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Elementary Differential Equations and Boundary Value Problems

Eleventh Edition

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To Elsa, Betsy, and in loving memory of Maureen

*To Siobhan, James, Richard Jr., Carolyn, Ann, Stuart,
Michael, Marybeth, and Bradley*

*And to the next generation:
Charles, Aidan, Stephanie, Veronica, and Deirdre*

The Authors

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DOUGLAS B. MEADE received B.S. degrees in Mathematics and Computer Science from Bowling Green State University, an M.S. in Applied Mathematics from Carnegie Mellon University, and a Ph.D. in mathematics from Carnegie Mellon University. After a two-year stint at Purdue University, he joined the mathematics faculty at the University of South Carolina, where he is currently an Associate Professor of mathematics and the Associate Dean for Instruction, Curriculum, and Assessment in the College of Arts and Sciences. He is a member of the American Mathematical Society, Mathematics Association of America, and Society for Industrial and Applied Mathematics; in 2016 he was named an ICTCM Fellow at the International Conference on Technology in Collegiate Mathematics (ICTCM). His primary research interests are in the numerical solution of partial differential equations arising from wave propagation problems in unbounded domains and from population models for infectious diseases. He is also well-known for his educational uses of computer algebra systems, particularly Maple. These include *Getting Started with Maple* (with M. May, C-K. Cheung, and G. E. Keough, Wiley, 2009, ISBN 978-0-470-45554-8), *Engineer’s Toolkit: Maple for Engineers* (with E. Bourkoff, Addison-Wesley, 1998, ISBN 0-8053-6445-5), and numerous Maple supplements for numerous calculus, linear algebra, and differential equations textbooks - including previous editions of this book. He was a member of the MathDL New Collections Working Group for Single Variable Calculus, and chaired the Working Groups for Differential Equations and Linear Algebra. The NSF is partially supporting his work, together with Prof. Philip Yasskin (Texas A&M), on the Maplets for Calculus project.

Preface

As we have prepared an updated edition our first priorities are to preserve, and to enhance, the qualities that have made previous editions so successful. In particular, we adopt the viewpoint of an applied mathematician with diverse interests in differential equations, ranging from quite theoretical to intensely practical—and usually a combination of both. Three pillars of our presentation of the material are methods of solution, analysis of solutions, and approximations of solutions. Regardless of the specific viewpoint adopted, we have sought to ensure the exposition is simultaneously correct and complete, but not needlessly abstract.

The intended audience is undergraduate STEM students whose degree program includes an introductory course in differential equations during the first two years. The essential prerequisite is a working knowledge of calculus, typically a two- or three-semester course sequence or an equivalent. While a basic familiarity with matrices is helpful, Sections 7.2 and 7.3 provide an overview of the essential linear algebra ideas needed for the parts of the book that deal with systems of differential equations (the remainder of Chapter 7, Section 8.5, and Chapter 9).



A strength of this book is its appropriateness in a wide variety of instructional settings. In particular, it allows instructors flexibility in the selection of and the ordering of topics and in the use of technology. The essential core material is Chapter 1, Sections 2.1 through 2.5, and Sections 3.1 through 3.5. After completing these sections, the selection of additional topics, and the order and depth of coverage are generally at the discretion of the instructor. Chapters 4 through 11 are essentially independent of each other, except that Chapter 7 should precede Chapter 9, and Chapter 10 should precede Chapter 11.

