1

Membrane Theory of Shells of Revolution

1.1 Introduction

All thin cylindrical shells, spherical and ellipsoidal heads, and conical transition sections are generally analyzed and designed in accordance with the general membrane theory of shells of revolution. These components include those designed in accordance with the ASME pressure vessel code (Section VIII), boiler code (Section I), and nuclear code (Section III). Some adjustments are sometimes made to the calculated thicknesses when the ratio of radius to thickness is small or when other factors such as creep or plastic analysis enter into consideration. The effect of these factors is discussed in later chapters, whereas assumptions and derivation of the basic membrane equations needed to analyze shells of revolution due to various loading conditions are described here.

1.2 Basic Equations of Equilibrium

The membrane shell theory is used extensively in designing such structures as flat bottom tanks, pressure vessel components (Figure 1.1), and vessel heads. The membrane theory assumes that equilibrium in the shell is achieved by having the in-plane membrane forces resist all applied loads without any bending moments. The theory gives accurate results as long as the applied loads are distributed over a large area of the shell such as pressure and wind loads. The membrane forces by themselves cannot resist local concentrated loads. Bending moments are needed to resist such loads as discussed in Chapters 3 and 5. The basic assumptions made in deriving the membrane theory (Gibson 1965) are as follows:

1. The shell is homogeneous and isotropic.
2. The thickness of the shell is small compared with its radius of curvature.

Stress in ASME Pressure Vessels, Boilers, and Nuclear Components, First Edition. Maan H. Jawad.

Figure 1.1 Pressure vessels. Source: Courtesy of the Nooter Corporation, St. Louis, MO.

3. The bending strains are negligible and only strains in the middle surface are considered.
4. The deflection of the shell due to applied loads is small.

In order to derive the governing equations for the membrane theory of shells, we need to define the shell geometry. The middle surface of a shell of constant thickness may be considered a surface of revolution. A surface of revolution is obtained by rotating a plane curve about an axis lying in the plane of the curve. This curve is called a meridian (Figure 1.2). Any point in the middle surface can be described first by specifying the meridian on which it is located and second by specifying a quantity, called a parallel circle, that varies along the meridian and is constant on a circle around the axis of the shell. The meridian is defined by the angle θ and the parallel circle by ϕ as shown in Figure 1.2.

Define r (Figure 1.3) as the radius from the axis of rotation to any given point o on the surface; r_1 as the radius from point o to the center of curvature of the meridian; and r_2 as the radius from the axis of revolution to point o, and it is perpendicular to the meridian. Then from Figure 1.3,

$$r = r_2 \sin\phi, \quad ds = r_1\, d\phi, \quad \text{and} \quad dr = ds\cos\phi. \tag{1.1}$$

The interaction between the applied loads and resultant membrane forces is obtained from statics and is shown in Figure 1.4. Shell forces N_ϕ and N_θ are membrane forces in the meridional

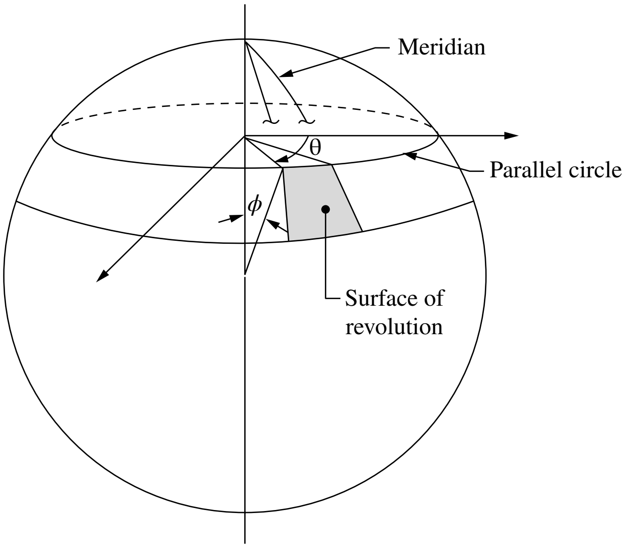


Figure 1.2 Surface of revolution

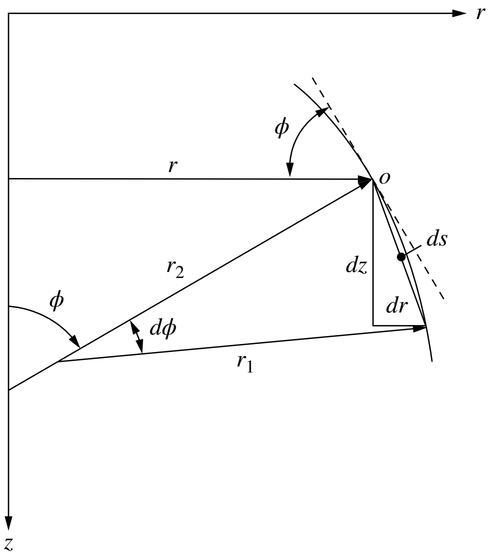


Figure 1.3 Shell geometry

and circumferential directions, respectively. Shearing forces $N_{\phi\theta}$ and $N_{\theta\phi}$ are as shown in Figure 1.4. Applied load p_{r} is perpendicular to the surface of the shell, load p_ϕ is in the meridional direction, and load p_θ is in the circumferential direction. All forces are positive as shown in Figure 1.4.

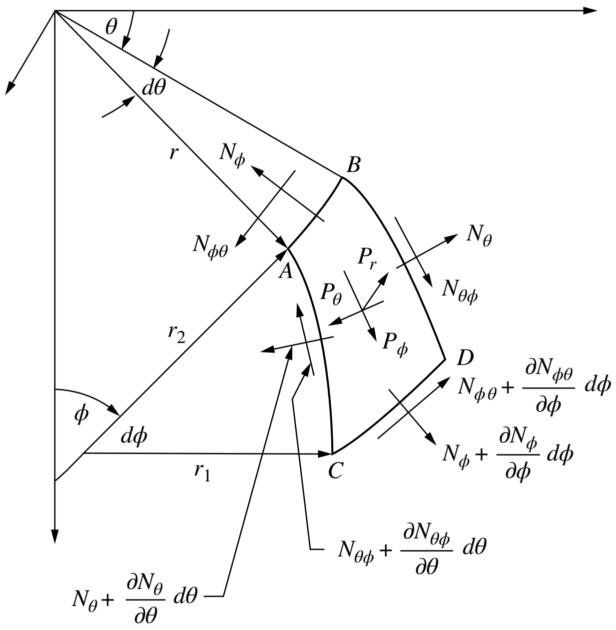


Figure 1.4 Membrane forces and applied loads

The first equation of equilibrium is obtained by summing forces parallel to the tangent at the meridian. This yields

$$
\begin{aligned}
&N_{\theta\phi} r_1\, d\phi - \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta\right) r_1\, d\phi - N_\phi r\, d\theta \\
&\quad + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi\right)\left(r + \frac{\partial r}{\partial \phi} d\phi\right) d\theta \\
&\quad + p_\phi r\ d\theta r_1\ d\phi - N_\theta r_1\ d\phi d\theta\ \cos\phi = 0.
\end{aligned} \tag{1.2}
$$

