one dimension (described by the axial coordinate x_1) is much larger than the other two dimensions (described by the cross-sectional coordinates x_2 and x_3).

The kinematics of the Euler–Bernoulli beam model includes four displacement variables ($\bar{u}_1, \bar{u}_2, \bar{u}_3, \theta_1$) with $\bar{u}_1, \bar{u}_2, \bar{u}_3$ describing displacements in three directions and θ_1 the twist angle. This model also involves four strain variables (also known as classical beam strain measures): the axial strain (ϵ_{11}), the twist rate (κ_{11}), and the curvatures (κ_{12}, κ_{13}) about x_2 and x_3, respectively. The strain–displacement relations are given as

$$\epsilon_{11} = \bar{u}_1', \qquad \kappa_{11} = \theta_1', \qquad \kappa_{12} = -\bar{u}_3'', \qquad \kappa_{13} = \bar{u}_2'' \tag{1.15}$$

Here, the prime denotes differentiation with respect to the axial coordinate x_1.

The kinetics of the Euler–Bernoulli beam model involves four stress resultants (F_1, M_1, M_2, M_3) with F_1 denoting the axial force and M_1, M_2, M_3 the moments about the three directions, respectively. These stress resultants are governed by the following four equilibrium equations:

$$F_1' + p_1 = 0, \quad M_1' + q_1 = 0, \quad M_2'' + p_3 + q_2' = 0, \quad M_3'' - p_2 + q_3' = 0 \tag{1.16}$$

Here, p_1, p_2, p_3 and q_1, q_2, q_3 are equivalent forces and moments per unit length in three directions, respectively, distributed along x_1.

The constitutive relations for the Euler–Bernoulli beam model are expressed as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} C_{11}^b & C_{14}^b & C_{15}^b & C_{16}^b \\ C_{14}^b & C_{44}^b & C_{45}^b & C_{46}^b \\ C_{15}^b & C_{45}^b & C_{55}^b & C_{56}^b \\ C_{16}^b & C_{46}^b & C_{56}^b & C_{66}^b \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \kappa_{11} \\ \kappa_{12} \\ \kappa_{13} \end{Bmatrix} \tag{1.17}$$

This 4×4 matrix is called the stiffness matrix and its inverse is the compliance matrix for the Euler-Bernoulli model.

For isotropic homogeneous beams, the constitutive relations for the Euler–Bernoulli model are commonly expressed using *engineering beam properties* as

$$F_1 = EA\epsilon_{11}, \quad M_1 = GJ\kappa_{11}, \quad M_2 = EI_{22}\kappa_{12}, \quad M_3 = EI_{33}\kappa_{13} \tag{1.18}$$

Here, EA denotes the extension stiffness, GJ the torsion stiffness, and EI_{22} and EI_{33} the bending stiffness about x_2 and x_3, respectively. For a single isotropic material, E is Young's modulus, G is the shear modulus, A is the cross-sectional area, J is the torsion constant, and I_{22} and I_{33} are the area moments of inertia about x_2 and x_3, respectively. If the beam is made of multiple materials (possibly anisotropic), the extension stiffness EA can not be interpreted as a single Young's modulus times the cross-sectional area (i.e., $E \times A$). Because each material has its own Young's moduli along different directions, it is unlikely to obtain the extension stiffness using the Young's modulus of a real material multiplied by the area. Instead, it must be calculated using a cross-sectional analysis code such as VABS considering the material properties and geometry. The same comments apply to GJ, EI_{22}, and EI_{33}.

For isotropic homogeneous beams, the constitutive relations in Eq. (1.18) can also be written in matrix form:

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_{22} & 0 \\ 0 & 0 & 0 & EI_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \kappa_{11} \\ \kappa_{12} \\ \kappa_{13} \end{Bmatrix} \tag{1.19}$$

Equation (1.19) implies that the four classical deformation modes (extension, torsion, and bending in two directions) are completely decoupled. This is true for isotropic homogeneous beams if x_2 and x_3 have the origin at the tension center of the cross-section and align with the principal bending axes of the beam. If these conditions are not met, even for isotropic homogeneous beams, coupling occurs, and off-diagonal terms in the stiffness matrix may be nonzero. Therefore, the formulas in Eq. (1.18) should not be used blindly. Proper offsets, including both positional and orientational offsets, are necessary.

Equations (1.15), (1.16), and (1.17) form a system of 12 equations underpinning the Euler–Bernoulli model to be solved along with appropriate boundary conditions for 12 unknowns: four displacement variables, four strain variables, and four stress resultants. Kinematics and kinetics remain unchanged for homogeneous or heterogeneous beams made of isotropic or anisotropic materials. The only difference is that the beam stiffness matrix in Eq. (1.17) could be fully populated for anisotropic homogeneous or heterogeneous beams.

Although the Euler–Bernoulli model was developed using a priori assumptions (e.g., cross-sections remain rigid, planar, and perpendicular to the reference line), such assumptions are not strictly necessary. Advanced modeling techniques, such as MSG, can derive the same model. Thus, the Euler–Bernoulli model described here only refers to the system of 12 field variables governed by the 12 equations in Eqs. (1.15), (1.16), and (1.17), regardless of the derivation method.

1.6.5 Timoshenko Model

When the cross-section of a beam is not very small compared to its length, or when deformation cannot be fully captured using extension, twist, and two bending curvatures, the Euler–Bernoulli model becomes inadequate. In such cases, a more refined model, such as the Timoshenko model, is required.

The kinematics of the Timoshenko beam model includes six displacement variables ($\bar{u}_1, \bar{u}_2, \bar{u}_3, \theta_1, \theta_2, \theta_3$) with $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}_3$ describing displacements in three directions and θ_1, θ_2, and θ_3 describing rotations about the respective coordinate axes. The Timoshenko beam model features six strain variables: the axial strain (ϵ_{11}), the transverse shear strains ($\epsilon_{12}, \epsilon_{13}$), the twist rate ($\kappa_{11}$), and the curvatures ($\kappa_{12}, \kappa_{13}$). The strain–displacement relations are given as

$$\begin{aligned} &\epsilon_{11} = \bar{u}_1', \qquad 2\epsilon_{12} = -\theta_3 + \bar{u}_2', \qquad 2\epsilon_{13} = \theta_2 + \bar{u}_3' \\ &\kappa_{11} = \theta_3', \qquad \kappa_{12} = \theta_2', \qquad \kappa_{13} = \theta_3' \end{aligned} \tag{1.20}$$

The kinetics of the Timoshenko beam model contains six stress resultants F_1, F_2, F_3, M_1, M_2, and M_3, with F_2 and F_3 denoting the transverse shear forces, governed by the following six equilibrium equations:

$$\begin{aligned} &F_1' + p_1 = 0, \qquad F_2' + p_2 = 0, \qquad F_3' + p_3 = 0 \\ &M_1' + q_1 = 0, \qquad M_2' - F_3 + q_2 = 0, \qquad M_3' + F_2 + q_3 = 0 \end{aligned} \tag{1.21}$$

The constitutive relations for the Timoshenko beam model are

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} C_{11}^b & C_{12}^b & C_{13}^b & C_{14}^b & C_{15}^b & C_{16}^b \\ C_{12}^b & C_{22}^b & C_{23}^b & C_{24}^b & C_{25}^b & C_{26}^b \\ C_{13}^b & C_{23}^b & C_{33}^b & C_{34}^b & C_{35}^b & C_{36}^b \\ C_{14}^b & C_{24}^b & C_{34}^b & C_{44}^b & C_{45}^b & C_{46}^b \\ C_{15}^b & C_{25}^b & C_{35}^b & C_{45}^b & C_{55}^b & C_{56}^b \\ C_{16}^b & C_{26}^b & C_{36}^b & C_{46}^b & C_{56}^b & C_{66}^b \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ \kappa_{11} \\ \kappa_{12} \\ \kappa_{13} \end{Bmatrix} \tag{1.22}$$

This 6×6 matrix is called the stiffness matrix and its inverse is the compliance matrix for the Timoshenko model. Note that C_{ij}^b, for $i = 1, 4, 5, 6$; $j = 1, 4, 5, 6$ in this stiffness matrix could differ from those in Eq. (1.17) due to possible coupling between transverse shear strains and classical beam strain measures.

For isotropic homogeneous beams, the following simplified relations are often used:

$$\begin{aligned} &F_1 = EA\epsilon_{11}, \quad F_2 = c_2 GA(2\epsilon_{12}), \quad F_3 = c_3 GA(2\epsilon_{13}) \\ &M_1 = GJ\kappa_{11}, \quad M_2 = EI_{22}\kappa_{12}, \quad M_3 = EI_{33}\kappa_{13} \end{aligned} \tag{1.23}$$

with c_2 and c_3 denoting the shear correction factors. These constitutive relations can be written in matrix form with a diagonal matrix:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2GA & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3GA & 0 & 0 & 0 \\ 0 & 0 & 0 & GJ & 0 & 0 \\ 0 & 0 & 0 & 0 & EI_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & EI_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ \kappa_{11} \\ \kappa_{12} \\ \kappa_{13} \end{Bmatrix} \tag{1.24}$$

This equation indicates that the six deformation modes of the Timoshenko model are completely decoupled. The diagonal stiffness matrix in Eq. (1.24) is achievable under the following four conditions:

1. The origin of x_2 and x_3 is at the tension center of the cross-section.
2. x_2 and x_3 align with the principal bending axes of the beam.
3. The origin of x_2 and x_3 is at the shear center.
4. x_2 and x_3 align with the principal shear axes of the beam.

