CHAPTER 1

Systems of Linear Equations and Matrices

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Introduction

Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

and that the solution of the system can be obtained by performing appropriate operations on this matrix. This is particularly important in developing computer programs for solving systems of equations because computers are well suited for manipulating arrays of numerical information. However, matrices are not simply a notational tool for solving systems of equations; they can be viewed as mathematical objects in their own right, and there is a rich and important theory associated with them that has a multitude of practical applications. It is the study of matrices and related topics that forms the mathematical field that we call "linear algebra." In this chapter we will begin our study of matrices.

1.1 Introduction to Systems of Linear Equations

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. In this first section we will introduce some basic terminology and discuss a method for solving such systems.

Linear Equations

Recall that in two dimensions a line in a rectangular *xy*-coordinate system can be represented by an equation of the form

$$ax + by = c \quad (a, b \text{ not both } 0)$$

and in three dimensions a plane in a rectangular *xyz*-coordinate system can be represented by an equation of the form

$$ax + by + cz = d \quad (a, b, c \text{ not all } 0)$$

These are examples of "linear equations," the first being a linear equation in the variables x and y and the second a linear equation in the variables x, y, and z. More generally, we define a ***linear equation*** in the n variables $x_1, x_2, \dots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

where $a_1, a_2, \dots, a_n$ and b are constants, and the a's are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$$a_1x + a_2y = b \tag{2}$$

$$a_1x + a_2y + a_3z = b \tag{3}$$

In the special case where $b = 0$, Equation (1) has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \tag{4}$$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \dots, x_n$.

EXAMPLE 1 | Linear Equations

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

$$x + 3y = 7 \qquad x_1 - 2x_2 - 3x_3 + x_4 = 0$$

$$\tfrac{1}{2}x - y + 3z = -1 \qquad x_1 + x_2 + \cdots + x_n = 1$$

The following are not linear equations:

$$x + 3y^2 = 4 \qquad 3x + 2y - xy = 5$$

$$\sin x + y = 0 \qquad \sqrt{x_1} + 2x_2 + x_3 = 1$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns***. For example, system (5) that follows has unknowns x and y, and system (6) has unknowns x_1, x_2, and x_3.

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \begin{aligned} 4x_1 - x_2 + 3x_3 &= -1 \\ 3x_1 + x_2 + 9x_3 &= -4 \end{aligned} \tag{5–6}$$

A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{7}$$

The double subscripting on the coefficients a_{ij} of the unknowns gives their location in the system—the first subscript indicates the equation in which the coefficient occurs, and the second indicates which unknown it multiplies. Thus, a_{12} is in the first equation and multiplies x_2.

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

makes each equation a true statement. For example, the system in (5) has the solution

$$x = 1, \quad y = -2$$

and the system in (6) has the solution

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$$

These solutions can be written more succinctly as

$$(1, -2) \quad \text{and} \quad (1, 2, -1)$$

in which the names of the variables are omitted. This notation allows us to interpret these solutions geometrically as points in two-dimensional and three-dimensional space. More generally, a solution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

of a linear system in n unknowns can be written as

$$(s_1, s_2, \ldots, s_n)$$

which is called an ***ordered n-tuple***. With this notation it is understood that all variables appear in the same order in each equation. If $n = 2$, then the n-tuple is called an ***ordered pair***, and if $n = 3$, then it is called an ***ordered triple***.

Linear Systems in Two and Three Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities (**Figure 1.1.1**):

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

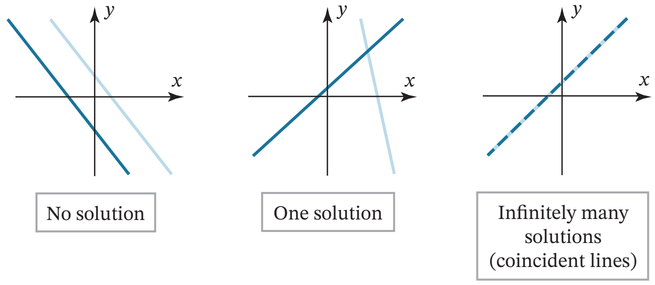


FIGURE 1.1.1

In general, we say that a linear system is ***consistent*** if it has at least one solution and ***inconsistent*** if it has no solutions. Thus, a *consistent* linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions (**Figure 1.1.2**).

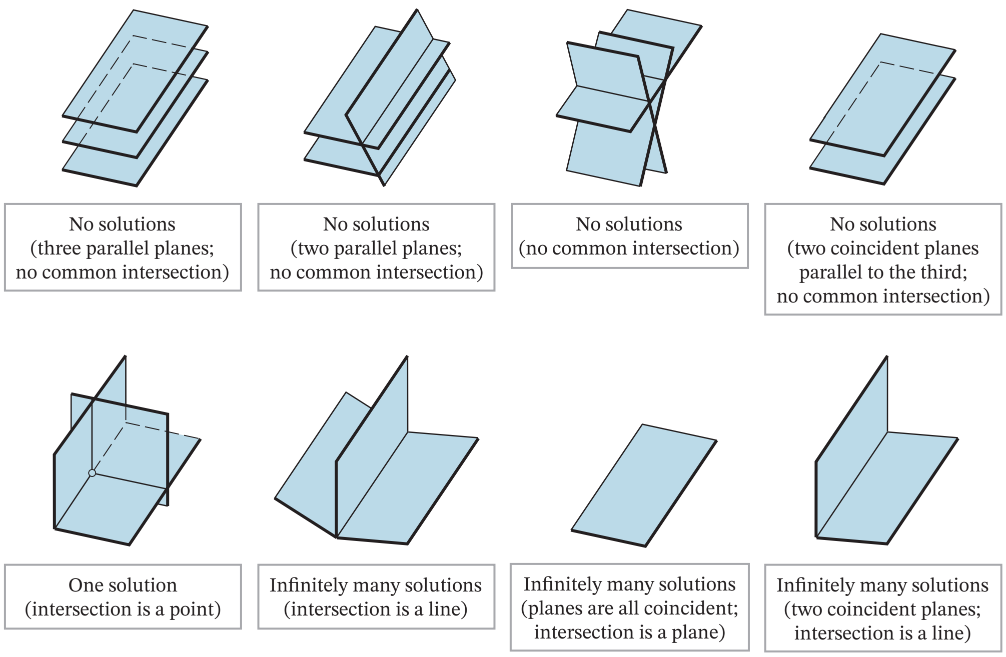


FIGURE 1.1.2

We will prove later that our observations about the number of solutions of linear systems of two equations in two unknowns and linear systems of three equations in three unknowns actually hold for *all* linear systems. That is:

> *Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.*

EXAMPLE 2 | A Linear System with One Solution

Solve the linear system

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -2 times the first equation to the second. This yields the simplified system

$$\begin{aligned} x - y &= 1 \\ 3y &= 4 \end{aligned}$$

From the second equation we obtain $y = \frac{4}{3}$, and on substituting this value in the first equation we obtain $x = 1 + y = \frac{7}{3}$. Thus, the system has the unique solution

$$x = \tfrac{7}{3}, \quad y = \tfrac{4}{3}$$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point $\left(\frac{7}{3}, \frac{4}{3}\right)$. We leave it for you to check this by graphing the lines.

EXAMPLE 3 | A Linear System with No Solutions

Solve the linear system

$$\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -3 times the first equation to the second equation. This yields the simplified system

$$\begin{aligned} x + y &= 4 \\ 0 &= -6 \end{aligned}$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. We leave it for you to check this by graphing the lines or by showing that they have the same slope but different y-intercepts.

EXAMPLE 4 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

Solution We can eliminate x from the second equation by adding -4 times the first equation to the second. This yields the simplified system

$$\begin{aligned} 4x - 2y &= 1 \\ 0 &= 0 \end{aligned}$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \tag{8}$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide. One way to describe the solution set is to solve this equation for x in terms of y to

In Example 4 we could have also obtained parametric equations for the solutions by solving (8) for y in terms of x and letting $x = t$ be the parameter. The resulting parametric equations would look different but would define the same solution set.

obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value t (called a ***parameter***) to y. This allows us to express the solution by the pair of equations (called ***parametric equations***)

$$x = \tfrac{1}{4} + \tfrac{1}{2}t, \quad y = t$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For example, $t = 0$ yields the solution $\left(\frac{1}{4}, 0\right)$, $t = 1$ yields the solution $\left(\frac{3}{4}, 1\right)$, and $t = -1$ yields the solution $\left(-\frac{1}{4}, -1\right)$. You can confirm that these are solutions by substituting their coordinates into the given equations.

EXAMPLE 5 | A Linear System with Infinitely Many Solutions

Solve the linear system

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \tag{9}$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of (9). We can do this by first solving this equation for x in terms of y and z, then assigning arbitrary values r and s (parameters) to these two variables, and then expressing the solution by the three parametric equations

$$x = 5 + r - 2s, \quad y = r, \quad z = s$$

Specific solutions can be obtained by choosing numerical values for the parameters r and s. For example, taking $r = 1$ and $s = 0$ yields the solution $(6, 1, 0)$.

Augmented Matrices and Elementary Row Operations

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra involved in finding solutions. The required computations can be made more manageable by simplifying notation and standardizing procedures. For example, by mentally keeping track of the location of the +'s, the x's, and the ='s in the linear system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

we can abbreviate the system by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

As noted in the introduction to this chapter, the term "matrix" is used in mathematics to denote a rectangular array of numbers. In a later section we will study matrices in detail, but for now we will only be concerned with augmented matrices for linear systems.

This is called the ***augmented matrix*** for the system. For example, the augmented matrix for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned} \quad \text{is} \quad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Historical Note

Maxime Bôcher (1867–1918)

The first known use of augmented matrices appeared between 200 B.C. and 100 B.C. in a Chinese manuscript entitled *Nine Chapters of Mathematical Art*. The coefficients were arranged in columns rather than in rows, as today, but remarkably the system was solved by performing a succession of operations on the columns. The actual use of the term *augmented matrix* appears to have been introduced by the American mathematician Maxime Bôcher in his book *Introduction to Higher Algebra*, published in 1907. In addition to being an outstanding research mathematician and an expert in Latin, chemistry, philosophy, zoology, geography, meteorology, art, and music, Bôcher was an outstanding expositor of mathematics whose elementary textbooks were greatly appreciated by students and are still in demand today.

[Image: HUP Bocher, Maxime (1), olvwork650836]

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are. Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 | Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add -2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1,\ y = 2,\ z = 3$ is now evident.

The solution in this example can also be expressed as the ordered triple $(1, 2, 3)$ with the understanding that the numbers in the triple are in the same order as the variables in the system, namely, x, y, z.

Exercise Set 1.1

1. In each part, determine whether the equation is linear in x_1, x_2, and x_3.

a. $x_1 + 5x_2 - \sqrt{2}x_3 = 1$ **b.** $x_1 + 3x_2 + x_1x_3 = 2$

c. $x_1 = -7x_2 + 3x_3$ **d.** $x_1^{-2} + x_2 + 8x_3 = 5$

e. $x_1^{3/5} - 2x_2 + x_3 = 4$ **f.** $\pi x_1 - \sqrt{2}x_2 = 7^{1/3}$

2. In each part, determine whether the equation is linear in x and y.

a. $2^{1/3}x + \sqrt{3}y = 1$ **b.** $2x^{1/3} + 3\sqrt{y} = 1$

c. $\cos\left(\frac{\pi}{7}\right)x - 4y = \log 3$ **d.** $\frac{\pi}{7}\cos x - 4y = 0$

e. $xy = 1$ **f.** $y + 7 = x$

3. Using the notation of Formula (7), write down a general linear system of

a. two equations in two unknowns.

b. three equations in three unknowns.

c. two equations in four unknowns.