The last term in Eq. (1.2) is the component of N_θ parallel to the tangent at the meridian (Jawad 2004). It is obtained from Figure 1.5. Simplifying Eq. (1.2) and neglecting terms of higher order results in

$$
\frac{\partial}{\partial \phi}\left(r N_\phi\right) - r_1 \frac{\partial N_{\theta\phi}}{\partial \theta} - r_1 N_\theta \cos\phi + p_\phi r r_1 = 0. \tag{1.3}
$$

The second equation of equilibrium is obtained from summation of forces in the direction of parallel circles. Referring to Figure 1.4,

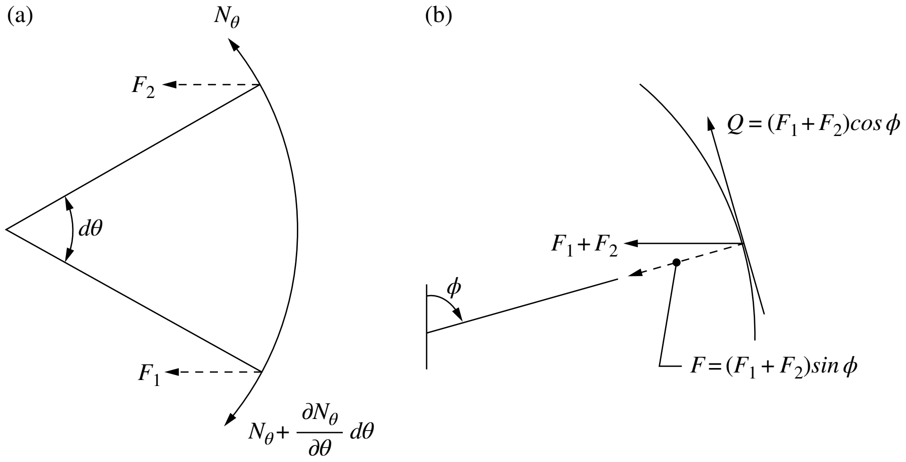


Figure 1.5 Components of N_θ: (a) circumferential cross section and (b) longitudinal cross section

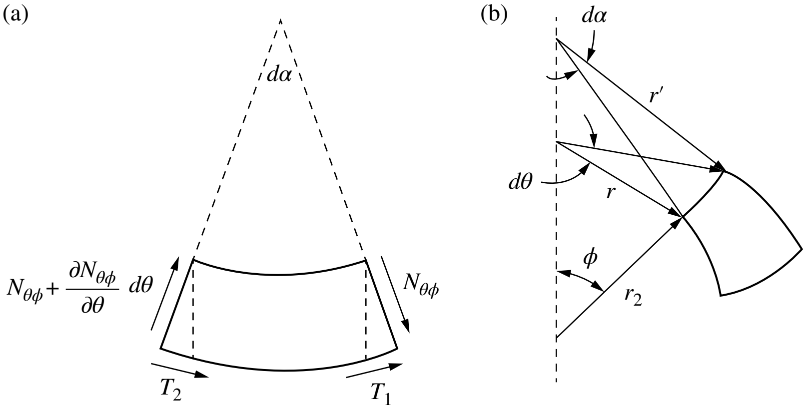


Figure 1.6 Components of $N_{\theta\phi}$: (a) side view and (b) three-dimensional view

$$\begin{aligned} &N_{\phi\theta} r\, d\theta - \left(N_{\phi\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi\right)\left(r + \frac{\partial r}{\partial \phi} d\phi\right) d\theta \\ &\quad - N_\theta r_1 d\phi + \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta\right)(r_1 d\phi) \\ &\quad + p_\theta r\, d\theta\, r_1\, d\phi - N_{\theta\phi} r_1 d\phi \frac{\cos\phi\, d\theta}{2} \\ &\quad - \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta\right)(r_1\, d\phi)\frac{\cos\phi\, d\theta}{2} = 0. \end{aligned} \tag{1.4}$$

The last two expressions in this equation are obtained from Figure 1.6 (Jawad 2004) and are the components of $N_{\theta\phi}$ in the direction of the parallel circles. Simplifying this equation results in

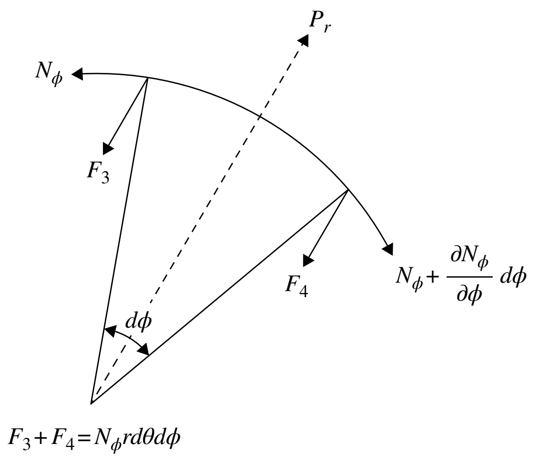


Figure 1.7 Components of N_ϕ

$$\frac{\partial}{\partial \phi}\left(rN_{\phi\theta}\right) - r_1 \frac{\partial N_\theta}{\partial \theta} + r_1 N_{\theta\phi}\cos\phi - p_\theta r r_1 = 0. \tag{1.5}$$

This is the second equation of equilibrium of the infinitesimal element shown in Figure 1.4. The last equation of equilibrium is obtained by summing forces perpendicular to the middle surface. Referring to Figures 1.4, 1.5, and 1.7,

$$(N_\theta r_1 d\phi d\theta)\sin\phi - p_\mathrm{r} r d\theta r_1 d\phi + N_\phi r d\theta d\phi = 0$$

or

$$N_\theta r_1 \sin\phi + N_\phi r = p_\mathrm{r} r r_1. \tag{1.6}$$

Equations (1.3), (1.5), and (1.6) are the three equations of equilibrium of a shell of revolution subjected to axisymmetric loads.

1.3 Spherical and Ellipsoidal Shells Subjected to Axisymmetric Loads

In many structural applications, loads such as deadweight, snow, and pressure are symmetric around the axis of the shell. Hence, all forces and deformations must also be symmetric around the axis. Accordingly, all loads and forces are independent of θ and all derivatives with respect to θ are zero. Equation (1.3) reduces to

$$\frac{\partial}{\partial \phi}\left(rN_\phi\right) - r_1 N_\theta \cos\phi = -p_\phi r r_1. \tag{1.7}$$

Equation (1.5) becomes

$$\frac{\partial}{\partial \phi}\left(rN_{\theta\phi}\right) + r_1 N_{\theta\phi}\cos\phi = p_\theta r r_1. \tag{1.8}$$

In this equation, we let the cross shears $N_{\phi\theta} = N_{\theta\phi}$ in order to maintain equilibrium.

Equation (1.6) can be expressed as

$$\frac{N_\theta}{r_2} + \frac{N_\phi}{r_1} = p_\mathrm{r}. \tag{1.9}$$

Equation (1.8) describes a torsion condition in the shell. This condition produces deformations around the axis of the shell. However, the deformation around the axis is zero due to axisymmetric loads. Hence, we must set $N_{\theta\phi} = p_\theta = 0$ and we disregard Eq. (1.8) from further consideration.