For the Euler–Bernoulli beam model, only the first two conditions are necessary for achieving a diagonal stiffness matrix in Eq. (1.19). However, for the Timoshenko model, all four conditions must be satisfied simultaneously, which is possible only for certain idealized cross-sections where the shear center coincides with the tension center, and the principal shear axes align with the principal bending axes. For most beams, even those made from isotropic materials, achieving a diagonal stiffness matrix for the Timoshenko model is generally infeasible. In such cases, some of the six deformation modes are coupled. Therefore, proper offsets—both positional and orientational—must be applied before using the formulas in Eq. (1.23).

Equations (1.20), (1.21), and (1.22) form a system of 18 equations to be solved along with appropriate boundary conditions for 18 unknowns: six displacement variables, six strain variables, and six stress resultants. Kinematics and kinetics remain unchanged for homogeneous or heterogeneous beams made of isotropic or anisotropic materials. The only difference lies in the beam stiffness matrix in Eq. (1.22), which could be fully populated for anisotropic homogeneous or heterogeneous beams.

Although the Timoshenko beam model was originally developed based on a set of a priori assumptions—commonly called the Timoshenko assumptions—these include the cross-section being rigid in its own plane, remaining plane during deformation, and experiencing special stress states. These assumptions are not strictly necessary for deriving the Timoshenko model. Using more advanced modeling techniques, such as MSG, this model can be derived without relying on these assumptions. Thus, the Timoshenko beam model described here is defined by the 18 unknowns governed by the 18 equations in Eqs. (1.20), (1.21), and (1.22), regardless of the derivation method used.

1.6.6 Vlasov Model

For thin-walled beams with open sections, cross-sectional distortion varies significantly along the beam axis. As a result, the Euler–Bernoulli beam model requires refinement into the Vlasov beam model, which incorporates the derivative of the twist rate as an additional strain variable. While the displacement variables remain the same as in the Euler–Bernoulli beam model, the strain variables now include ϵ_{11}, κ_{11}, κ_{12}, κ_{13}, and κ'_{11}. These are defined by the following strain–displacement relations:

$$\epsilon_{11} = \bar{u}'_1, \qquad \kappa_{11} = \theta'_1, \qquad \kappa_{12} = -\bar{u}''_3, \qquad \kappa_{13} = \bar{u}''_2, \qquad \kappa'_1 = \theta''_1 \tag{1.25}$$

The kinetic variables of the Vlasov beam model include five stress resultants $F_1, M_1, M_2, M_3, M_\omega$, where M_ω is referred to as the bi-moment, conjugate to κ'_{11}. These stress resultants are governed by the equilibrium equations in Eq. (1.16), with a revised form for the second equation as

$$M'_1 - M''_\omega + q_1 = 0 \tag{1.26}$$

The constitutive relations for the Vlasov beam model are expressed as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \\ M_\omega \end{Bmatrix} = \begin{bmatrix} C^b_{11} & C^b_{14} & C^b_{15} & C^b_{16} & C^b_{17} \\ C^b_{14} & C^b_{44} & C^b_{45} & C^b_{46} & C^b_{47} \\ C^b_{15} & C^b_{45} & C^b_{55} & C^b_{56} & C^b_{57} \\ C^b_{16} & C^b_{46} & C^b_{56} & C^b_{66} & C^b_{67} \\ C^b_{17} & C^b_{47} & C^b_{57} & C^b_{67} & C^b_{77} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \kappa_{11} \\ \kappa_{12} \\ \kappa_{13} \\ \kappa'_{11} \end{Bmatrix} \tag{1.27}$$

This 5×5 symmetric matrix is referred to as the stiffness matrix, while its inverse is the compliance matrix for the Vlasov model. Note that C^b_{ij}, for $i = 1, 4, 5, 6;\ j = 1, 4, 5, 6$ in this stiffness matrix may differ from those in Eqs. (1.17) and (1.22), owing to possible coupling between κ'_{11} and the classical beam strain measures.

Equations (1.25), (1.16) (modified with (1.26)), and (1.27) constitute a system of 14 equations. These equations are solved alongside appropriate boundary conditions for 14 unknowns: four displacement variables, five strain variables, and five stress resultants. The kinematics and kinetics remain unchanged for homogeneous or heterogeneous beams made of isotropic or anisotropic materials. The only difference lies in the beam stiffness matrix in Eq. (1.27), which may be fully populated for anisotropic homogeneous or heterogeneous beams.

Although the Vlasov beam model was originally developed using set of a priori assumptions, these assumptions are not strictly necessary. By employing advanced modeling techniques such as MSG, the Vlasov beam model can be derived without reliance on traditional assumptions. The Vlasov model described here refers to the system of the aforementioned 14 equations to be solved for the 14 unknowns.

1.7 Composite Materials

The primary focus of this book is on the mechanics of anisotropic heterogeneous structures and materials, with conventional composite materials serving as representative examples. While we provide a concise introduction to conventional composite materials, topics such as the material science of constituents and manufacturing methods are only briefly touched upon. Readers seeking more comprehensive information on these aspects are encouraged to consult specialized books on these aspects of composite materials.

According to most textbooks on composites, a *composite* material is traditionally defined a material consisting of two or more different constituents, each idealized as a homogeneous material. By this definition, a composite material is inherently heterogeneous, as the material properties vary among the different constituents within the composite continuum.

Composites are often anisotropic, exhibiting different behaviors along different directions. However, composites can also be designed to achieve isotropic behavior, such as randomly dispersed particle-reinforced composites and quasi-isotropic composite laminates (isotropic only in in-plane directions). A composite material may exhibit anisotropic behavior even when composed of isotropic constituents, such as aluminum reinforced with tungsten fibers. Conversely, a composite material can exhibit isotropic behavior even when made from anisotropic constituents, such as polymers reinforced with randomly distributed short carbon fibers.

Composites are as ancient as nature. Most natural materials, such as wood and human bones, are composites. Even man-made composites are not new, as evidenced in what is written in the Holy Bible (Exodus 5:7, New International Version): "You are no longer to supply the people with straw for making bricks; let them go and gather their own straw." Here, the straw acts as a reinforcing material, improving the properties of the mud brick—a primitive composite. Although composites have existed for millennia, the field of composites as an engineering discipline is relatively recent. The invention of high-performance fibers in the 1960s marked a turning point, motivating engineers and scientists to explore the immense potential of composites in various engineering applications. These high-performance fiber-based materials are often referred to as advanced composite materials. As sketched in Figure 1.7, a composite consists of two main phases:

1. Reinforcing Phase: This can take the form of particles, fibers, or other geometries. The reinforcing phase is typically much stiffer and stronger than the matrix, serving as the primary load carrier.
2. Matrix Phase: The matrix binds the reinforcing elements together, transfers the load to the reinforcing phases, and provides protection. In addition to support the reinforcement, the matrix also shares part of the load.

In some cases, an *interphase* may form between the reinforcing phase and the matrix due to chemical or processing effects. While the interphase is often modeled as a zero-thickness interface, modeling it with finite thickness requires a very fine mesh due to its typically thin geometry.

Depending on the form of the reinforcements, composites can be categorized as either particle-reinforced composites (PRCs) or fiber-reinforced composites (FRCs). Because fibers can be used in a continuous form or chopped into various lengths, FRCs can be further

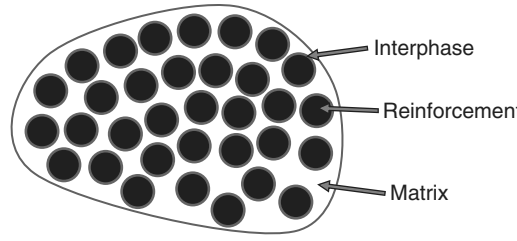


Figure 1.7 Typical phases of a composite material.

Table 1.1 Typical properties of fibers used in advanced composites.

Material	Fiber diameter (μm)	Density (g/cm^3)	Stiffness (GPa)	Poisson's ratio (No unit)	Strength (MPa)	CTE (μ/°C)
E-Glass	5–20	2.54	77	0.22	3500	5.4
S-Glass	8–14	2.48	85	0.23	4500	1.6
Carbon (IM*)	5–8	1.80	230	0.20	2500	−0.3
Carbon (HM*)	5–10	1.80	350	0.20	2600	−0.8
Boron	140	2.65	385	0.21	3500	8.3
Kevlar-49	12	1.45	130	0.34	3800	−2.0

*IM stands for Intermediate Modulus; HM stands for High Modulus.

categorized as continuous fiber-reinforced composites (CFRCs) or short fiber-reinforced composites (SFRCs), respectively. A particular CFRC is supplied in a tape form with fibers aligned in a single direction. This type of composite is called unidirectional fiber-reinforced composites (UDFRCs).