4. Write down the augmented matrix for each of the linear systems in Exercise 3.

In each part of Exercises 5–6, find a system of linear equations in the unknowns $x_1, x_2, x_3, \ldots,$ that corresponds to the given augmented matrix.

5. a. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 & -2 & 5 \\ 7 & 1 & 4 & -3 \\ 0 & -2 & 1 & 7 \end{bmatrix}$

6. a. $\begin{bmatrix} 0 & 3 & -1 & -1 & -1 \\ 5 & 2 & 0 & -3 & -6 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 0 & 1 & -4 & 3 \\ -4 & 0 & 4 & 1 & -3 \\ -1 & 3 & 0 & -2 & -9 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

In each part of Exercises 7–8, find the augmented matrix for the linear system.

7. a. $\begin{aligned} -2x_1 &= 6 \\ 3x_1 &= 8 \\ 9x_1 &= -3 \end{aligned}$ **b.** $\begin{aligned} 6x_1 - x_2 + 3x_3 &= 4 \\ 5x_2 - x_3 &= 1 \end{aligned}$

c. $\begin{aligned} 2x_2 - 3x_4 + x_5 &= 0 \\ -3x_1 - x_2 + x_3 &= -1 \\ 6x_1 + 2x_2 - x_3 + 2x_4 - 3x_5 &= 6 \end{aligned}$

8. a. $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \\ 7x_1 + 3x_2 &= 2 \end{aligned}$ **b.** $\begin{aligned} 2x_1 + 2x_3 &= 1 \\ 3x_1 - x_2 + 4x_3 &= 7 \\ 6x_1 + x_2 - x_3 &= 0 \end{aligned}$

c. $\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$

9. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

a. $(3, 1, 1)$ **b.** $(3, -1, 1)$ **c.** $(13, 5, 2)$

d. $\left(\frac{13}{2}, \frac{5}{2}, 2\right)$ **e.** $(17, 7, 5)$

10. In each part, determine whether the given 3-tuple is a solution of the linear system

$$\begin{aligned} x + 2y - 2z &= 3 \\ 3x - y + z &= 1 \\ -x + 5y - 5z &= 5 \end{aligned}$$

a. $\left(\frac{5}{7}, \frac{8}{7}, 1\right)$ **b.** $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$ **c.** $(5, 8, 1)$

d. $\left(\frac{5}{7}, \frac{10}{7}, \frac{2}{7}\right)$ **e.** $\left(\frac{5}{7}, \frac{22}{7}, 2\right)$

11. In each part, solve the linear system, if possible, and use the result to determine whether the lines represented by the equations in the system have zero, one, or infinitely many points of intersection. If there is a single point of intersection, give its coordinates, and if there are infinitely many, find parametric equations for them.

a. $\begin{aligned} 3x - 2y &= 4 \\ 6x - 4y &= 9 \end{aligned}$ **b.** $\begin{aligned} 2x - 4y &= 1 \\ 4x - 8y &= 2 \end{aligned}$ **c.** $\begin{aligned} x - 2y &= 0 \\ x - 4y &= 8 \end{aligned}$

12. Under what conditions on a and b will the linear system have no solutions, one solution, infinitely many solutions?

$$\begin{aligned} 2x - 3y &= a \\ 4x - 6y &= b \end{aligned}$$

In each part of Exercises 13–14, use parametric equations to describe the solution set of the linear equation.

13. a. $7x - 5y = 3$

b. $3x_1 - 5x_2 + 4x_3 = 7$

c. $-8x_1 + 2x_2 - 5x_3 + 6x_4 = 1$

d. $3v - 8w + 2x - y + 4z = 0$

14. a. $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

In Exercises 15–16, each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

15. a. $\begin{aligned} 2x - 3y &= 1 \\ 6x - 9y &= 3 \end{aligned}$

b. $\begin{aligned} x_1 + 3x_2 - x_3 &= -4 \\ 3x_1 + 9x_2 - 3x_3 &= -12 \\ -x_1 - 3x_2 + x_3 &= 4 \end{aligned}$

16. a. $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$ **b.** $\begin{aligned} 2x - y + 2z &= -4 \\ 6x - 3y + 6z &= -12 \\ -4x + 2y - 4z &= 8 \end{aligned}$

In Exercises 17–18, find a single elementary row operation that will create a 1 in the upper left corner of the given augmented matrix and will not create any fractions in its first row.

17. a. $\begin{bmatrix} -3 & -1 & 2 & 4 \\ 2 & -3 & 3 & 2 \\ 0 & 2 & -3 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -1 & -5 & 0 \\ 2 & -9 & 3 & 2 \\ 1 & 4 & -3 & 3 \end{bmatrix}$

18. a. $\begin{bmatrix} 2 & 4 & -6 & 8 \\ 7 & 1 & 4 & 3 \\ -5 & 4 & 2 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & -4 & -2 & 2 \\ 3 & -1 & 8 & 1 \\ -6 & 3 & -1 & 4 \end{bmatrix}$

In Exercises 19–20, find all values of k for which the given augmented matrix corresponds to a consistent linear system.

19. a. $\begin{bmatrix} 1 & k & -4 \\ 4 & 8 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & k & -1 \\ 4 & 8 & -4 \end{bmatrix}$

20. a. $\begin{bmatrix} 3 & -4 & k \\ -6 & 8 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} k & 1 & -2 \\ 4 & -1 & 2 \end{bmatrix}$

21. The curve $y = ax^2 + bx + c$ shown in the accompanying figure passes through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the coefficients a, b, and c form a solution of the system of linear equations whose augmented matrix is

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

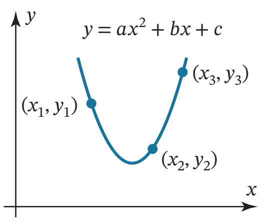


FIGURE Ex-21

22. Explain why each of the three elementary row operations does not affect the solution set of a linear system.

23. Show that if the linear equations

$$x_1 + kx_2 = c \qquad \text{and} \qquad x_1 + lx_2 = d$$

have the same solution set, then the two equations are identical (i.e., $k = l$ and $c = d$).