Substituting Eq. (1.9) into Eq. (1.7) gives

$$N_\phi = \frac{1}{r_2 \sin^2\phi}\left[\int r_1 r_2 \left(p_\mathrm{r}\cos\phi - p_\phi \sin\phi\right)\sin\phi \, d\phi + C\right]. \tag{1.10}$$

The constant of integration C in Eq. (1.10) is additionally used to take into consideration the effect of any additional applied loads that cannot be defined by p_r and p_ϕ such as weight of contents.

Equations (1.9) and (1.10) are the two governing equations for designing double-curvature shells under membrane action.

1.3.1 Spherical Shells Subjected to Internal Pressure

For spherical shells under axisymmetric loads, the differential equations can be simplified by letting $r_1 = r_2 = R$. Equations (1.9) and (1.10) become

$$N_\phi + N_\theta = p_\mathrm{r} R \tag{1.11}$$

and

$$N_\phi = \frac{R}{\sin^2\phi}\left[\int \left(p_\mathrm{r}\cos\phi - p_\phi \sin\phi\right)\sin\phi \, d\phi + C\right]. \tag{1.12}$$

These two expressions form the basis for developing solutions to various loading conditions in spherical shells. For any loading condition, expressions for p_r and p_ϕ are first determined and then the previous equations are solved for N_ϕ and N_θ.

For a spherical shell under internal pressure, $p_\mathrm{r} = P$ and $p_\phi = 0$. Hence, from Eqs. (1.11) and (1.12),

$$N_\phi = N_\theta = \frac{PR}{2} = \frac{PD}{4} \tag{1.13}$$

where D is the diameter of the sphere. The required thickness is obtained from

$$t = \frac{N_\phi}{S} = \frac{N_\theta}{S} \tag{1.14}$$

where S is the allowable stress.

Equation (1.14) is accurate for design purposes as long as $R/t \geq 10$. If $R/t < 10$, then thick shell equations, described in Chapter 3, must be used.

1.3.2 Spherical Shells under Various Loading Conditions

The following examples illustrate the use of Eqs. (1.11) and (1.12) for determining forces in spherical segments subjected to various loading conditions.

Example 1.1

A storage tank roof with thickness t has a dead load of γ psf. Find the expressions for N_ϕ and N_θ.

Solution

From Figure 1.8a and Eq. (1.12),

$$p_\mathrm{r} = -\gamma\cos\phi \quad \text{and} \quad p_\phi = \gamma\sin\phi$$

$$N_\phi = \frac{R}{\sin^2\phi}\left[\int\left(-\gamma\cos^2\phi - \gamma\sin^2\phi\right)\sin\phi\, d\phi + C\right]$$

$$N_\phi = \frac{R}{\sin^2\phi}(\gamma\cos\phi + C). \tag{1}$$

As ϕ approaches zero, the denominator in Eq. (1) approaches zero. Accordingly, we must let the bracketed term in the numerator equal zero. This yields $C = -\gamma$. Equation (1) becomes

$$N_\phi = \frac{-R\gamma(1-\cos\phi)}{\sin^2\phi}. \tag{2}$$

The convergence of Eq. (2) as ϕ approaches zero can be checked by l'Hopital's rule. Thus,

$$N_\phi\Big|_{\phi=0} = \frac{-R\gamma\sin\phi}{2\sin\phi\cos\phi}\Bigg|_{\phi=0} = \frac{-\gamma R}{2}.$$

Equation (2) can be written as

$$N_\phi = \frac{-\gamma R}{1+\cos\phi}. \tag{3}$$

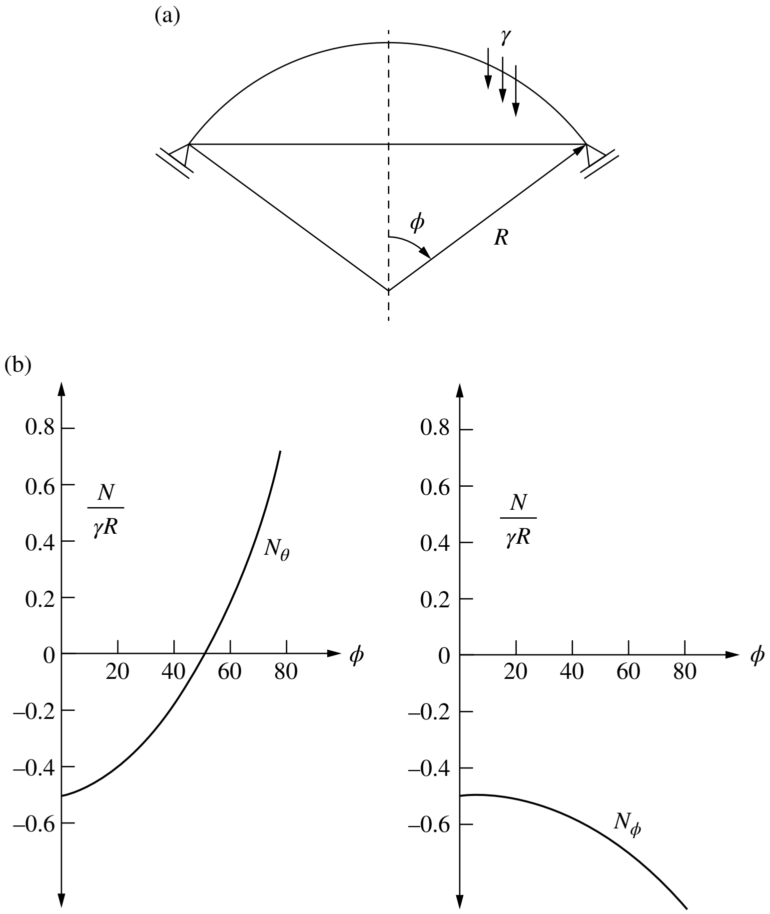


Figure 1.8 Membrane forces in a head due to deadweight: (a) dead load and (b) force patterns

From Eq. (1.11), N_θ is given by

$$N_\theta = \gamma R\left(\frac{1}{1+\cos\phi} - \cos\phi\right). \tag{4}$$

A plot of N_ϕ and N_θ for various values of ϕ is shown in Figure 1.8b, showing that for angles ϕ greater than 52°, the hoop force, N_θ, changes from compression to tension and special attention is needed in using the appropriate allowable stress values.

Example 1.2
Find the forces in a spherical head due to a vertical load P_o applied at an angle $\phi = \phi_o$ as shown in Figure 1.9a.