Composites can also be categorized according to the matrix material, such as polymer matrix composites (PMCs), metal matrix composites (MMCs), and ceramic matrix composites (CMCs) if the matrix material is polymer, metal, or ceramic, respectively. It is also noted that a composite material could have more than one type of reinforcement. It could contain different fibers or particles, or reinforcements in both particle and fiber forms. Such composite materials are also commonly called hybrid composites.

Advanced composites frequently use fibers made from materials like glass, boron, carbon, and Kevlar. These fibers are often made with carefully controlled, sophisticated manufacturing processes, resulting in fine diameters that improve material properties. As summarized in Table 1.1, except for boron fibers, many fibers have a diameter of a few microns, which is approximately ten times smaller than a human hair (see Figure 1.8).

Fibers, due to their small diameters, contain fewer flaws compared to bulk forms of the same material. As a result, they exhibit significantly greater stiffness and strength, especially along the fiber direction. Fiber extrusion typically aligns material along the fiber axis, further enhancing material properties. The properties listed in Table 1.1—including

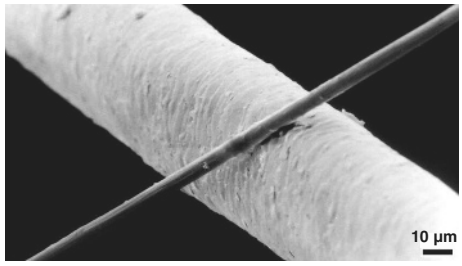


Figure 1.8 Relative size of carbon fiber and human hair.

Table 1.2 Typical properties of conventional metals.

Material	Density (g/cm^3)	Stiffness (GPa)	Poisson's ratio (No unit)	Strength (MPa)	CTE ($\mu/°C$)
Steel	7.80	210	0.32	340–2100	12.8
Aluminum	2.70	70	0.33	140–620	23.4
Titanium	4.46	110	0.36	920	8.8

density, stiffness, strength, and CTE—are measured along the fiber direction, as transverse properties are challenging to evaluate. Measuring the compressive strength and stiffness of fibers is also difficult, so the strength values provided in the table represent tensile strength in the fiber direction. Compressive stiffness is often assumed to equal tensile stiffness.

The CTE of fibers along its length is typically very small and may even be negative, as shown in Table 1.1. For comparison purposes, Table 1.2 lists the typical properties of conventional metals like aluminum, steel, and titanium, which are often considered isotropic. We can observe that metals, particularly steel, are much heavier than composite fibers and exhibit larger CTE values.

Composite fibers possess superior *specific stiffness* (stiffness-to-density ratio) and *specific strength* (strength-to-density ratio) compared to metals, enabling significantly lighter structures to bear equivalent loads. This advantage makes composites particularly attractive for weight-sensitive applications, such as aerospace systems or fuel-efficient cars.

Continuous fibers are supplied in two main forms: dry fibers and pre-impregnated fibers (prepregs). Dry fibers lack any matrix precursor and are available as tow (fiber bundles), yarn (twisted fiber bundles), and fabric (a unit with fibers running in different directions). Prepregs, on the other hand, are fibers pre-impregnated with a matrix binder, commonly supplied as tape (e.g., UDFRC) or fabric.

Common matrix materials include thermoset resins such as epoxy, polyester, and polyimide, and thermoplastic resins such as polyether ether ketone (PEEK). These polymer-based materials have a limited temperature range for use. Polyimide-based composites can sustain temperatures up to 350° C. For applications requiring higher temperature performance, metal or ceramic matrices are used. Table 1.3 lists typical properties of some common matrix materials. Note that matrices are often considered isotropic. Given the

Table 1.3 Typical properties of matrix materials used in advanced composites.

Material	Density (g/cm^3)	Stiffness (GPa)	Poisson's Ratio (No unit)	Strength (MPa)	CTE ($\mu/°C$)
Epoxy	1.20	4.6	0.36	58.6	63
Polyester	1.30	2.8	0.30	65	130
Polyimide	1.46	3.5	0.35	103	36
PEEK	1.32	5.0	0.30	96	78
Copper	8.90	117	0.33	400	17
Silicon carbide	3.2	400	0.25	310	4.8

sensitivity of material properties to manufacturing processes, the values provided in this table are representative. For real design and analysis, it is recommended to use supplier-provided data that are carefully characterized for a specific material.

Fiber and matrix can be combined and processed to form composites through various manufacturing methods, including:

1. Autoclave Molding: High precision and quality, but costly.
2. Injection Molding: High production rates for complex geometries.
3. Compression Molding: Efficient for simple, high-strength parts.
4. Filament Winding: Ideal for cylindrical structures like tanks and pipes.
5. Resin Transfer Molding (RTM): Cost-effective for mid-to-high production.
6. Continuous Pultrusion: Suitable for long, uniform profiles.

The choice of a particular method depends on factors such as the part geometry, cost, and the desired production rate.

This introduction focuses on fiber-reinforced composites, which are the most commonly used composite materials at the time of writing. However, the mechanics covered in this book also apply to a broader range of anisotropic and heterogeneous materials, such as:

1. Metallic Foams: Heterogeneous due to pores, potentially anisotropic depending on the microstructure.
2. Functionally Graded Materials: Properties change continuously with position.
3. 3D-printed materials: Heterogeneous due to varying thermal histories during manufacturing. Even if a 3D-printed part is composed of a single material, such as aluminum, it is heterogeneous because of spatial variations thermal history, resulting in different material properties.

While composite laminates (layered structures made of composites) are the primary focus of composite structures studied in this book, the principles we will explore apply to other structures with general anisotropy and heterogeneity, including corrugated structures, sandwich structures with truss or honeycomb cores, and stiffened structures.

1.8 Benefits of Using Composites

Composites have consistently outperformed traditional materials in numerous aspects:

1. Lightweight Structures: Composites are highly attractive for creating lightweight structures due to their high specific stiffness and strength.
2. Engineered Microstructure: The microstructure of a composite can be tailored to meet specific application demands.
3. Design Freedom: Engineers have significantly more design freedom with composites than with traditional monolithic materials. They can choose from various constituent materials and fabrication methods, adjust constituent volume fractions, and vary the number of layers, layer orientations and thickness, and processing parameters.
4. Anisotropic Properties: Composites can be anisotropic, allowing engineers to achieve desired properties in a specific direction without over-designing in other directions.
5. Fatigue Life: Composites generally have much better fatigue life than metals, as demonstrated by the success of composite rotor blades in the helicopter industry. Fatigue life is crucial for structures experiencing cyclic loading, such as aircraft, helicopters, wind turbines, bridges, and industrial equipment components.
6. Thermal Stability: Engineers can design composites to have very small or zero thermal expansion for dimensional stability in fluctuating temperature environments, such as space antennae and highway bridges.
7. Thermal Stress Reduction: Composites can minimize CTE mismatches to reduce thermal stresses generated within a part made of different materials due to temperature changes.
8. Corrosion Resistance: Composites can be more resistant to moisture and chemical corrosion, leading to lower maintenance costs for structures.
9. Cost-Effective Fabrication: The fabrication of composites can be more cost-effective than metals, with minimal material waste and the ability to produce large parts directly, reducing the number of parts needed for assembly.
10. Elastic Coupling : Composites offer elastic coupling, not available in isotropic materials. For example, a structure made of an isotropic homogeneous material will elongate along the direction it is pulled, whereas a composite structure might bend, twist, or shear when pulled. A successful example is the forward-swept wings of the Grumman X-29A, which were aeroelastically tailored using composites.

The applications of composites have growing exponentially since the 1960s. With the success of the Boeing 787 and Airbus 350, which feature extensive use of composites, these materials are now widely used in aerospace systems. Composites are also making significant inroads into other industries, such as athletic equipment, infrastructure, automobiles, and large wind turbines.

The primary limitations to the broader application of composites are the lack of engineers with the necessary knowledge and experience to confidently design and manufacture with these materials and their relatively high cost compared to conventional materials. However, with advancements in manufacturing methods, more predictive engineering tools, improved knowledge within the engineering community, increased material usage

volume, and decreased costs, composites are becoming more affordable and are expected to be used in many more new products.

1.9 Mechanics of Composite Materials

Mechanics of composite materials focus on methods to predict stresses, strains, deformation, and failure of composite materials and structures under external loads and environmental changes such as temperature and moisture. The predicted deformation, stresses, and strains can guide the determination of whether a structure can safely perform its intended design purpose. A structure fails if it deforms excessively or fractures.