24. Consider the system of equations

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \\ ex + fy &= m \end{aligned}$$

Discuss the relative positions of the lines $ax + by = k$, $cx + dy = l$, and $ex + fy = m$ when

a. the system has no solutions.

b. the system has exactly one solution.

c. the system has infinitely many solutions.

25. Suppose that a certain diet calls for 7 units of fat, 9 units of protein, and 16 units of carbohydrates for the main meal, and suppose that an individual has three possible foods to choose from to meet these requirements:

Food 1: Each ounce contains 2 units of fat, 2 units of protein, and 4 units of carbohydrates.

Food 2: Each ounce contains 3 units of fat, 1 unit of protein, and 2 units of carbohydrates.

Food 3: Each ounce contains 1 unit of fat, 3 units of protein, and 5 units of carbohydrates.

Let x, y, and z denote the number of ounces of the first, second, and third foods that the dieter will consume at the main meal. Find (but do not solve) a linear system in x, y, and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

26. Suppose that you want to find values for a, b, and c such that the parabola $y = ax^2 + bx + c$ passes through the points $(1, 1)$, $(2, 4)$, and $(-1, 1)$. Find (but do not solve) a system of linear equations whose solutions provide values for a, b, and c. How many solutions would you expect this system of equations to have, and why?

27. Suppose you are asked to find three real numbers such that the sum of the numbers is 12, the sum of two times the first plus the second plus two times the third is 5, and the third number is one more than the first. Find (but do not solve) a linear system whose equations describe the three conditions.

True-False Exercises

TF. In parts **(a)–(h)** determine whether the statement is true or false, and justify your answer.

a. A linear system whose equations are all homogeneous must be consistent.

b. Multiplying a row of an augmented matrix through by zero is an acceptable elementary row operation.

c. The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

cannot have a unique solution, regardless of the value of k.

d. A single linear equation with two or more unknowns must have infinitely many solutions.

e. If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent.

f. If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

g. Elementary row operations permit one row of an augmented matrix to be subtracted from another.

h. The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent.

Working with Technology

T1. Solve the linear systems in Examples 2, 3, and 4 to see how your technology utility handles the three types of systems.

T2. Use the result in Exercise 21 to find values of a, b, and c for which the curve $y = ax^2 + bx + c$ passes through the points $(-1, 1, 4)$, $(0, 0, 8)$, and $(1, 1, 7)$.

1.2 Gaussian Elimination

In this section we will develop a systematic procedure for solving systems of linear equations. The procedure is based on the idea of performing certain operations on the rows of the augmented matrix that simplify it to a form from which the solution of the system can be ascertained by inspection.

Considerations in Solving Linear Systems

When considering methods for solving systems of linear equations, it is important to distinguish between large systems that must be solved by computer and small systems that can be solved by hand. For example, there are many applications that lead to linear systems in thousands or even millions of unknowns. Large systems require special techniques to deal with issues of memory size, roundoff errors, solution time, and so forth. Such techniques are studied in the field of ***numerical analysis*** and will only be touched on in this text. However, almost all of the methods that are used for large systems are based on the ideas that we will develop in this section.

Echelon Forms

In Example 6 of the last section, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution $x = 1$, $y = 2$, $z = 3$ became evident. This is an example of a matrix that is in ***reduced row echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

EXAMPLE 1 | Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 | More on Row Echelon and Reduced Row Echelon Form

As Example 1 illustrates, a matrix in row echelon form has zeros below each leading 1, whereas a matrix in reduced row echelon form has zeros below *and above* each leading 1. Thus, with any real numbers substituted for the $*$'s, all matrices of the following types are in row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

All matrices of the following types are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

If, by a sequence of elementary row operations, the augmented matrix for a system of linear equations is put in *reduced* row echelon form, then the solution set can be obtained either by inspection or by converting certain linear equations to parametric form. Here are some examples.

EXAMPLE 3 | Unique Solution

Suppose that the augmented matrix for a linear system in the unknowns x_1, x_2, x_3, and x_4 has been reduced by elementary row operations to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

This matrix is in reduced row echelon form and corresponds to the equations

$$\begin{aligned} x_1 \qquad\qquad\qquad &= 3 \\ x_2 \qquad\qquad &= -1 \\ x_3 \qquad &= 0 \\ x_4 &= 5 \end{aligned}$$

Thus, the system has a unique solution, namely, $x_1 = 3, x_2 = -1, x_3 = 0, x_4 = 5$, which can also be expressed as the 4-tuple $(3, -1, 0, 5)$.

EXAMPLE 4 | Linear Systems in Three Unknowns

In each part, suppose that the augmented matrix for a linear system in the unknowns x, y, and z has been reduced by elementary row operations to the given reduced row echelon form. Solve the system.

$$(a)\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (c)\ \begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution (*a*) The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 1$$

Since this equation is not satisfied by any values of x, y, and z, the system is inconsistent.