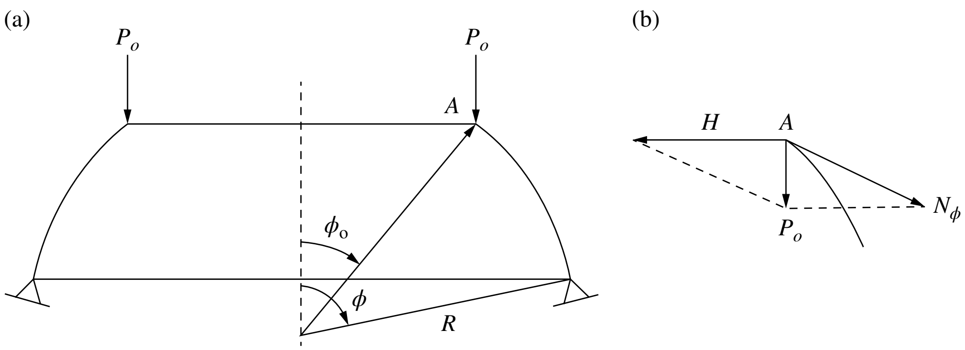


Figure 1.9 Edge loads in a spherical head: (a) edge load and (b) forces due to edge load

Solution

Since $p_r = p_\phi = 0$, Eq. (1.12) becomes

$$N_\phi = \frac{RC}{\sin^2\phi}. \tag{1}$$

From statics at $\phi = \phi_o$, we get from Figure 1.9b

$$N_\phi = \frac{P_o}{\sin\phi_o}.$$

Substituting this expression into Eq. (1), and keeping in mind that it is a compressive membrane force, gives

$$C = \frac{-P_o}{R}\sin\phi_o$$

and Eq. (1) yields

$$N_\phi = -P_o\frac{\sin\phi_o}{\sin^2\phi}.$$

From Eq. (1.11),

$$N_\theta = P_o\frac{\sin\phi_o}{\sin\phi}.$$

In this example there is another force that requires consideration. Referring to Figure 1.9b, it is seen that in order for P_o and N_ϕ to be in equilibrium, another horizontal force, H, must be considered. The direction of H is inward in order for the force system to have a net resultant force P_o downward. This horizontal force is calculated as

$$H = \frac{-P_o\cos\phi_o}{\sin\phi_o}.$$

A compression ring is needed at the inner edge in order to contain force H. The required area, A, of the ring is given by

$$A = \frac{H(R\sin\phi_o)}{\sigma}$$

where σ is the allowable compressive stress of the ring.

Example 1.3

The sphere shown in Figure 1.10a is filled with a liquid of density γ. Hence, p_r and p_ϕ can be expressed as

$$p_r = \gamma R(1 - \cos\phi)$$
$$p_\phi = 0.$$

a. Determine the expressions for N_ϕ and N_θ throughout the sphere.
b. Plot N_ϕ and N_θ for various values of ϕ when $\phi_o = 110°$.
c. Plot N_ϕ and N_θ for various values of ϕ when $\phi_o = 130°$.
d. If $\gamma = 62.4$ pcf, $R = 30$ ft, and $\phi_o = 110°$, determine the magnitude of the unbalanced force H at the cylindrical shell junction. Design the sphere, the support cylinder, and the junction ring. Let the allowable stress in tension be 20 ksi and that in compression be 10 ksi.

(a)

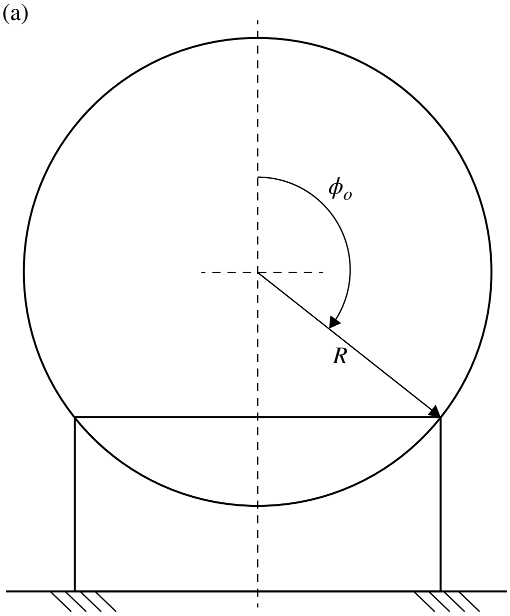


Figure 1.10 Spherical tank: (a) spherical tank, (b) support at 110°, (c) support at 130°, and (d) forces at support junction

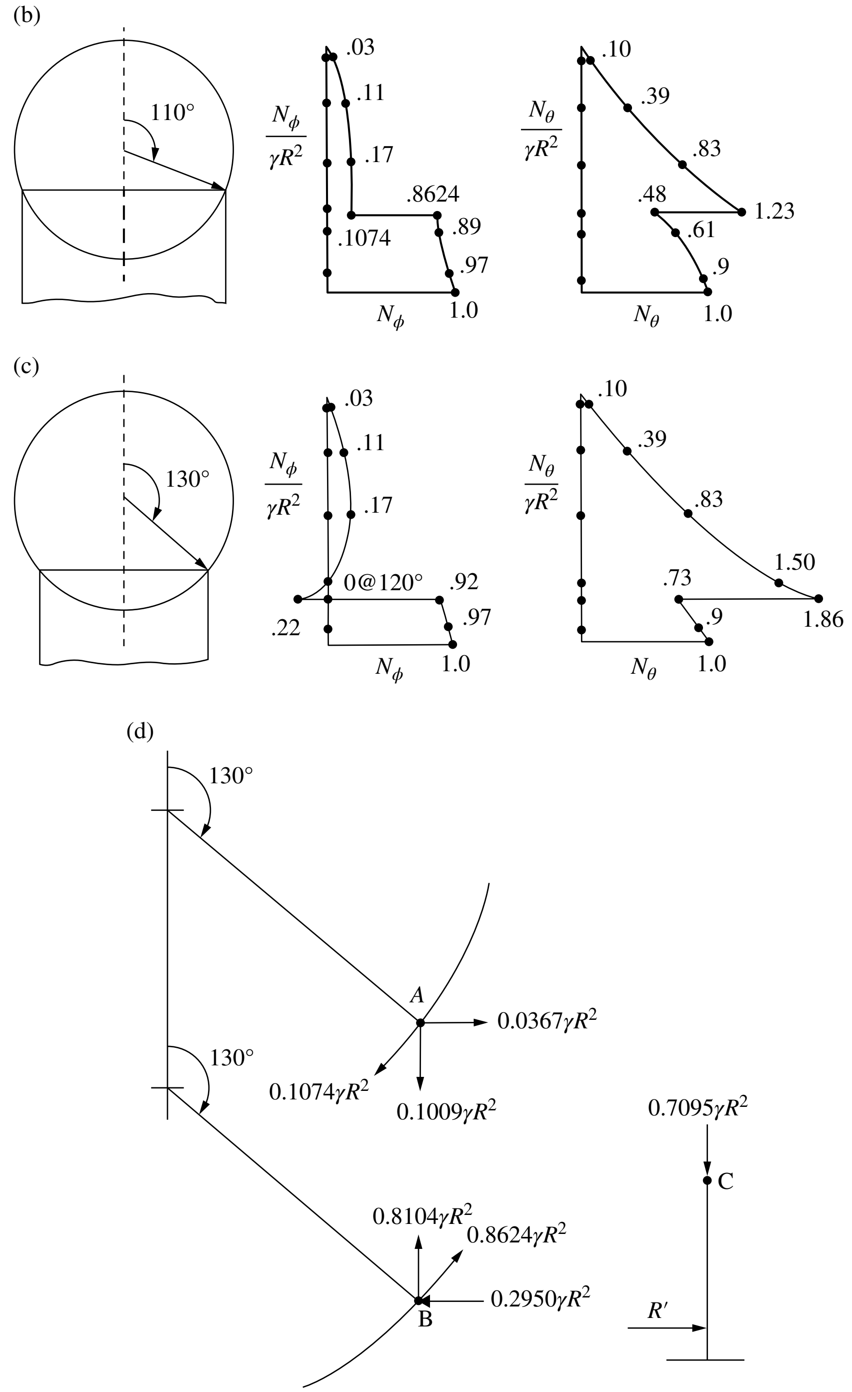


Figure 1.10 (Continued)