Similar to other continuum mechanics models, a mechanics of composite materials model consists of three main components:

1. **Kinetics:** Kinetics deals with the forces and stresses generated within the material. Stress is a concept derived from force. The governing equations, known as the *equations of motion*, can be directly derived from Newton's laws of motion. For static behavior, these equations become the equilibrium equations.
2. **Kinematics:** Kinematics uses differential geometry to describe the deformation and motion of the material in terms of displacement and strain fields. Strain describes the relative motion of material points within the body. The governing equations, known as the *compatibility equations*, are formulated based on the fact that the displacement field is unique and can be integrated from a given strain field.
3. **Constitutive Relations:** The concepts of stress and strain, along with their corresponding equations, apply to any continuous body. To close the formulation, constitutive relations are needed to relate the stress and strain fields. These relations are typically assumed to follow a specific functional form, with constants determined experimentally or through *constitutive modeling*. Constitutive relations also describe how material failure initiates and evolves. For composites, the material properties are anisotropic, making their constitutive relations more complex than those of isotropic materials.

If a material returns to its original shape and position upon the removal of external loading, it is considered elastic and governed by the theory of elasticity.

1. A problem is *geometrically linear* if the strain field is linearly defined in terms of the displacement field. If not, it is classified as geometrically nonlinear. Typical geometrically nonlinear problems include large deformation, structural instability, and limit cycle vibration.
2. A problem is *materially linear* if the constitutive models express stress-strain relations linearly. Otherwise, it is material nonlinear, which includes phenomena such as plasticity, viscoelasticity, and energy dissipation with hysteresis.

It is possible to encounter geometrically linear and material nonlinear problems, such as linear plasticity. While real engineering problems often involve both geometric and material nonlinearities, they can reasonably be approximated as geometrically and materially linear problems, especially in early design stages. This simplification greatly aids both solution generation and interpretation.

Most engineering analyses, particularly during early design stages, assumed geometric and material linearity. For composites, these problems are rigorously addressed using the theory of linear anisotropic elasticity, which emphasizes the anisotropic nature of constitutive relations. Many mechanics models for composites are constructed based on this theory, applying the principles of continuum mechanics to anisotropic solids such as composites.

Most of this book focuses on geometric and material linearities for composite materials and structures. However, many dimensionally reducible structures exhibit large deformations, necessitating the inclusion of geometrical nonlinearity in the analysis. Fortunately, these structures typically experience large global rotations and displacements, while local rotations and displacements remain small. In Chapter 7, we utilize MSG to rigorously split the original, complex 3D geometrically nonlinear problem into a linear constitutive modeling procedure and a geometrically exact global analysis. The MSG-based constitutive modeling provides a robust framework for effectively analyzing such scenarios.

1.10 Challenges for Modeling Composites

Mechanics of composite materials remain essentially the same as other solid mechanics subjects. Both kinetics and kinematics remain unchanged, regardless of whether the solid is a composite or an isotropic homogeneous solid. The key distinction lies in the constitutive relations, where the primary challenges in composites modeling arise. Specifically, these challenges are rooted in anisotropy and heterogeneity.

1.10.1 Anisotropy

Most composites are *anisotropic*, requiring a larger set of material constants to describe their constitutive relations compared to isotropic materials, significantly increasing their complexity. Additionally, anisotropy makes material behavior less intuitive, as most materials encountered in daily life are isotropic or designed to behave isotropically. While materials like wood exhibit anisotropic properties, we often focus only on their behavior in specific directions.

For composites, however, their anisotropic *feature* should be leveraged in design. Unfortunately, many composite structures, such as quasi-isotropic laminates, are currently designed to minimize anisotropy, treating it as a *defect* rather than an asset. To fully exploit the advantages of composites, engineers must embrace and utilize their anisotropic characteristics intelligently.

Mathematically, handling the anisotropy of composites involves tensor algebra, which may be unfamiliar to many engineers. However, practical applications generally require only index notation and coordinate transformation, which are accessible to engineers with standard undergraduate training. Chapter 2 covers the foundational knowledge necessary to address anisotropy effectively, enabling engineers to master these concepts with relative ease.

While the mathematical aspects of anisotropy are straightforward to learn, the less intuitive behavior of anisotropic materials poses a greater challenge. To overcome this, engineers must think, simulate, test, and visualize the anisotropic responses of composites

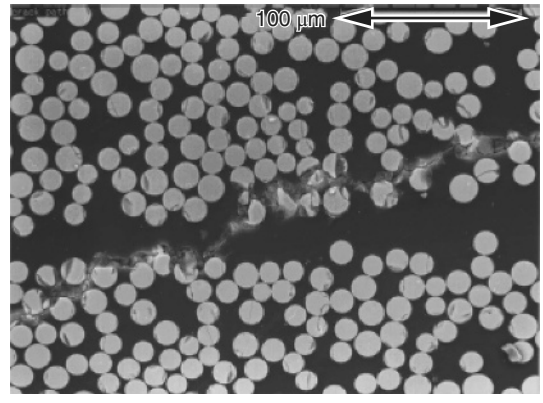


Figure 1.9 Microstructure of a typical UDFRC.

regularly. This iterative process can help develop an intuitive understanding of anisotropic behaviors. Though demanding, this challenge is not insurmountable.

1.10.2 Heterogeneity

Composites, by definition, are *heterogeneous*, representing a significant challenge for mathematical modeling. As shown in Figure 1.9, the microstructure of a typical UDFRC comprises both fiber and matrix, with fibers distributed in a random pattern. While experiments can characterize composite behavior without accounting for the microstructural details—similar to how the continuum hypothesis neglects the atomic details of homogeneous materials—this approach is often expensive and time-consuming. The vast design space available for composites further complicates experimental exploration.

An alternative is to employ models, including both *micromechanics* and *structural mechanics* models, to approximate composite behavior by capturing key microstructural details. Micromechanics models allow us to replace the original heterogeneous material with an equivalent imaginary homogeneous material that matches the performance characteristics by analyzing the microstructure. Structural mechanics models then use the effective properties computed by micromechanics models to reduce the computing cost of detailed 3D structural analyses, enabling simplified 2D plate or shell analyses or 1D beam analyses.

Certain composite behaviors, such as deformation, buckling, and vibration, are relatively straightforward to predict, as they depend primarily on effective stiffness, which is less sensitive to microstructural details. For instance, the longitudinal Young's modulus of a UDFRC can be accurately predicted by almost all micromechanics models, including simple approaches like the rule of mixtures [8].

However, predicting complex behaviors such as strength, failure, fracture, and fatigue is far more challenging. These behaviors depend not only on effective stiffness but also on the local stress and strain fields, which are highly sensitive to microstructural details. This necessitates models capable of accurately predicting both effective stiffness and local stress/strain distributions based on the heterogeneous microstructure.

Furthermore, heterogeneity significantly complicates failure prediction in composite structures, which can exhibit many different failure modes. Although many theories have

been proposed, the failure of composites remains an active area of research, with ongoing efforts to develop improved models. Traditionally, failure analysis of composite structures (e.g., laminate) has been performed by assuming that the composite structure is made of homogeneous materials, with the strength of the composite material (e.g., lamina strength) characterized by specific failure criteria. However, current trends favor multiscale models, which propagate the failure down to the constituent level, such as fiber or matrix. For these models to be predictive, they must rigorously establish relations between the local fields at the constituent scale with the macroscopic structural responses.

1.11 Multiscale Modeling*

Due to the heterogeneity at the continuum level, analyzing composite structures often necessitates consideration of features at two or more scales. For instance, fibers used in composites typically have diameters measured in microns, while composite structures, such as aircraft wings, have dimensions measured in meters. Engineering decisions made at the fiber scale can significantly influence the performance at the system scale. Therefore, effective structural design must appropriately account for all relevant length scales, ranging from microns to meters.

In principle, the entire system could be analyzed using a direct numerical simulation (DNS) of the composite structure, capturing all microstructural details through FEA. However, this approach would require an extremely fine-scale finite element mesh, resulting in an enormous number of degrees of freedom (DOFs). For example, to analyze UDFRC while directly resolving the fiber details, the element size would need not to be larger than the diameter of the carbon fiber, typically a few microns. Such discretization would necessitate millions of DOFs for a material block as small as one mm^3. For most composite structures encountered in engineering, the DOF count would soar into trillions, far exceeding the capabilities of most available computing hardware and software.

Moreover, employing DNS to capture fiber details is often an unnecessary expenditure of engineering time and computing resources. Properly constructed models can achieve nearly the same level of accuracy for behaviors of interest to engineers at a fraction of the computational cost. The resulting savings in time and resources can then be allocated to exploring advanced designs and fostering technological innovations.

This efficiency is made possible through multiscale modeling, which can be broadly classified into two approaches: bottom-up and top-down multiscale modeling.