Solution (*b*) The equation that corresponds to the last row of the augmented matrix is

$$0x + 0y + 0z = 0$$

This equation can be omitted since it imposes no restrictions on x, y, and z; hence, the linear system corresponding to the augmented matrix is

$$\begin{aligned} x \qquad + 3z &= -1 \\ y - 4z &= \ \ 2 \end{aligned}$$

In general, the variables in a linear system that correspond to the leading l's in its augmented matrix are called the ***leading variables***, and the remaining variables are called the ***free variables***. In this case the leading variables are x and y, and the variable z is the only free variable. Solving for the leading variables in terms of the free variables gives

$$\begin{aligned} x &= -1 - 3z \\ y &= 2 + 4z \end{aligned}$$

From these equations we see that the free variable z can be treated as a parameter and assigned an arbitrary value t, which then determines values for x and y. Thus, the solution set can be represented by the parametric equations

$$x = -1 - 3t, \quad y = 2 + 4t, \quad z = t$$

By substituting various values for t in these equations we can obtain various solutions of the system. For example, setting $t = 0$ yields the solution

$$x = -1, \quad y = 2, \quad z = 0$$

and setting $t = 1$ yields the solution

$$x = -4, \quad y = 6, \quad z = 1$$

Solution (*c*) As explained in part (b), we can omit the equations corresponding to the zero rows, in which case the linear system associated with the augmented matrix consists of the single equation

$$x - 5y + z = 4 \tag{1}$$

from which we see that the solution set is a plane in three-dimensional space. Although (1) is a valid form of the solution set, there are many applications in which it is preferable to express the solution set in parametric form. We can convert (1) to parametric form by solving for the leading variable x in terms of the free variables y and z to obtain

$$x = 4 + 5y - z$$

From this equation we see that the free variables can be assigned arbitrary values, say $y = s$ and $z = t$, which then determine the value of x. Thus, the solution set can be expressed parametrically as

$$x = 4 + 5s - t, \quad y = s, \quad z = t \tag{2}$$

We will usually denote parameters in a general solution by the letters $r, s, t, \ldots$, but any letters that do not conflict with the names of the unknowns can be used. For systems with more than three unknowns, subscripted letters such as $t_1, t_2, t_3, \ldots$ are convenient.

Formulas, such as (2), that express the solution set of a linear system parametrically have some associated terminology.

Definition 1

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a ***general solution*** of the system.

Thus, for example, Formula (2) is a general solution of system (iii) in the previous example.

Elimination Methods

We have just seen how easy it is to solve a system of linear equations once its augmented matrix is in reduced row echelon form. Now we will give a step-by-step *algorithm* that can be used to reduce any matrix to reduced row echelon form. As we state each step in the algorithm, we will illustrate the idea by reducing the following matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

Step 3. If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $1/a$ in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The first row of the preceding matrix was multiplied by $\frac{1}{2}$.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the first row of the preceding matrix was added to the third row.

Step 5. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until the *entire* matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

–5 times the first row of the submatrix was added to the second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again to Step 1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submatrix was multiplied by 2 to introduce a leading 1.

The *entire* matrix is now in row echelon form. To find the reduced row echelon form we need the following additional step.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\frac{7}{2}$ times the third row of the preceding matrix was added to the second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

The last matrix is in reduced row echelon form.

The algorithm we have just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***. It consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and a ***backward phase*** in which zeros are introduced above the leading 1's. If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***. For example, in the preceding computations a row echelon form was obtained at the end of Step 5.

Historical Note

Carl Friedrich Gauss (1777–1855)

Wilhelm Jordan (1842–1899)

Although versions of Gaussian elimination were known much earlier, its importance in scientific computation became clear when the great German mathematician Carl Friedrich Gauss used it to help compute the orbit of the asteroid Ceres from limited data. What happened was this: On January 1, 1801 the Sicilian astronomer and Catholic priest Giuseppe Piazzi (1746–1826) noticed a dim celestial object that he believed might be a "missing planet." He named the object Ceres and made a limited number of positional observations but then lost the object as it neared the Sun. Gauss, then only 24 years old, undertook the problem of computing the orbit of Ceres from the limited data using a technique called "least squares," the equations of which he solved by the method that we now call "Gaussian elimination." The work of Gauss created a sensation when Ceres reappeared a year later in the constellation Virgo at almost the precise position that he predicted! The basic idea of the method was further popularized by the German engineer Wilhelm Jordan in his book on geodesy (the science of measuring Earth shapes) entitled *Handbuch der Vermessungskunde* and published in 1888.

[Images: Photo Inc/Photo Researchers/Getty Images (Gauss); https://en.wikipedia.org/wiki/Andrey_Markov#/media/File:Andrei_Markov.jpg. Public domain. (Jordan)]

EXAMPLE 5 | Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination.

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0\\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1\\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5\\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6
\end{array}$$

Solution The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Adding -2 times the first row to the second and fourth rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -2 & 0 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Multiplying the second row by -1 and then adding -5 times the new second row to the third row and -4 times the new second row to the fourth row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the forward phase since there are zeros below the leading 1's.

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the reduced row echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This completes the backward phase since there are zeros above the leading 1's.

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad\qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned} \tag{3}$$

Note that in constructing the linear system in (3) we ignored the row of zeros in the corresponding augmented matrix. Why is this justified?

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Finally, we express the general solution of the system parametrically by assigning the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively. This yields

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & 0 \end{array}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

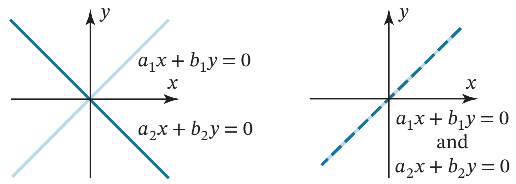
Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

In the special case of a homogeneous linear system of two equations in two unknowns, say

$$\begin{aligned} a_1x + b_1y &= 0 \quad [\boldsymbol{a_1, b_1} \textbf{ not both zero}] \\ a_2x + b_2y &= 0 \quad [\boldsymbol{a_2, b_2} \textbf{ not both zero}] \end{aligned}$$

the graphs of the equations are lines through the origin, and the trivial solution corresponds to the point of intersection at the origin (**Figure 1.2.1**).



Only the trivial solution

Infinitely many solutions

FIGURE 1.2.1

There is one case in which a homogeneous system is assured of having nontrivial solutions—namely, whenever the system involves more unknowns than equations. To see why, consider the following example of four equations in six unknowns.