Solution

a. From Eq. (1.12), we obtain

$$N_\phi = \frac{\gamma R^2}{\sin^2\phi}\left(\frac{1}{2}\sin^2\phi + \frac{1}{3}\cos^3\phi + C\right). \tag{1}$$

As ϕ approaches zero, the denominator approaches zero. Hence, the bracketed term in the numerator must be set to zero. This gives $C = -1/3$ and Eq. (1) becomes

$$N_\phi = \frac{\gamma R^2}{6\sin^2\phi}\left(3\sin^2\phi + 2\cos^3\phi - 2\right). \tag{2}$$

The corresponding N_θ from Eq. (1.11) is

$$N_\theta = \gamma R^2\left[\frac{1}{2} - \cos\phi - \frac{1}{3\sin^2\phi}\left(\cos^3\phi - 1\right)\right]. \tag{3}$$

As ϕ approaches π, we need to evaluate Eq. (1) at that point to ensure a finite solution. Again the denominator approaches zero and the bracketed term in the numerator must be set to zero. This gives $C = 1/3$ and Eq. (1) becomes

$$N_\phi = \frac{\gamma R^2}{6\sin^2\phi}\left(3\sin^2\phi + 2\cos^3\phi + 2\right). \tag{4}$$

The corresponding N_θ from Eq. (1.11) is

$$N_\theta = \gamma R^2\left[\frac{1}{2} - \cos\phi - \frac{1}{3\sin^2\phi}\left(\cos^3\phi + 1\right)\right]. \tag{5}$$

Equations (2) and (3) are applicable between $0 < \phi < \phi_o$, and Eqs. (4) and (5) are applicable between $\phi_o < \phi < \pi$.

b. A plot of Eqs. (2) through (5) for $\phi_o = 110°$ is shown in Figure 1.10b. N_ϕ below circle $\phi_o = 110°$ is substantially larger than that above circle 110°. This is due to the fact that most of the weight of the contents is supported by the spherical portion that is below the circle $\phi_o = 110°$. Also, because N_ϕ does not increase in proportion to the increase in pressure as ϕ increases, Eq. (1.11) necessitates a rapid increase in N_θ in order to maintain the relationship between the left- and right-hand sides. This is illustrated in Figure 1.10b.

A plot of N_ϕ and N_θ for $\phi_o = 130°$ is shown in Figure 1.10c. In this case, N_ϕ is in compression just above the circle $\phi_o = 130°$. This indicates that as the diameter of the supporting cylinder gets smaller, the weight of the water above circle $\phi_o = 130°$ must be supported by the sphere in compression. This results in a much larger N_θ value just above $\phi_o = 130°$. Buckling of the sphere becomes a consideration in this case.

c. From Figure 1.10b for $\phi_o = 110°$, the maximum force in the sphere is $N_\theta = 1.23\gamma R^2$. The required thickness of the sphere is

$$t = \frac{1.23(62.4)(30)^2/12}{20,000}$$
$$= 0.29 \text{ inch.}$$

A free-body diagram of the spherical and cylindrical junction at $\phi_o = 110°$ is shown in Figure 1.10d. The values of N_ϕ at points A and B are obtained from Eqs. (2) and (4), respectively. The vertical and horizontal components of these forces are shown at points A and B in Figure 1.10d. The unbalanced vertical forces result in a downward force at point C of magnitude $0.7095\gamma R^2$. The total force on the cylinder is $(0.7095\gamma R^2)(2\pi)(R)(\sin(180 - 110))$. This total force is equal to the total weight of the contents in the sphere given by (4/3) $(\pi R^3)\gamma$. The required thickness of the cylinder is

$$t = \frac{0.7095(62.4)(30)^2/12}{10,000}$$
$$= 0.33 \text{ inch.}$$

Summation of horizontal forces at points A and B results in a compressive force of magnitude $0.2583\gamma R^2$. The needed area of compression ring at the cylinder to sphere junction is

$$A = \frac{Hr}{\sigma} = \frac{0.2583 \times 62.4 \times 30^2(30 \sin 70)}{10,000}$$
$$= 40.89 \text{ inch}^2.$$

This area is furnished by a large ring added to the sphere or an increase in the thickness of the sphere at the junction.

1.3.3 ASME Code Equations for Spherical Shells under Various Loading Conditions

Loading conditions such as those shown in Examples 1.1 through 1.3 are not specifically covered by equations in the boiler and pressure vessel codes. However, they are addressed in paragraph PG-16.1 of Section I, paragraph U-2(g) of Section VIII-1, and paragraph 4.1.1 of Section VIII-2 using special analysis.

In the nuclear code, paragraph NC-3932.2 of Section NC and ND-3932.2 of Section ND provide equations for calculating forces at specific locations in a shell due to loading conditions similar to those shown in Examples 1.1 through 1.5. This procedure is discussed further in Chapter 2.

1.3.4 Ellipsoidal Shells under Internal Pressure

Ellipsoidal heads of all sizes and shapes are used in the ASME code as end closure for pressure components. The general configuration is shown in Figure 1.11.

Small-size heads are formed by using dyes shaped to a true ellipse. However, large diameter heads formed from plate segments are in the shapes of spherical and torispherical geometries that simulate ellipses as shown in Figures 1.12 and 1.13. Figure 1.12 shows an ASME

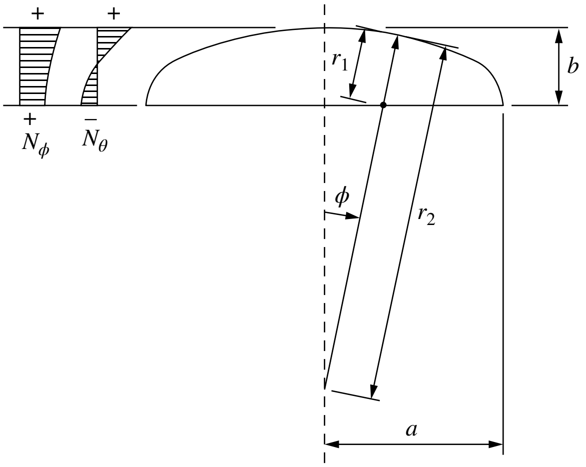


Figure 1.11 Ellipsoidal head under internal pressure

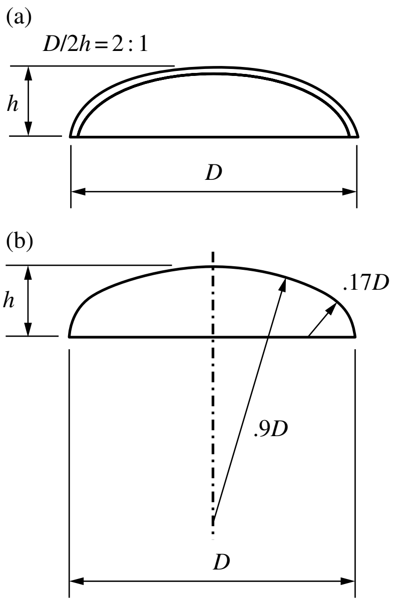


Figure 1.12 2 : 1 elliptical head: (a) exact configuration and (b) approximate configuration