1.11.1 Bottom-up Multiscale Modeling

Most multiscale approaches are bottom-up, which can be illustrated through the example of analyzing composite laminates made of UDFRCs. As shown in Figure 1.10, a composite laminate consists of multiple layers of composite laminas with different fiber orientations. The composite laminas themselves are made of UDFRCs, which comprise carbon fibers bonded together using a matrix material, such as epoxy. To avoid capturing the intricate fiber-level details during laminate design, engineers typically begin by creating a unit cell (UC) that represents the microstructure. This UC contains two distinct phases, fiber

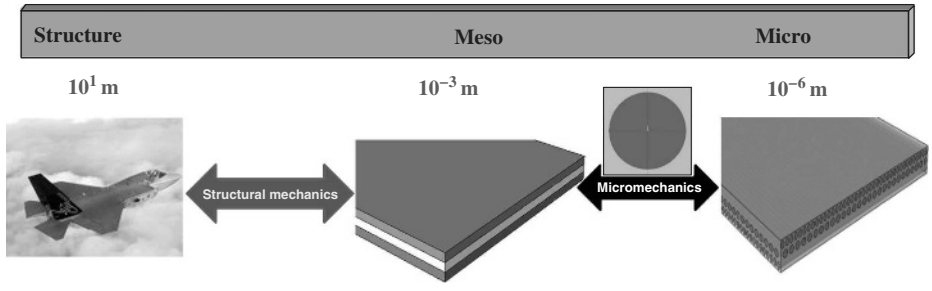


Figure 1.10 Bottom-up multiscale approach to analyze composite laminates.

and matrix, with the fiber volume fraction in the UC designed to match the overall fiber volume fraction in the composite lamina. The UC serves as an idealized representation of the composite's actual microstructure, as shown in Figure 1.9.

A micromechanics model is first employed to replace the two-phase material with a single, imaginary material possessing effective properties. After this step, each layer in the composite laminate is treated as if it were made from this imaginary, anisotropic, homogeneous material. However, even this model proves computationally intensive for effective structural design of systems made from composites. Each layer would need to be meshed with one or more 3D solid elements of FEA, while the layer thickness is typically very small (around 0.125 mm for an advanced composite lamina). A real structure may comprise numerous layers, and to accurately predict the stress and strain fields within the laminates, several solid elements would be needed in the thickness direction of each layer [9]. Clearly, such an approach remains computationally prohibitive for analyzing real composite structures in production environments.

One alternative is to employ structural mechanics models, such as CLT, to further homogenize the layerwise details along the thickness. This results in a 2D continuum with effective structural properties defined for a reference surface. The final 2D analysis over the reference surface is typically used in the structural design and analysis of composite laminates. A separate theory or custom code for the 2D analysis is unnecessary, as plate and shell elements in commercial FEA software (e.g., Abaqus) can handle it effectively when provided with the appropriate effective plate and shell properties as inputs. The focus should instead be on computing these effective structural properties for structural analysis and recovering local fields in the original heterogeneous structure based on macroscopic responses obtained from the structural analysis. This process, known as *structural modeling*, bridges the gap between the original 3D heterogeneous structure and the macroscopic structural analysis (e.g., the 2D surface analysis for composite laminates) [10].

This two-scale approach, combining micromechanics and structural mechanics, is widely used in the analysis of composite laminates. For more complex structures, additional scales may be required. For example, analyzing a composite stiffened panel made of woven laminas might involve four scales: fiber, yarn, woven composite, and panel (see Figure 1.11). The process begins by computing yarn properties based on fiber and matrix properties and yarn geometry. Next, woven composite properties are derived using yarn and matrix properties and the weave pattern. Laminate properties are then computed in terms of the woven

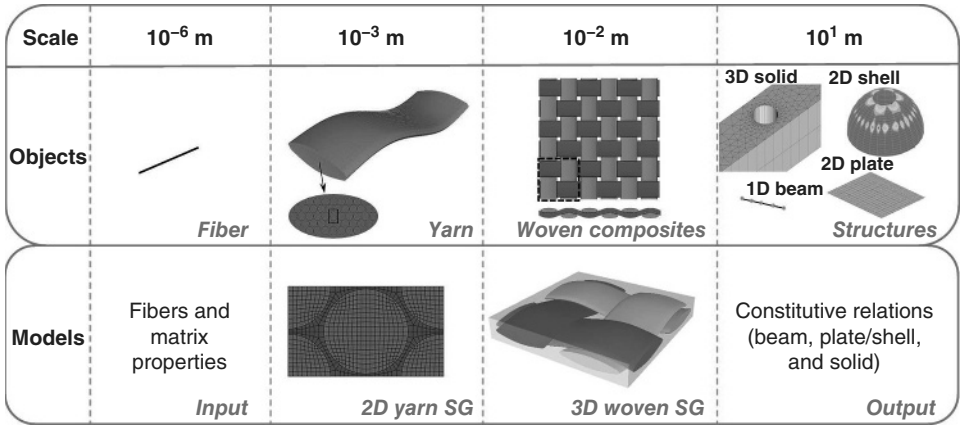


Figure 1.11 Bottom-up multiscale approach to analyze woven composites.

composite properties. Finally, a panel-level analysis is performed based on the laminate properties.

Various assumptions are often employed in the models used to bridge different scales, such as the iso-strain and iso-stress assumptions in rule of mixtures (ROM)-type micromechanics models or the Kirchhoff–Love assumptions in CLT. These assumptions might significantly influence the predictive capability of a model. Generally, reducing the number of assumptions enhances the accuracy and robustness of predictions.

Furthermore, several fundamental issues are associated with the bottom-up multiscale modeling approach. Current bottom-up multiscale modeling approaches link single-scale models through information passing between scales. Beyond the assumptions used at each individual scale, there is a more fundamental assumption inherent in these techniques, known as the *scale separation assumption*.

The scale separation assumption treats the microstructure as effectively being a material point in the macroscopic analysis. This simplification introduces artificial discontinuities at the boundaries of microstructures within the real heterogeneous structure. For example, consider a two-scale approach to model a composite laminate, as shown in Figure 1.12. In this case, the original heterogeneous panel (composed of fibers and matrix, shown in the left figure) is replaced by a panel with homogeneous layers (depicted in the right figure).

If we assume perfect bonding between laminas, the interfaces between different layers in the original structure are continuous, as the same matrix material spans across the interfaces. However, in homogenized representation (the right figure), material discontinuities are introduced at the layer interfaces where different layup orientations meet. This artificial discontinuity causes approximations in the stress and strain fields near the interfaces, directly affecting the accuracy of predictions for damage and failure in composite laminates.

The scale separation assumption is generally reasonable for predicting macroscopic behavior when the macroscopic scale is significantly larger than the microscopic scale. However, there are situations where this assumption fails, even for predicting macroscopic behavior. For example, consider analyzing a sandwich panel composed of two face sheets separated by a honeycomb core. In a typical bottom-up multiscale modeling approach,

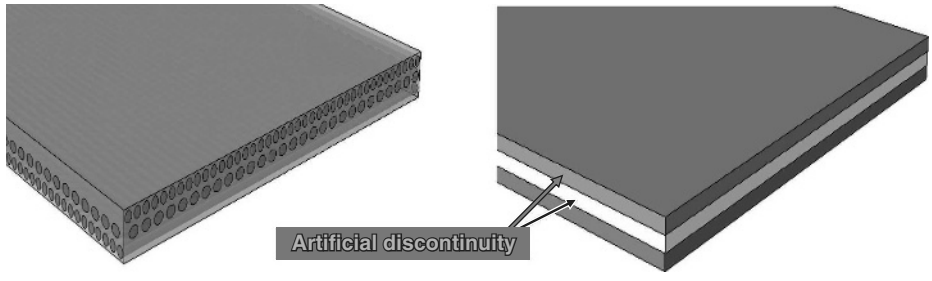


Figure 1.12 Artificial discontinuity created by the scale separation assumption.

a micromechanical analysis is first performed to derive the effective properties of the honeycomb core. The sandwich panel is then modeled as a laminate with homogeneous layers. However, the approach introduces substantial inaccuracies, even for predicting macroscopic behavior, because the honeycomb microstructure often has dimensions comparable to the panel thickness. As a result, the scale separation assumption becomes invalid. For accurate constitutive modeling in such cases, the honeycomb scale and the thickness scale must be simultaneously considered.

Additionally, determining the number of scales involved in an analysis and selecting the appropriate model for the each scale are complex challenges in conventional bottom-up multiscale modeling approaches.

1.11.2 Top-down Multiscale Modeling

The limitations of bottom-up multiscale modeling motivate a fundamentally different approach. Despite the complexity of composite structures, the mechanical behavior of engineering systems can often be represented as a combination of typical structural components such as 3D structures, plates, shells, beams, and thin-walled beams (see Figure 1.3). Each structural component has a corresponding structural model governing its behavior. According to continuum mechanics, every structural model comprises three sets of equations: kinematics, kinetics, and constitutive relations. Both kinematics and kinetics can be formulated exactly within the continuum mechanics framework for linear or nonlinear problems. These equations remain unchanged regardless of the materials used. Constitutive relations, however, are inherently approximate and material-specific. Multiscale modeling, therefore, should focus on predicting accurate constitutive relations for the relevant structural models—a process referred to as *multiscale constitutive modeling*.