EXAMPLE 6 | A Homogeneous System

Use Gauss–Jordan elimination to solve the homogeneous linear system

$$\begin{array}{rcrcrcrcrcrcl} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & 0 \\ & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 0 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 0 \end{array} \tag{4}$$

Solution Observe that this system is the same as that in Example 5 except for the constants on the right side, which in this case are all zero. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \tag{5}$$

which is the same as that in Example 5 except for the entries in the last column, which are all zeros in this case. Thus, the reduced row echelon form of this matrix will be the same as that of the augmented matrix in Example 5, except for the last column. However, a moment's reflection will make it evident that a column of zeros is not changed by an elementary row operation, so the reduced row echelon form of (5) is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

The corresponding system of equations is

$$\begin{array}{rcrcrcrcrcl} x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & = 0 \\ & & & & x_3 & + & 2x_4 & & & & = 0 \\ & & & & & & & & & x_6 & = 0 \end{array}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned} \tag{7}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

Note that the trivial solution results when $r = s = t = 0$.

Free Variables in Homogeneous Linear Systems

Example 6 illustrates two important points about solving homogeneous linear systems:

1. Elementary row operations do not alter columns of zeros in a matrix, so the reduced row echelon form of the augmented matrix for a homogeneous linear system has a final column of zeros. This implies that the linear system corresponding to the reduced row echelon form is homogeneous, just like the original system.
2. When we constructed the homogeneous linear system corresponding to augmented matrix (6), we ignored the row of zeros because the corresponding equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 = 0$$

does not impose any conditions on the unknowns. Thus, depending on whether or not the reduced row echelon form of the augmented matrix for a homogeneous linear system has any zero rows, the linear system corresponding to that reduced row echelon form will either have the same number of equations as the original system or it will have fewer.

Now consider a general homogeneous linear system with n unknowns, and suppose that the reduced row echelon form of the augmented matrix has r nonzero rows. Since each nonzero row has a leading 1, and since each leading 1 corresponds to a leading variable, the homogeneous system corresponding to the reduced row echelon form of the augmented matrix must have r leading variables and $n - r$ free variables. Thus, this system is of the form

$$\begin{array}{llll} x_{k_1} & & & + \sum(\;) = 0 \\ & x_{k_2} & & + \sum(\;) = 0 \\ & & \ddots & \quad\vdots \\ & & x_{k_r} & + \sum(\;) = 0 \end{array} \tag{8}$$

where in each equation the expression $\sum(\;)$ denotes a sum that involves the free variables, if any [see (7), for example]. In summary, we have the following result.

Theorem 1.2.1

Free Variable Theorem for Homogeneous Systems

If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem 1.2.1 has an important implication for homogeneous linear systems with more unknowns than equations. Specifically, if a homogeneous linear system has m equations in n unknowns, and if $m < n$, then it must also be true that $r < n$ (why?). This being the case, the theorem implies that there is at least one free variable, and this implies that the system has infinitely many solutions. Thus, we have the following result.

Note that Theorem 1.2.2 applies only to homogeneous systems—a *nonhomogeneous* system with more unknowns than equations need not be consistent. However, we will prove later that if a nonhomogeneous system with more unknowns than equations is consistent, then it has infinitely many solutions.

Theorem 1.2.2

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

In retrospect, we could have anticipated that the homogeneous system in Example 6 would have infinitely many solutions since it has four equations in six unknowns.

Gaussian Elimination and Back-Substitution

For small linear systems that are solved by hand (such as most of those in this text), Gauss–Jordan elimination (reduction to reduced row echelon form) is a good procedure to use. However, for large linear systems that require a computer solution, it is generally more efficient to use Gaussian elimination (reduction to row echelon form) followed by a technique known as ***back-substitution*** to complete the process of solving the system. The next example illustrates this technique.

EXAMPLE 7 | Example 5 Solved by Back-Substitution

From the computations in Example 5, a row echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

we proceed as follows:

Step 1. Solve the equations for the leading variables.

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Step 2. Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Step 3. Assign arbitrary values to the free variables, if any.

If we now assign x_2, x_4, and x_5 the arbitrary values r, s, and t, respectively, the general solution is given by the formulas

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

This agrees with the solution obtained in Example 5.

EXAMPLE 8 | Existence and Uniqueness of Solutions

Suppose that the matrices below are augmented matrices for linear systems in the unknowns x_1, x_2, x_3, and x_4. These matrices are all in row echelon form but not reduced row echelon form. Discuss the existence and uniqueness of solutions to the corresponding linear systems

$$(a)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (b)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(c)\ \begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution (*a*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

from which it is evident that the system is inconsistent.

Solution (*b*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

which has no effect on the solution set. In the remaining three equations the variables x_1, x_2, and x_3 correspond to leading 1's and hence are leading variables. The variable x_4 is a free variable. With a little algebra, the leading variables can be expressed in terms of the free variable, and the free variable can be assigned an arbitrary value. Thus, the system must have infinitely many solutions.

Solution (*c*) The last row corresponds to the equation

$$x_4 = 0$$

which gives us a numerical value for x_4. If we substitute this value into the third equation, namely,

$$x_3 + 6x_4 = 9$$

we obtain $x_3 = 9$. You should now be able to see that if we continue this process and substitute the known values of x_3 and x_4 into the equation corresponding to the second row, we will obtain a unique numerical value for x_2; and if, finally, we substitute the known values of x_4, x_3, and x_2 into the equation corresponding to the first row, we will produce a unique numerical value for x_1. Thus, the system has a unique solution.

Some Facts About Echelon Forms

There are three facts about row echelon forms and reduced row echelon forms that are important to know but we will not prove:

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss–Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.*
2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.

*A proof of this result can be found in the article "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," by Thomas Yuster, *Mathematics Magazine*, Vol. 57, No. 2, 1984, pp. 93–94.

3. Although row echelon forms are not unique, the reduced row echelon form and all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions. Those are called the ***pivot positions*** of A. The columns containing the leading 1's in a row echelon or reduced row echelon form of A are called the ***pivot columns*** of A, and the rows containing the leading 1's are called the ***pivot rows*** of A. A *nonzero* entry in a pivot position of A is called a ***pivot*** of A.