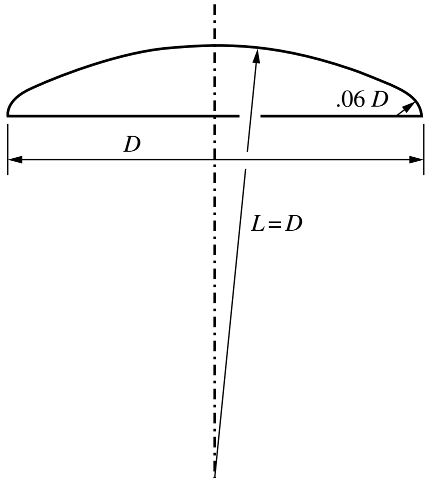


Figure 1.13 Elliptical head with $a/b = 2.96$ ratio

equivalent 2 : 1 ellipsoidal head. It consists of a spherical segment with $R = 0.9D$ and a knuckle with $r = 0.17D$ where D is the base diameter of the head. Figure 1.13 shows a shallow head (2.96 : 1 ratio) referred to as flanged and dished (F&D) head consisting of a spherical segment with $R = D$ and a knuckle section with $r = 0.06D$.

For internal pressure we define $p_r = p$ and $p_\phi = 0$. Then from Eqs. (1.1) and (1.10),

$$N_\phi = \frac{1}{r_2 \sin^2\phi}\left(p\int r\,dr + C\right)$$

$$N_\phi = \frac{1}{r_2 \sin^2\phi}\left(\frac{pr^2}{2} + C\right). \tag{1.15}$$

The constant C is obtained from the following boundary condition:

$$\text{At}\ \ \phi = \frac{\pi}{2},\ \ r_2 = r\ \ \text{and}\ \ N_\phi = \frac{pr}{2}.$$

Hence, from Eq. (1.15) we get $C = 0$ and N_ϕ can be expressed as

$$N_\phi = \frac{pr^2}{2r_2 \sin^2\phi}$$

or

$$N_\phi = \frac{pr_2}{2}. \tag{1.16}$$

From Eq. (1.9),

$$N_\theta = pr_2\left(1 - \frac{r_2}{2r_1}\right). \tag{1.17}$$

From analytical geometry, the relationship between the major and minor axes of an ellipse and r_1 and r_2 is given by

$$r_1 = \frac{a^2b^2}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{3/2}}$$

$$r_2 = \frac{a^2}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{1/2}}.$$

Substituting these expressions into Eqs. (1.16) and (1.17) gives the following expressions for membrane forces in ellipsoidal shells due to internal pressure:

$$N_\phi = \frac{pa^2}{2}\frac{1}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{1/2}} \tag{1.18}$$

$$N_\theta = \frac{pa^2}{2b^2}\frac{b^2 - (a^2 - b^2)\sin^2\phi}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{1/2}}. \tag{1.19}$$

The maximum tensile force in an ellipsoidal head is at the apex as shown in Figure 1.11. The maximum value is obtained from Eqs. (1.18) and (1.19) by letting $\phi = 0$. This gives

$$N_\phi = N_\theta = \frac{Pa^2}{2b} \tag{1.20}$$

For a 2 : 1 head with $a/b = 2$ and $a = D/2$, Eq. (1.18) becomes

$$N_\phi = N_\theta = 0.5PD \tag{1.21}$$

where D is the base diameter. Comparing this equation with Eq. (1.13) for spherical heads shows that the force, and thus the stress, in a 2 : 1 ellipsoidal head is twice that of a spherical head having the same base diameter.

For an F&D head with $a/b = 2.96$ and $a = D/2$, Eq. (1.18) becomes

$$N_\phi = N_\theta = 0.74PD \tag{1.22}$$

Comparing this equation with Eq. (1.13) for spherical heads shows the stress of a 2.95 : 1 F&D head at the apex is 2.96 times that of a spherical head having the same base diameter.

A plot of Eqs. (1.18) and (1.19) is shown in Figure 1.11. Equation (1.18) for the longitudinal force, N_ϕ, is always in tension regardless of the a/b ratio. Equation (1.19) for N_θ on the other hand gives compressive circumferential forces near the equator when the value $a/b \geq \sqrt{2}$.

For large *a/b* ratios under internal pressure, the compressive circumferential force tends to increase in magnitude, whereas instability may occur for large *a/t* ratios. This extreme care must be exercised by the engineer to avoid buckling failure. The ASME code contains design rules that take into account the instability of shallow ellipsoidal shells due to internal pressure as described in Chapter 5.

Example 1.4
Determine the required thickness of a 2 : 1 ellipsoidal head with $a = 30$ inches, $b = 15$ inches, and $P = 500$ psi, and allowable stress in tension is $S = 20{,}000$ psi.

Solution
At the apex, $\phi = 0$, and from Eqs. (1.18) and (1.19),

$$N_\phi = Pa \quad \text{and} \quad N_\theta = Pa$$

At the equator, $\phi = 90°$, and from Eqs. (1.18) and (1.19),

$$N_\phi = \frac{Pa}{2} \quad \text{and} \quad N_\theta = -Pa$$

Thus, the required thickness is $t = Pa/S = 500(30)/20{,}000 = 0.75$ inch.

Notice at the equator, N_θ is compressive and may result in instability as discussed in Chapter 5.

1.4 Conical Shells

Equations (1.9) and (1.10) cannot readily be used for analyzing conical shells because the angle ϕ in a conical shell is constant. Hence, the two equations have to be modified accordingly. Referring to Figure 1.14, it can be shown that

$$\left.\begin{aligned} \phi &= \beta = \text{constant} \\ r_1 &= \infty \quad r_2 = s\tan\alpha \quad r = s\sin\alpha \\ N_\phi &= N_s. \end{aligned}\right] \tag{1.23}$$

Equation (1.9) can be written as

$$\frac{N_s}{r_1} + \frac{N_\theta}{s\tan\alpha} = p_r$$

or since $r_1 = \infty$,

$$\left.\begin{aligned} N_\theta &= p_r s\tan\alpha \\ &= p_r r_2 \\ N_\theta &= \frac{p_r r}{\cos\alpha} \end{aligned}\right]. \tag{1.24}$$

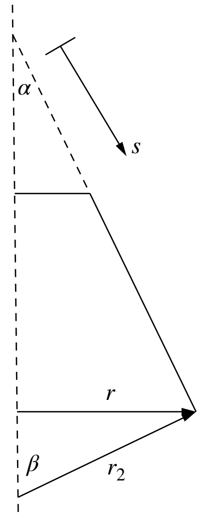


Figure 1.14 Conical shell

Similarly from Eqs. (1.1) and (1.7),

$$\frac{d}{ds} r_1 (s \sin\alpha N_s) - r_1 N_\theta \sin\alpha = -p_s s \sin\alpha r_1.$$

Substituting Eq. (1.24) into this equation results in

$$N_s = \frac{-1}{s}\left[\int (p_s - p_r \tan\alpha) s ds + C\right]. \tag{1.25}$$

It is of interest to note that while N_θ is a function of N_ϕ for shells with double curvature, it is independent of N_ϕ for conical shells as shown in Eqs. (1.24) and (1.25). Also, as α approaches 0°, Eq. (1.25) becomes

$$N_\theta = p_r r_2,$$

which is the expression for the circumferential hoop force in a cylindrical shell.