Top-down multiscale modeling begins at the system level design and asks: what are the relevant behaviors for a particular design? For instance, when designing a composite helicopter rotor blade for specific dynamic performance, the relevant constitutive properties would include extension, bending, and torsion stiffness.

The second key question to consider is: at which scale should the design process begin? If the design begins with a UDFRC lamina characterized by calibrated material properties (usually called lamina constants), the blade properties must be computed in terms of these lamina constants. If the design starts from the fiber and matrix properties, a top-down multiscale modeling approach is required to compute the blade properties directly in terms of

these constituent properties. If the length scale of the fiber is much smaller than the thickness of the composite lamina, the scale separation assumption can be applied. In this case, a separate micromechanics model can be used to compute the lamina constants for the composite lamina in terms of fiber and matrix properties and the lamina's microstructure. These lamina constants then serve as inputs for calculating the blade properties. If the length scale of the fiber is similar to the thickness of the lamina, the scale separation assumption should be avoided. In this scenario, the blade properties must be computed directly from the fiber and matrix properties without relying on separate micromechanics models.

1.11.3 Mechanics of Structure Genome

The mechanics of structure genome (MSG) [11, 12] can be regarded as a top-down multiscale modeling approach. Unlike conventional bottom-up approaches, which depend exclusively on direct information passing between scales, MSG provides a formal framework that begins with homogenized behavior while still capturing detailed scale-specific phenomena relevant to design requirements.

MSG emphasizes multiscale modeling tailored to behaviors pertinent at micron and larger scales, aligning closely with design objectives. While atomistic materials modeling establishes the link between a composite's constituent properties and its atomistic or molecular structure, MSG focuses on scaling up to address behaviors and properties at the structural level, ensuring applicability for engineering design at macroscopic scales.

As illustrated in Figure 1.13, the MSG-based multiscale modeling framework begins with the original model, typically formulated using 3D continuum mechanics. The process involves identifying the *structure gene* (SG), defined as the smallest mathematical building block of the structure. Using the principle of minimum information loss (PMIL) along with the variational asymptotic method (VAM) [13], the original problem is mathematically decoupled into two components: a constitutive modeling procedure over the SG

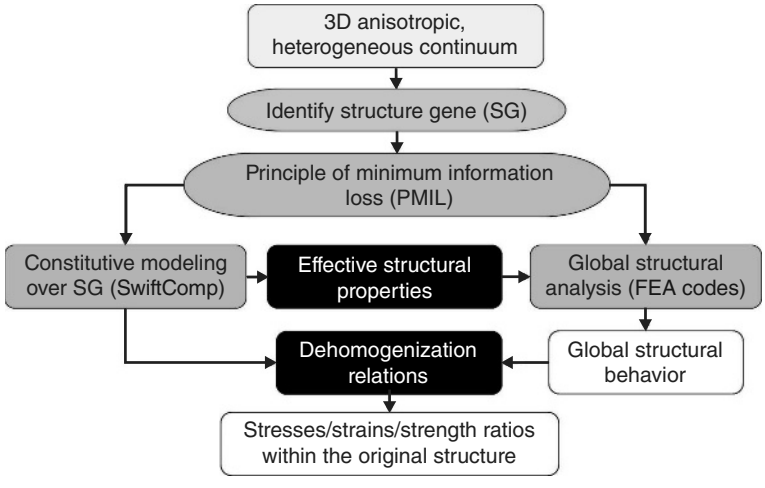


Figure 1.13 The basic idea of mechanics of structure genome (MSG).

and a corresponding macroscopic structural analysis. The constitutive modeling has been implemented in SwiftComp™, which performs homogenization to obtain the constitutive models needed for the macroscopic structural analysis, as well as dehomogenization to compute pointwise local fields within the original structure, essential for strength and failure predictions.

The structural analysis can be formulated as a geometrically exact continuum theory, with all approximations confined to the constitutive modeling. This ensures that the models' accuracy is guaranteed to be the best by PMIL. The global structural behavior can then be predicted using conventional FEA tools. Subsequently, dehomogenization relations derived from the constitutive modeling along with the global structural behavior can be used to determine local fields, such as displacements, stresses, strains, and failure status, within the original structure.

To apply MSG effectively, three fundamental questions must be answered:

1. What is the macroscopic model required for the structural analysis? The answer to this question depends on the desired behavior to capture for a particular design, computing efficiency, and the structural geometry.
2. What is the original model? The answer to this question is determined by the physics that must be accurately captured. The original model is treated as the "true" model.
3. What is the SG? The SG serves as the domain for constitutive modeling, including homogenization and dehomogenization.

Different answers to these questions lead to different multiscale models tailored to specific needs.

Viewing multiscale modeling through the MSG framework provides a systematic approach for validating multiscale models. First, the original model should be validated against experimental data to ensure that it captures the correct physics. Observations in a lab or operational environment provide the necessary reference. Second, the multiscale models derived from the original model should be validated against the original model itself, rather than experiments. Although it is a routine practice in the field to validate multiscale models using experiments, it is not necessarily the best practice. The whole objective of multiscale modeling is to avoid the computational cost of the original model. Thus, we can perform a limited number of computationally expensive DNS of the original model to validate the multiscale model derived from the original model. These simulations can be performed on artificial structures composed of representative microstructures. While DNS for real structures may be too computationally intensive to be practical, this approach ensures the robustness of multiscale models while managing computational resources effectively.

1.11.4 Structure Gene and Structure Genome

SG serves as the crucial link between the original structure, which includes microscopic details, and the macroscopic structural analysis. In this context, the terms "microstructure" and "microscopic details" are used in a general sense: any details explicitly used to describe the SG but not included in the macroscopic structural analysis are termed microscopic details. The actual structure with these microscopic details is referred to as the original structure, whereas the structure used in macroscopic structural analysis, with effective structural properties, is referred to as the macroscopic structure.

1.11.4.1 SG for Beams

Consider the design of a composite rotor blade. The slenderness of the structure suggests the use of a beam model to simplify the analysis. In this scenario, a chosen beam model becomes the desired macroscopic model for the rotor blade. Assuming the focus is on elastic behavior and each composite layer is homogeneous, the original model for the composite rotor blade would be governed by the 3D anisotropic elasticity theory with layerwise heterogeneities.

The SG of the structure is the cross-section of the composite rotor blade, provided that it can be approximated as having a uniform cross-section (see Figure 1.14a). Alternatively, if the rotor blade consists of segments, each with a uniform cross-section, the SG would correspond to the cross-section of each segment. Constitutive modeling performed over the cross-section yields equivalent beam properties for the 1D beam analysis and provides dehomogenization relations for predicting local fields within the original 3D structure.

If the slender structure exhibits spanwise heterogeneity, the SG changes accordingly. In this case, the SG becomes a 3D block that captures the relevant heterogeneity (see Figure 1.14b). The concept of SG thus generalizes the notion of a cross-section to account for slender structures with spanwise heterogeneities.

Furthermore, changing the answer to the second question of the original model expands the application domain. For example, if we change the original model to be the 3D anisotropic elasticity theory with pointwise heterogeneities, we have the possibility to directly compute equivalent beam properties in terms of fiber and matrix properties and predict local fields at the fiber and matrix level.

The concept of SG inspires a *new perspective* in *beam modeling*, a traditional branch of *structural mechanics*. If the beam's reference line is treated as a 1D continuum, each material point of this continuum is associated with an SG that represents its microstructure. In this view, *constitutive modeling for beams, traditionally part of structural mechanics, can be reinterpreted as a specialized application of micromechanics to slender structures.*

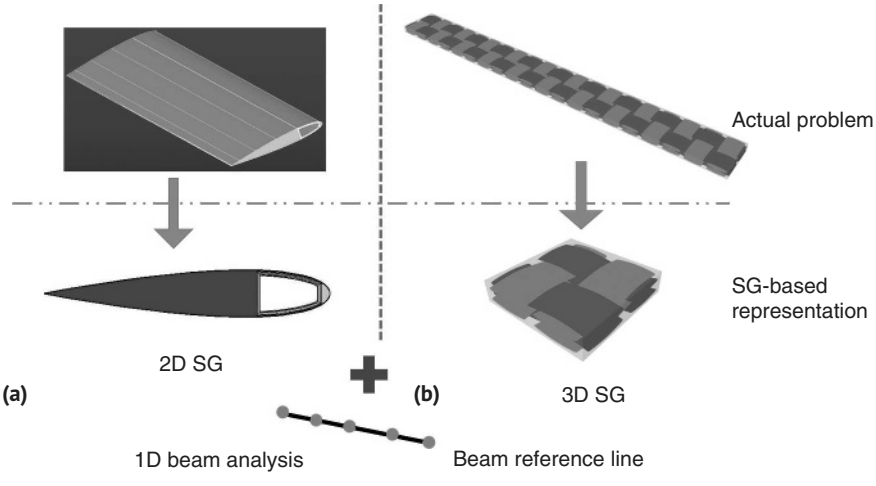


Figure 1.14 Analysis of slender structures approximated using constitutive modeling over an SG and a corresponding 1D beam analysis.