A particularly appealing aspect of differential equations is that even the simplest differential equations have a direct correspondence to realistic physical phenomena: exponential growth and decay, spring-mass systems, electrical circuits, competitive species, and wave propagation. More complex natural processes can often be understood by combining and building upon simpler and more basic models. A thorough knowledge of these basic models, the differential equations that describe them, and their solutions—either explicit solutions or qualitative properties of the solution—is the first and indispensable step toward analyzing the solutions of more complex and realistic problems. The modeling process is detailed in Chapter 1 and Section 2.3. Careful constructions of models appear also in Sections 2.5, 3.7, 9.4, 10.5, and 10.7 (and the appendices to Chapter 10). Various problem sets throughout the book include problems that involve modeling to formulate an appropriate differential equation, and then to solve it or to determine some qualitative properties of its solution. The primary purposes of these applied problems are to provide students with hands-on experience in the derivation of differential equations, and to convince them that differential

equations arise naturally in a wide variety of real-world applications.

Another important concept emphasized repeatedly throughout the book is the transportability of mathematical knowledge. While a specific solution method applies to only a particular class of differential equations, it can be used in any application in which that particular type of differential equation arises. Once this point is made in a convincing manner, we believe that it is unnecessary to provide specific applications of every method of solution or type of equation that we consider. This decision helps to keep this book to a reasonable size, and allows us to keep the primary emphasis on the development of more solution methods for additional types of differential equations.

From a student's point of view, the problems that are assigned as homework and that appear on examinations define the course. We believe that the most outstanding feature of this book is the number, and above all the variety and range, of the problems that it contains. Many problems are entirely straightforward, but many others are more challenging, and some are fairly open-ended and can even serve as the basis for independent student projects. The observant reader will notice that there are fewer problems in this edition than in previous editions; many of these problems remain available to instructors via the WileyPlus course. The remaining 1600 problems are still far more problems than any instructor can use in any given course, and this provides instructors with a multitude of choices in tailoring their course to meet their own goals and the needs of their students. The answers to almost all of these problems can be found in the pages at the back of the book; full solutions are in either the Student's Solution Manual or the Instructor's Solution Manual.

While we make numerous references to the use of technology, we do so without limiting instructor freedom to use as much, or as little, technology as they desire. Appropriate technologies include advanced graphing calculators (TI Nspire), a spreadsheet (Excel), web-based resources (applets), computer algebra systems, (Maple, Mathematica, Sage), scientific computation systems (MATLAB), or traditional programming (FORTRAN, Javascript, Python). Problems marked with a  are ones we believe are best approached with a graphical tool; those marked with a  are best solved with the use of a numerical tool. Instructors should consider setting their own policies, consistent with their interests and intents about student use of technology when completing assigned problems.

Many problems in this book are best solved through a combination of analytic, graphic, and numeric methods. Pencil-and-paper methods are used to develop a model that is best solved (or analyzed) using a symbolic or graphic tool. The quantitative results and graphs, frequently produced using computer-based resources, serve to illustrate and to clarify conclusions that might not be readily apparent from a complicated explicit solution formula. Conversely, the

implementation of an efficient numerical method to obtain an approximate solution typically requires a good deal of preliminary analysis—to determine qualitative features of the solution as a guide to computation, to investigate limiting or special cases, or to discover ranges of the variables or parameters that require an appropriate combination of both analytic and numeric computation. Good judgment may well be required to determine the best choice of solution methods in each particular case. Within this context we point out that problems that request a “sketch” are generally intended to be completed without the use of any technology (except your writing device).

We believe that it is important for students to understand that (except perhaps in courses on differential equations) the goal of solving a differential equation is seldom simply to obtain the solution. Rather, we seek the solution in order to obtain insight into the behavior of the process that the equation purports to model. In other words, the solution is not an end in itself. Thus, we have included in the text a great many problems, as well as some examples, that call for conclusions to be drawn about the solution. Sometimes this takes the form of finding the value of the independent variable at which the solution has a certain property, or determining the long-term behavior of the solution. Other problems ask for the effect of variations in a parameter, or for the determination of all values of a parameter at which the solution experiences a substantial change. Such problems are typical of those that arise in the applications of differential equations, and, depending on the goals of the course, an instructor has the option of assigning as few or as many of these problems as desired.