EXAMPLE 9 | Pivot Positions and Columns

Earlier in this section (immediately after Definition 1) we found a row echelon form of

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad \text{to be} \quad \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The leading 1's occur in (row 1, column 1), (row 2, column 3), and (row 3, column 5). These are the pivot positions of A. The pivot columns of A are 1, 3, and 5, and the pivot rows are 1, 2, and 3. The pivots of A are the nonzero numbers in the pivot positions. These are marked by shaded rectangles in the following diagram.

If A is the augmented matrix for a linear system, then the pivot columns identify the leading variables. As an illustration, in Example 5 the pivot columns are 1, 3, and 6, and the leading variables are x_1, x_3, and x_6.

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & \boxed{-10} & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & \boxed{-5} & -1 \end{bmatrix}$$

Pivot columns

Roundoff Error and Instability

There is often a gap between mathematical theory and its practical implementation—Gauss–Jordan elimination and Gaussian elimination being good examples. The problem is that computers generally approximate numbers, thereby introducing ***roundoff*** errors, so unless precautions are taken, successive calculations may degrade an answer to a degree that makes it useless. Algorithms in which this happens are called ***unstable***. There are various techniques for minimizing roundoff error and instability. For example, it can be shown that for large linear systems Gauss–Jordan elimination involves roughly 50% more operations than Gaussian elimination, so most computer algorithms are based on the latter method. Some of these matters will be considered in Chapter 9.

Exercise Set 1.2

In Exercises 1–2, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

1. a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ **e.** $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ **g.** $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **e.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 7 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **g.** $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

In Exercises 3–4, suppose that the augmented matrix for a linear system has been reduced by row operations to the given row echelon form. Identify the pivot rows and columns and solve the system.

3. a. $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. a. $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 5–8, solve the system by Gaussian elimination.

5. $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

6. $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

7. $\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$

8. $\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$

In Exercises 9–12, solve the system by Gauss–Jordan elimination.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

In Exercises 13–14, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper).

13. $\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$

14. $\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$

In Exercises 15–22, solve the given linear system by any method.

15. $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 \qquad &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$

16. $\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$

17. $\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$

18. $\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$

19. $\begin{aligned} 2x + 2y + 4z &= 0 \\ w \qquad - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$

20. $\begin{aligned} x_1 + 3x_2 \qquad + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 \qquad &= 0 \\ -2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$

21. $\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 \qquad - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$

22. $\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 \qquad - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 \qquad + Z_5 &= 0 \end{aligned}$

In each part of Exercises 23–24, the augmented matrix for a linear system is given in which the asterisk represents an unspecified real number. Determine whether the system is consistent, and if so whether the solution is unique. Answer "inconclusive" if there is not enough information to make a decision.

23. a. $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}$

b. $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 1 & * \end{bmatrix}$

24. a. $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & * \\ * & 1 & 0 & * \\ * & * & 1 & * \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & * & * & * \end{bmatrix}$

d. $\begin{bmatrix} 1 & * & * & * \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 25–26, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

25. $\begin{aligned} x + 2y - \qquad 3z &= 4 \\ 3x - y + \qquad 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$

26.
$$\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$$

In Exercises 27–28, what condition, if any, must a, b, and c satisfy for the linear system to be consistent?

27.
$$\begin{aligned} x + 3y - z &= a \\ x + y + 2z &= b \\ 2y - 3z &= c \end{aligned}$$

28.
$$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

In Exercises 29–30, solve the following systems, where a, b, and c are constants.

29.
$$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned}$$

30.
$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma \le 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

[*Hint:* Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

36. Solve the following system for x, y, and z.

$$\begin{aligned} \tfrac{1}{x} + \tfrac{2}{y} - \tfrac{4}{z} &= 1 \\ \tfrac{2}{x} + \tfrac{3}{y} + \tfrac{8}{z} &= 0 \\ -\tfrac{1}{x} + \tfrac{9}{y} + \tfrac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

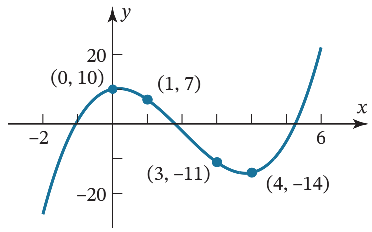


FIGURE Ex-37

38. Find the coefficients a, b, c, and d so that the circle shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

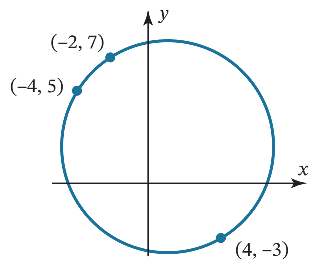


FIGURE Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x - b_2y + c_2z &= 0 \\ a_3x + b_3y - c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x - b_2y + c_2z &= 7 \\ a_3x + b_3y - c_3z &= 11 \end{aligned}$$

40. a. If A is a matrix with three rows and five columns, then what is the maximum possible number of leading 1's in its reduced row echelon form?

b. If B is a matrix with three rows and six columns, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

c. If C is a matrix with five rows and three columns, then what is the minimum possible number of rows of zeros in any row echelon form of C?

41. Describe all possible reduced row echelon forms of

a. $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

b. $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when the system has only the trivial solution and when it has nontrivial solutions.

Working with Proofs

43. **a.** Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. Use the result in part (a) to prove that if $ad - bc \neq 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

True-False Exercises

TF. In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

a. If a matrix is in reduced row echelon form, then it is also in row echelon form.

b. If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

c. Every matrix has a unique row echelon form.

d. A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

e. All leading 1's in a matrix in row echelon form must occur in different columns.

f. If every column of a matrix in row echelon form has a leading 1, then all entries that are not leading 1's are zero.

g. If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

h. If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

i. If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Working with Technology

T1. Find the reduced row echelon form of the augmented matrix for the linear system

$$\begin{aligned} 6x_1 + x_2 + 4x_4 &= -3 \\ -9x_1 + 2x_2 + 3x_3 - 8x_4 &= 1 \\ 7x_1 - 4x_3 + 5x_4 &= 2 \end{aligned}$$

Use your result to determine whether the system is consistent and, if so, find its solution.