The concept of SG also provides a unified treatment of structural modeling and micromechanics modeling. It enables us to collapse a cross-section or a 3D beam segment into a material point on the reference line. The mechanical behavior of this material point is described using a fully populated stiffness matrix, which can simultaneously account for all the deformation modes of a beam model. The beam theory constructed this way can easily handle buildup structures as long as their external contours look like a beam (e.g., high aspect ratio wings), or the analyst chooses to approximate a structure as a beam.

1.11.4.2 SG for Plates or Shells

When modeling panels, the corresponding macroscopic structural model is typically a plate or shell model. The answer to the third question—the SG—is determined by the internal construction of the panel (see Figure 1.15).

1. If the panel is a laminate made of homogeneous layers, the SG is the 1D transverse normal line (see Figure 1.15a), with segments representing the corresponding layers. This is because the transverse normal line can be mathematically repeated in-plane to construct the entire laminate.
2. If the panel exhibits heterogeneity in one of its in-plane directions, the SG becomes a 2D domain (see Figure 1.15b).
3. If the panel has heterogeneity in both in-plane directions, the SG is a 3D block (see Figure 1.15c).

Regardless of the dimensionality of the SG, constitutive modeling over the SG predicts the equivalent plate and shell properties for the 2D plate and shell analysis and provides dehomogenization relations to predict the local fields within the original 3D structure.

Plate and shell theories are traditionally part of structural mechanics, but using the concept of SG, their constitutive modeling can be regarded as a specialized application of micromechanics. For a panel, if the reference surface is considered a 2D continuum, every

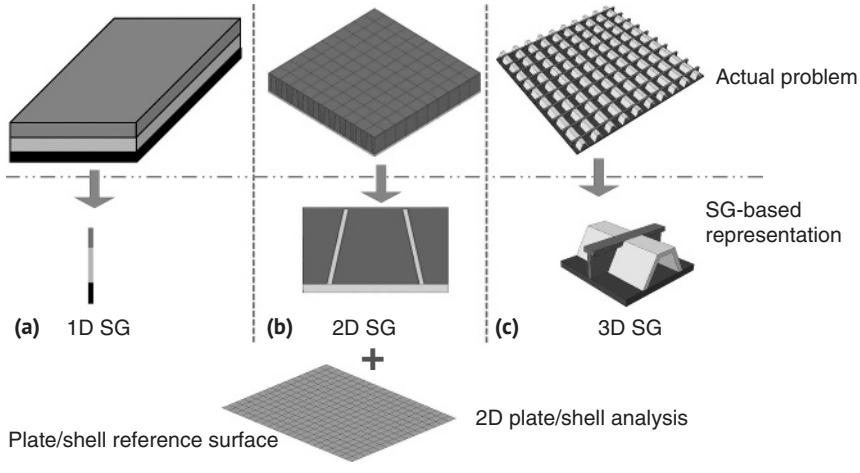


Figure 1.15 Analysis of panel structures approximated using constitutive modeling over an SG and a corresponding 2D plate analysis.

material point of this continuum has an associated SG that represents its microstructure. Plate and shell models constructed using the SG concept can handle buildup structures as long as their external contours look like a plate or shell, or the analyst chooses to approximate a structure using plate or shell models.

1.11.4.3 SG for 3D Structures

For structures where all three dimensions are of similar size (see Figure 1.16), the appropriate macroscopic model is typically a solid model, such as the 3D elasticity model. The SG for such structures is chosen based on their internal construction, serving a role analogous to a representative volume element (RVE) in micromechanics. However, SG and RVE differ significantly in their application and capabilities.

For composite structures with 1D heterogeneity (e.g., binary composites with alternating layers as in Figure 1.16a), the SG is a straight line with segments representing the individual layers. If one wants to model a laminate as an equivalent homogeneous solid, the transverse normal material line is the corresponding 1D SG. The constitutive modeling over the 1D SG can compute the complete set of 3D properties and local fields. Such applications of SG are not achievable by using RVE, which requires a 3D block to compute the complete set of 3D properties.

For composite structures with 2D heterogeneity (e.g., UDFRC as in Figure 1.16b), the SG is a 2D domain. While common 2D RVE models only yield in-plane properties and local fields, the SG approach can compute the complete 3D property set using just a 2D domain, avoiding the need for a full 3D RVE as in traditional RVE methods [14].

For composite structure with 3D heterogeneity (e.g., particle-reinforced composites in Figure 1.16c), the SG is a 3D volume, closely resembling an RVE. However, unlike RVE-based models, SG-based models do not require boundary conditions in terms of displacements and tractions, simplifying the modeling process.

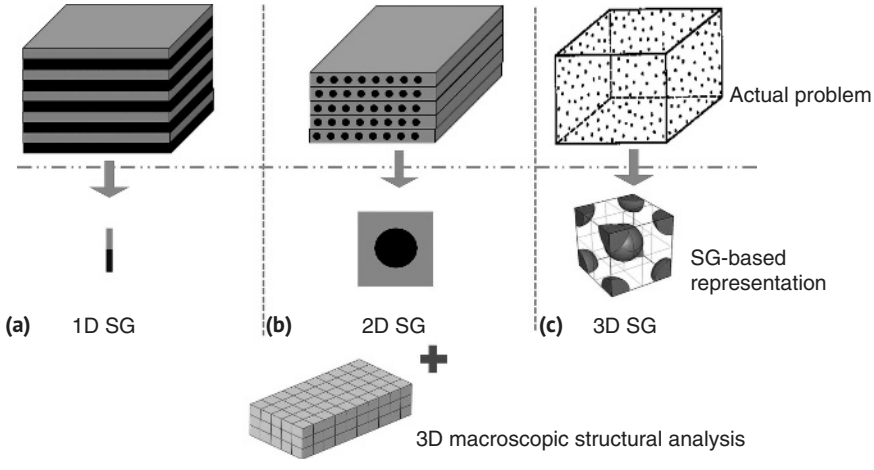


Figure 1.16 Analysis of 3D heterogeneous structures approximated using constitutive modeling over an SG and a corresponding 3D structural analysis.

The SG concept, when applied to 3D structures, offers a general-purpose micromechanics approach capable of addressing a wide range of microstructural configurations. This includes structures with 1D, 2D, or 3D heterogeneities, covering materials that are periodic, partially periodic, or even aperiodic [15].

1.11.4.4 Special Considerations of SG

It is easy to identify SGs for periodic structures, as shown in Figures 1.14—1.16. Real composite structures, however, are typically aperiodic. We often assume that the structure is at least periodic in the vicinity of a material point in the macroscopic structural analysis. This assumption, known as local periodicity, is implicit in all traditional multiscale modeling approaches [16].

For nonlinear behavior, particularly geometrically nonlinear behavior, the smallest mathematical building block of the structure may not suffice, as the characteristic length scale of nonlinear behavior can encompass multiple building blocks. In such cases, SG should be understood as the smallest mathematical building block required to capture the nonlinear behavior accurately. Nonlinear constitutive modeling couples homogenization and dehomogenization to compute nonlinear effective properties and the corresponding in-situ local fields.

In general, SG can be regarded as the block of material corresponding to an element in the finite element mesh of the macroscopic structural analysis. In extreme cases, a structure could have as many SGs as the number of elements in the macroscopic structural analysis. To be rigorous, such cases should be addressed using DNS, which incorporates all the details in the structural analysis, as the local periodicity assumption is invalid in these scenarios. Nevertheless, the multiscale approach remains widely practiced for such cases in real-world structural design and analysis. For example, in the design of wind turbine blades, the cross-section changes significantly along the span (see Figure 1.17). Industrial standards typically involve cutting the blade into many stations, with each cross-section treated

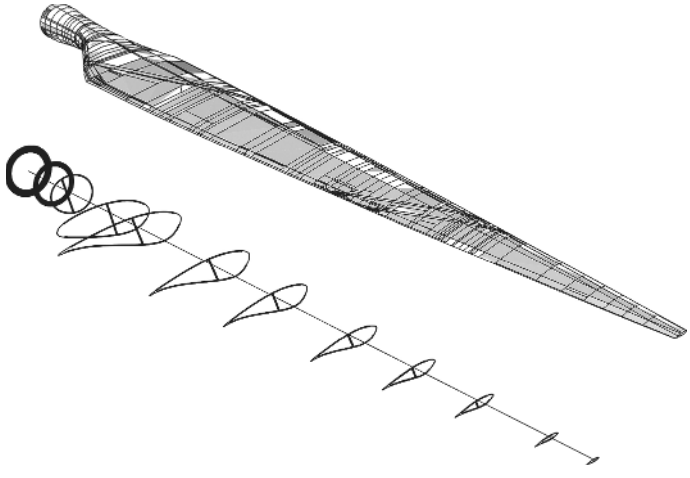


Figure 1.17 Analysis of wind turbine blades approximated by constitutive modeling over many cross-sections and a corresponding 1D beam analysis.

as an SG corresponding to a beam element. Beam elements are essential for the multibody dynamic analysis of the entire wind turbine. Drawing analogy to biological taxonomy, a genome is defined as the complete set of genes present in an organism. Similarly, the totality of all SGs (cross-sections) used in this analysis can be regarded as the *structure genome* of the wind turbine blade.