The analysis of conical shells consists of solving the forces in Eqs. (1.24) and (1.25) for any given loading condition. The thickness is then determined from the maximum forces and a given allowable stress.

Equations (1.24) and (1.25) and Figure 1.15 will be used to determine forces in a conical shell due to internal pressure.

From Eq. (1.24), the maximum N_θ occurs at the large end of the cone and is given by

$$N_\theta = p\left(\frac{r_o}{\sin\alpha}\right)\tan\alpha = \frac{p r_o}{\cos\alpha}. \tag{1.26}$$

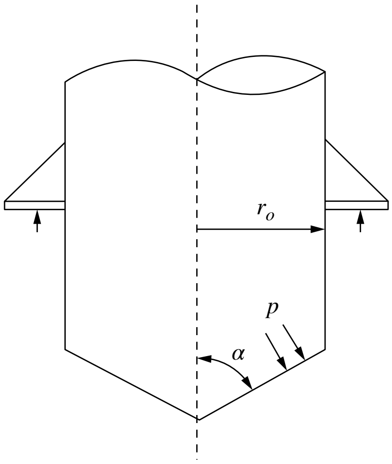


Figure 1.15 Conical bottom head

From Eq. (1.25),

$$\begin{aligned} N_s &= \frac{-1}{s}\left(\int -p\tan\alpha s ds + C\right) \\ &= \frac{-1}{s}\left(-p\tan\alpha\frac{s^2}{2}+C\right). \end{aligned} \tag{1.27}$$

$$\text{At } s = L, \quad N_s = \frac{pr_o}{2}\frac{1}{\cos\alpha}$$

Substituting this expression into Eq. (1.27), and using the relationships of Eq. (1.23), gives $C = 0$. Equation (1.27) becomes

$$\begin{aligned} N_s &= \frac{pr}{2\cos\alpha} \\ \text{and max } N_s &= \frac{pr_o}{2\cos\alpha}. \end{aligned} \tag{1.28}$$

It is of interest to note that the longitudinal and hoop forces are identical to those of a cylinder with equivalent radius of $r_o/\cos\alpha$.

All sections of the ASME code have equations for designing conical sections based on Eqs. (1.26) and (1.28).

1.5 Cylindrical Shells

Equipment consisting of cylindrical shells subjected to pressure and axial loads are frequently encountered in refineries and chemical plants. If the radius of the shell is designated by R, Figure 1.16a, then from Figure 1.3 $r_1 = \infty$, $\phi = 90°$, $P = p_r$, and $r = r_2 = R$. The value of the

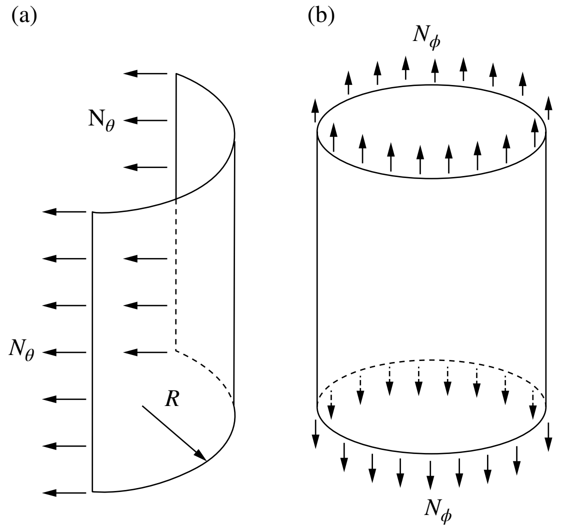


Figure 1.16 Cylindrical shell: (a) circumferential force and (b) longitudinal force

circumferential force N_θ can be obtained by equating the pressure acting on the cross section, Figure 1.16a, to the forces in the material at the cross section. This results in

$$N_\theta = p_r R \tag{1.29}$$

The required thickness, t, of a cylindrical shell due to internal pressure is obtained from Eq. (1.29) as

$$t = \frac{PR}{S} \tag{1.30}$$

where S is the allowable stress and t is the thickness.

The required thickness of cylindrical shells in the ASME code is obtained from a modified Eq. (1.30) that takes into consideration stress variation in the wall of the cylinder for small R/t ratios. This equation is described in Chapter 3.

Similarly, the value of the axial force N_ϕ is obtained by equating the pressure acting on the cross section, Figure 1.16b, to the forces in the material at the cross section. This yields

$$N_\phi = \frac{p_r R}{2} \tag{1.31}$$

The corresponding stress and thickness are obtained from Eq. (1.31) as

$$t = \frac{PR}{2S} \tag{1.32}$$

1.6 Cylindrical Shells with Elliptical Cross Section

When the cross section of a thin cylinder is elliptical, Figure 1.17, rather than circular in shape, then the relationship

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \tag{1.33}$$

is incorporated in the basic equations and the value of circumferential stress N_θ becomes

$$N_\theta = \frac{p_\mathrm{r} a^2 b^2}{\left(a^2 \sin^2\phi + b^2 \cos^2\phi\right)^{3/2}} \tag{1.34}$$

where a, b, and ϕ are as defined in Figure 1.17. The value of N_ϕ is obtained from a free-body diagram at a given location.

Equation (1.34) is utilized in cases where a fabricated cylinder is slightly out-of-round and the required thickness based on the obround geometry is needed. This is illustrated in the following example.

Example 1.5

A cylindrical pressure vessel is constructed of steel plates and has an internal pressure of 100 psi. The allowable stress of steel is 20 ksi. Find the required thickness of the cylinder when the cross section is (a) circular with a diameter of 96 inches and (b) elliptical with a minor diameter of 92 inches and a major diameter of 100 inches.

Solution

a. From Eq. (1.29),

$$N_\theta = (100)\left(\frac{96}{2}\right) = 4800\,\text{lbs/inch}.$$

The thickness is obtained from the relationship $t = N_\theta/\sigma$

$$t = \frac{4800}{20{,}000}$$

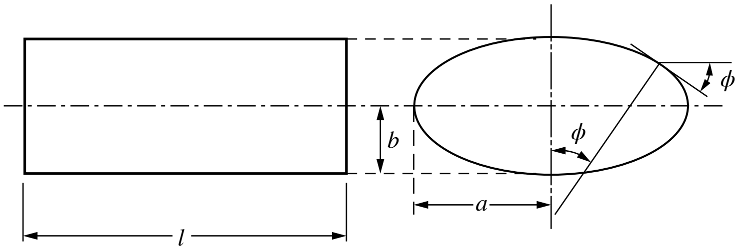


Figure 1.17 Elliptical shell

or

$$t = 0.24\ \text{inch.}$$

b. From Eq. (1.34), the maximum value of N_θ occurs at $\phi = 0°$.

$$N_\theta = \frac{(100)(50)^2(46)^2}{\left(50^2\sin^2 0 + 46^2\cos^2 0\right)^{3/2}}$$

$$N_\theta = \frac{529{,}000{,}000}{\left(46^2\right)^{3/2}}$$

$$N_\theta = 5435\ \text{lbs/inch}$$

or

$$t = \frac{5435}{20{,}000}$$
$$t = 0.27\ \text{inch.}$$

Hence, the thickness needs to be increased from 0.24 to 0.27 inch due to the obround shape of the cylinder.