Readers familiar with the preceding edition will observe that the general structure of the book is unchanged. The minor revisions that we have made in this edition are in many cases the result of suggestions from users of earlier editions. The goals are to improve the clarity and readability of our presentation of basic material about differential equations and their applications. More specifically, the most important revisions include the following:

1. Chapter 1 has been rewritten. Instead of a separate section on the History of Differential Equations, this material appears in three installments in the remaining three sections.
2. Additional words of explanation and/or more explicit details in the steps in a derivation have been added throughout each chapter. These are too numerous and widespread to mention individually, but collectively they should help to make the book more readable for many students.
3. There are about forty new or revised problems scattered throughout the book. The total number of problems has been reduced by about 400 problems, which are still available through WileyPlus, leaving about 1600 problems in print.
4. There are new examples in Sections 2.1, 3.8, and 7.5.
5. The majority (is this correct?) of the figures have been redrawn, mainly by the use full color to allow for easier identification of critical properties of the solution. In addition, numerous captions have been expanded to clarify the purpose of the figure without requiring a search of the surrounding text.
6. There are several new references, and some others have been updated.

The authors have found differential equations to be a never-ending source of interesting, and sometimes surprising, results and phenomena. We hope that users of this book, both students and instructors, will share our enthusiasm for the subject.

William E. Boyce and Douglas B. Meade
Watervliet, New York and Columbia, SC
29 August 2016

Supplemental Resources for Instructors and Students

An Instructor’s Solutions Manual, ISBN 978-1-119-16976-5, includes solutions for all problems not contained in the Student Solutions Manual.

A Student Solutions Manual, ISBN 978-1-119-16975-8, includes solutions for selected problems in the text.

A Book Companion Site, www.wiley.com/college/boyce, provides a wealth of resources for students and instructors, including

- PowerPoint slides of important definitions, examples, and theorems from the book, as well as graphics for presentation in lectures or for study and note taking.
- Chapter Review Sheets, which enable students to test their knowledge of key concepts. For further review, diagnostic feedback is provided that refers to pertinent sections in the text.
- Mathematica, Maple, and MATLAB data files for selected problems in the text providing opportunities for further exploration of important concepts.
- Projects that deal with extended problems normally not included among traditional topics in differential equations, many involving applications from a variety of disciplines. These vary in length and complexity, and they can be assigned as individual homework or as group assignments.

A series of supplemental guidebooks, also published by John Wiley & Sons, can be used with Boyce/DiPrima/Meade in order to incorporate computing technologies into the course. These books emphasize numerical methods and graphical analysis, showing how these methods enable us to interpret solutions of ordinary differential equations (ODEs) in the real world. Separate guidebooks cover each of the three major mathematical software formats, but the ODE subject matter is the same in each.

- Hunt, Lipsman, Osborn, and Rosenberg, *Differential Equations with MATLAB*, 3rd ed., 2012, ISBN 978-1-118-37680-5

- Hunt, Lardy, Lipsman, Osborn, and Rosenberg, *Differential Equations with Maple*, 3rd ed., 2008, ISBN 978-0-471-77317-7
- Hunt, Outing, Lipsman, Osborn, and Rosenberg, *Differential Equations with Mathematica*, 3rd ed., 2009, ISBN 978-0-471-77316-0

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- The E-book, which is an exact version of the print text but also features hyperlinks to questions, definitions, and supplements for quicker and easier support.
- Guided Online (GO) Exercises, which prompt students to build solutions step-by-step. Rather than simply grading an exercise answer as wrong, GO problems show students precisely where they are making a mistake.
- Homework management tools, which enable instructors easily to assign and grade questions, as well as to gauge student comprehension.
- QuickStart pre-designed reading and homework assignments. Use them as is, or customize them to fit the needs of your classroom.

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It is a pleasure to express my appreciation to the many people who have generously assisted in various ways in the preparation of this book.

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WILLIAM E. BOYCE AND DOUGLAS B. MEADE

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