Although the local periodicity assumption is implicit in most multiscale modeling approaches, MSG has the potential to eliminate this assumption entirely. To achieve this, an SG must be associated with a finite element rather than a point in the macroscopic structure. Using this approach, the model can be constructed using MSG to compute the effective stiffness matrix for the structural element, which is then assembled directly into the global stiffness matrix of the finite element model. This contrasts with the traditional method of computing a material stiffness matrix for each point in the structural model. Implementing this approach requires the development of a user-defined element within an FEA package, such as Abaqus. This element would take the effective element stiffness matrix computed by MSG as input for the macroscopic structural analysis. For more information on this innovative MSG-based multiscale modeling approach, please refer to Refs. [17, 18].

1.11.5 MSG-based Multiscale Structural Modeling

There are three basic steps to formulate an MSG-based model. First, the displacements of the original model ($\mathbf{u}$) must be expressed in terms of the displacements of the macroscopic model ($\bar{\mathbf{u}}$) and some unknown functions ($\mathbf{w}$), referred to as the fluctuating functions. These functions capture the deformation not accounted for by the simpler kinematics of the macroscopic model. They are also referred to as warping functions in traditional structural mechanics theories. This relationship can be symbolically written as

$$\mathbf{u} = \mathbf{u}(\bar{\mathbf{u}}, \mathbf{w}) \tag{1.28}$$

The macroscopic displacements ($\bar{\mathbf{u}}$) might not correspond to traditional displacements and rotations but instead represent the nodal displacements and rotations of the macroscopic model [17, 18].

Next, based on the strain definitions of the original model ($\boldsymbol{\varepsilon}$) and those of the macroscopic model ($\bar{\boldsymbol{\epsilon}}$), the following relations can be derived to connect these two sets of strain variables:

$$\boldsymbol{\varepsilon} = \boldsymbol{\epsilon}(\bar{\boldsymbol{\epsilon}}, \nabla\mathbf{w}) \tag{1.29}$$

It is important to note that the macroscopic strain ($\bar{\boldsymbol{\epsilon}}$) may not appear explicitly, depending on the specific macroscopic model [17, 18].

Since the macroscopic model is constructed out of the original model, the kinematic fields of the macroscopic model must be defined in terms of the kinematic fields of the original model. This effectively introduces constraints on the fluctuating functions $\mathbf{w}$.

For elastic behavior, the original model can be governed by the *principle of minimum total potential energy* (PMTPE). The total potential energy can be expressed in terms of displacements and strains as

$$\Pi(\mathbf{u}, \varepsilon) = \Pi(\bar{\mathbf{u}}, \bar{\boldsymbol{\epsilon}}, \mathbf{w}) \tag{1.30}$$

To minimize the loss of information (in this case, the total potential energy) between the original model and the macroscopic model, the following expression is obtained:

$$\min_{\mathbf{w}} [\Pi(\bar{\mathbf{u}}, \bar{\boldsymbol{\epsilon}}, \mathbf{w}) - \bar{\Pi}(\bar{\mathbf{u}}, \bar{\epsilon})] \tag{1.31}$$

Here, $\bar{\Pi}$ denotes the total potential energy of the macroscopic model.

The objective is to use the macroscopic model to approximate the behavior of the original model. This requires determining how the original model behaves under given macroscopic behavior $(\bar{\mathbf{u}}, \bar{\boldsymbol{\epsilon}})$ of the macroscopic model. Since $\bar{\mathbf{u}}$ and $\bar{\boldsymbol{\epsilon}}$ are given and cannot vary in Eq. (1.31), the resulting variational principle of the original model is

$$\min_{\mathbf{w}} \Pi(\bar{\mathbf{u}}, \bar{\boldsymbol{\epsilon}}, \mathbf{w}) \tag{1.32}$$

This principle is solved to determine $\mathbf{w}$ in terms of $\bar{\mathbf{u}}$ and $\bar{\boldsymbol{\epsilon}}$. However, directly solving this variational principle is usually challenging. To address this, the smallness of the SG compared to the macroscopic deformation is exploited, allowing an asymptotical analysis of this variational principle using VAM. The VAM-simplified variational principle can be solved analytically for some simple cases [19, 20]. For general cases, numerical techniques, such as the finite element method (FEM), are required.

Because macroscopic structural analysis can be easily handled by standard FEA software packages, MSG-based multiscale constitutive modeling enables FEA software packages to model composites as a "black aluminum" in macroscopic structural analysis with minimum loss of accuracy. In this context, "black aluminum" refers to an anisotropic homogeneous continuum (3D structure, plate, shell, beam, and structural elements), rather than an isotropic material traditionally implied by this term in the composites industry. The unique features of the MSG-based multiscale constitutive modeling approach are

1. Use of SG: SG bridges the gap between the materials genome and macroscopic structural analysis. Conceptually, SG enables us to treat the constitutive modeling of structures as an application of micromechanics. Technically, SG enables systematically modeling of all composite structures (beams, plates, shells, 3D structures, etc.), accommodating anisotropy and heterogeneity in arbitrary microstructures.
2. Use of PMIL and VAM: PMIL, combined with the powerful VAM, eliminates the need for a priori assumptions commonly used in other approaches. Thus, ensures maximum mathematical rigor and engineering generality.
3. Decoupling of Analysis: MSG mathematically splits the original problem into two sets of analyses: a constitutive modeling procedure over SG and a macroscopic structural analysis. This decoupling allows exact formulation of macroscopic structural analysis using general continuum models (1D, 2D, or 3D) readily available in commercial FEA software. All approximations are confined to the constitutive modeling, ensuring optimal accuracy through PMIL.
4. Engineering Simplicity and Legacy: MSG preserves engineering simplicity and legacy by reformatting refined asymptotically correct functionals into conventional engineering models. This ensures compatibility with standard FEA software packages.

As explored in later chapters, MSG is emerging as a powerful multiscale modeling approach that connects simple structural elements available in FEA software to microstructural details. It provides a formal framework for multiscale constitutive modeling, enabling analysts to incorporate details as needed and feasible. By unifying micromechanical and structural modeling, MSG avoids invalid scale separation assumptions and eliminates a priori assumptions specific to any scale.

After identifying the original model, the macroscopic model, and the SG, PMIL constructs the most accurate macroscopic model from the original model. We have demonstrated its applications to slender structures [21], free-edge stress analysis [22, 23], interfaces [24–26], textile composites [27–30], periodic heterogenous materials [31], damage of composite beams [32], nonlinear shear behavior [33], thermoelastic constitutive modeling [34, 35], time- and temperature-dependent behavior of composites [36], thin-walled structures [37–39], printed circuit board [40], aperiodic heterogeneous solids [15, 17], aperiodic heterogeneous beams [18], buckling [41, 42], metamaterials and metastructures [43], composites curing simulation [44], printed circuit Board [40], simulating column bending test of thin-ply composites [45], thermal conductivity of woven composites [46], etc.

In principle, MSG provides a versatile, unified framework for modeling all types of composite structures such as laminates, stiffened structures, sandwich structures, corrugated panels, and lattice structures, as long as the original model and corresponding SG are identified. Many additional applications of this versatile multiscale constitutive modeling framework remain to be explored.

Problems

1.1 For a beam with bending stiffness EI and subjected to a distributed load p_3, we know that the deflection $\bar{u}_3$ is governed by a fourth-order ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\left(EI\frac{\mathrm{d}^2\bar{u}_3}{\mathrm{d}x_1^2}\right)=p_3$$

where x_1 is the coordinate along the beam reference line, and both EI and p_3 are constants, $\bar{u}_3$ is a function of x_1. This beam has a length of L and is clamped at both ends. Solve this problem (i) by hand, (ii) using math software (e.g., Mathematica), and (iii) verify that both approaches yield the same solution.

1.2 Describe the differences between materials and structures.

1.3 Describe the meaning of four words used to describe a material or structure: isotropic, anisotropic, homogeneous, and heterogeneous.

1.4 Describe the major benefits of using composite materials in comparison to conventional materials such as metals.

1.5 For the fibers in Table 1.1, which fiber exhibits the highest specific stiffness, and which fiber exhibits the highest specific strength? Compare these two numbers to those of aluminum and steel with the properties given in Table 1.2.

1.6 What are the main challenges related to the mathematical modeling of composites? What are the corresponding techniques used to handle these challenges?

1.7 Write a 4-page (single-spaced, 12 point, 1-inch margin) reading report for this chapter to describe what you have learned from this chapter.