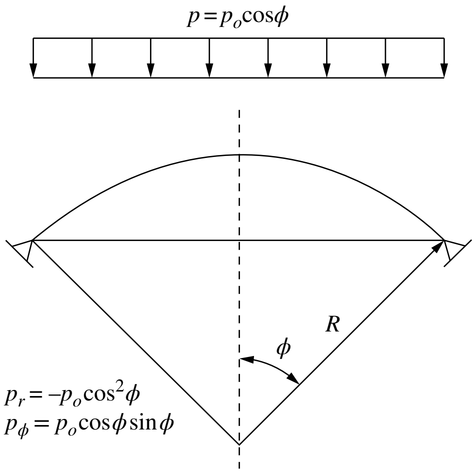
1.7 Design of Shells of Revolution

The maximum forces for various shell geometries subjected to commonly encountered loading conditions are listed in numerous references. One such reference is by NASA (Baker et al. 1968), where extensive tables and design charts are listed. Flugge (1967) contains a thorough coverage of a wide range of applications to the membrane theory, as does Roark's handbook (Young et al. 2012).

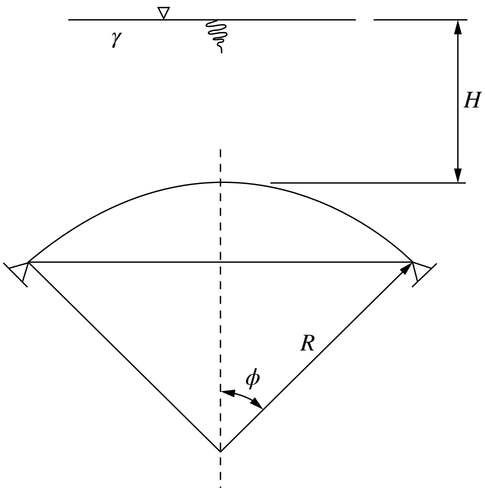
Problems

1.1 Derive Eq. (1.10).
1.2 Determine the forces in the spherical roof of a flat bottom tank due to the snow load shown.
1.3 Determine the values of N_ϕ and N_θ in the underwater spherical shell shown. For hydrostatic pressure, let

$$p_\phi = 0$$
$$p_r = -\gamma[H + R(1 - \cos\phi)]$$



Problem 1.2 Snow load on a spherical roof



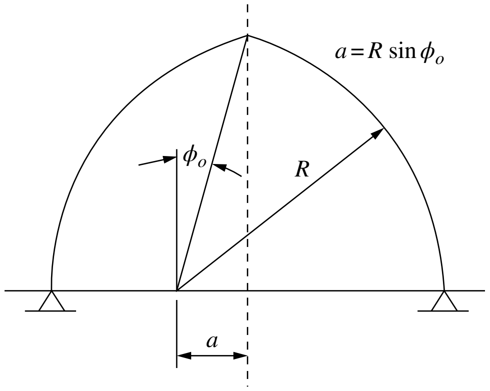
Problem 1.3 Underwater spherical shell

1.4 Determine the forces in the roof of the silo shown due to deadweight, γ. The equivalent pressure is expressed as

$$p_\phi = \gamma \sin\phi$$

$$p_{\mathrm{r}} = -\gamma \cos\phi.$$

1.5 Plot the values of N_ϕ and N_θ as a function of ϕ in an ellipsoidal shell with a ratio of 3 : 1 and subjected to internal pressure.

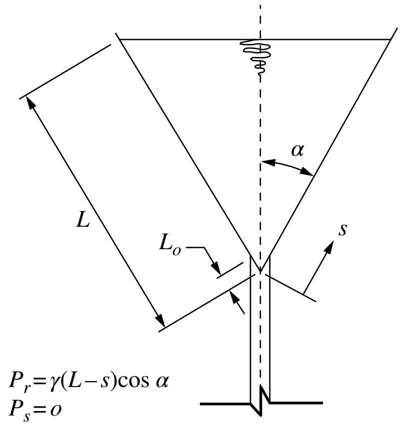


Problem 1.4 Roof of a silo

1.6 The nose of a submersible titanium vehicle is made of a 2 : 1 ellipsoidal head. Calculate the required thickness due to an external pressure of 300 psi. Let $a = 30$ inches, $b = 15$ inches, and the allowable compressive stress = 10 ksi.

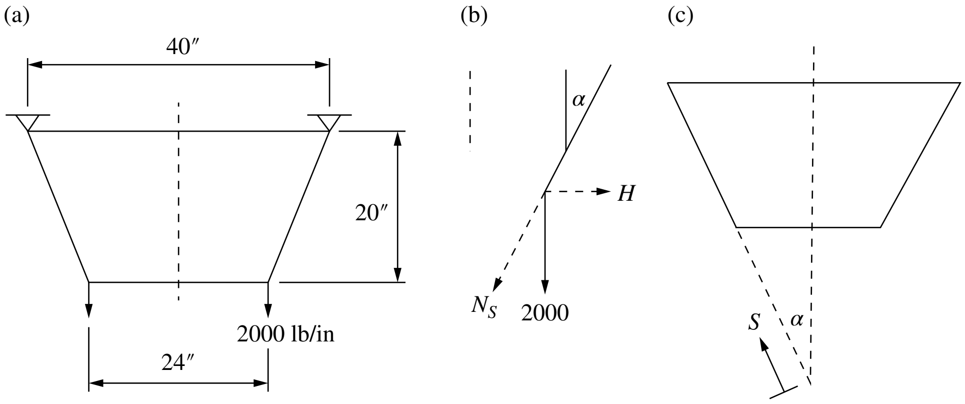
1.7 An ellipsoidal head with $a/b = 2$ has a maximum stress at the apex of 20 ksi. A nozzle is required at $\phi = 45°$ (Figure 1.14). What is the stress in the head at this location in order to properly reinforce the nozzle?

1.8 Determine the maximum values and locations of N_s and N_θ in the small holding tank shown. Let $L = 22$ ft, $L_o = 2$ ft, $\gamma = 50$ pcf, and $\alpha = 30°$.



Problem 1.8 Holding tank

1.9 The lower portion of a reactor is subjected to a radial nozzle load as shown. Determine the required thickness of the conical section. Use an allowable stress of 10 ksi. What is the required area of the ring at the point of application of the load?



Problem 1.9 Nozzle load in a conical shell: (a) axial force, (b) components of axial force, and (c) dimension origin