T2. Find values of the constants A, B, C, and D that make the following equation an identity (i.e., true for all values of x).

$$\frac{3x^3 + 4x^2 - 6x}{(x^2 + 2x + 2)(x^2 - 1)} = \frac{Ax + B}{x^2 + 2x + 2} + \frac{C}{x - 1} + \frac{D}{x + 1}$$

[*Hint:* Obtain a common denominator on the right, and then equate corresponding coefficients of the various powers of x in the two numerators. Students of calculus will recognize this as a problem in partial fractions.]

1.3 Matrices and Matrix Operations

Rectangular arrays of real numbers arise in contexts other than as augmented matrices for linear systems. In this section we will begin to study matrices as objects in their own right by defining operations of addition, subtraction, and multiplication on them.

Matrix Notation and Terminology

In Section 1.2 we used rectangular arrays of numbers, called *augmented matrices*, to abbreviate systems of linear equations. However, rectangular arrays of numbers occur in other contexts as well. For example, the following rectangular array with three rows and seven columns might describe the number of hours that a student spent studying three subjects during a certain week:

	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.	Sun.
Math	2	3	2	4	1	4	2
History	0	3	1	4	3	2	2
Language	4	1	3	1	0	0	2

If we suppress the headings, then we are left with the following rectangular array of numbers with three rows and seven columns, called a "matrix":

$$\begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 4 & 2 \\ 0 & 3 & 1 & 4 & 3 & 2 & 2 \\ 4 & 1 & 3 & 1 & 0 & 0 & 2 \end{bmatrix}$$

More generally, we make the following definition.

Definition 1

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** of the matrix.

EXAMPLE 1 | Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

Matrix brackets are often omitted from 1×1 matrices, making it impossible to tell, for example, whether the symbol 4 denotes the number "four" or the matrix [4]. This rarely causes problems because it is usually possible to tell which is meant from the context.

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2). In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4, 3×3, 2×1, and 1×1, respectively.

A matrix with only one row, such as the second in Example 1, is called a ***row vector*** (or a ***row matrix***), and a matrix with only one column, such as the fourth in that example, is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{or} \quad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

When discussing matrices, it is common to refer to numerical quantities as ***scalars***. Unless stated otherwise, *scalars will be real numbers*; complex scalars will be considered later in the text.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{1}$$

When a compact notation is desired, matrix (1) can be written as

$$A = [a_{ij}]_{m \times n} \quad \text{or} \quad A = [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix B we would generally use b_{ij} for the entry in row i and column j, and for a matrix C we would use the notation c_{ij}.

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

Row and column vectors are of special importance, and it is common practice to denote them by boldface lowercase letters rather than capital letters. For such matrices, double subscripting of the entries is unnecessary. Thus a general $1 \times n$ row vector $\mathbf{a}$ and a general $m \times 1$ column vector $\mathbf{b}$ would be written as

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Operations on Matrices

So far, we have used matrices to abbreviate the work in solving systems of linear equations. For other applications, however, it is desirable to develop an "arithmetic of matrices" in which matrices can be added, subtracted, and multiplied in a useful way. The remainder of this section will be devoted to developing this arithmetic.

Definition 2

Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

EXAMPLE 2 | Equality of Matrices

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are the same. There is no value of x for which $A = C$ since A and C have different sizes.

Definition 3

If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$\boxed{(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}}$$

The equality of two matrices

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

of the same size can be expressed either by writing

$$(A)_{ij} = (B)_{ij}$$

or by writing

$$a_{ij} = b_{ij}$$

EXAMPLE 3 | Addition and Subtraction

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.

Definition 4

If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

In matrix notation, if $A = [a_{ij}]$, then

$$\boxed{(cA)_{ij} = c(A)_{ij} = ca_{ij}}$$

EXAMPLE 4 | Scalar Multiples

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

Thus far we have defined multiplication of a matrix by a scalar but not the multiplication of two matrices. Since matrices are added by adding corresponding entries and subtracted by subtracting corresponding entries, it would seem natural to define multiplication of matrices by multiplying corresponding entries. However, it turns out that such a definition would not be very useful. Experience has led mathematicians to the following definition, the motivation for which will be given later in this chapter.

Definition 5

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add the resulting products.

EXAMPLE 5 | Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{26} & \boxed{} \end{bmatrix}$$

$$\boxed{(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26}$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{13} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

$$\boxed{(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13}$$

The computations for the remaining entries are

$$\begin{aligned}
(1\cdot 4) + (2\cdot 0) + (4\cdot 2) &= 12\\
(1\cdot 1) - (2\cdot 1) + (4\cdot 7) &= 27\\
(1\cdot 4) + (2\cdot 3) + (4\cdot 5) &= 30\\
(2\cdot 4) + (6\cdot 0) + (0\cdot 2) &= 8\\
(2\cdot 1) - (6\cdot 1) + (0\cdot 7) &= -4\\
(2\cdot 3) + (6\cdot 1) + (0\cdot 2) &= 12
\end{aligned}
\qquad
AB = \begin{bmatrix} 12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in (3), the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.

$$\begin{array}{ccc} A & B & AB \end{array}$$
$$\underbrace{m \times \underbrace{r \qquad r}_{\text{Inside}} \times n}_{\text{Outside}} = m \times n \tag{3}$$

EXAMPLE 6 | Determining Whether a Product Is Defined

Suppose that A, B, and C are matrices with the following sizes:

$$\begin{array}{ccc} A & B & C \\ 3\times 4 & 4\times 7 & 7\times 3 \end{array}$$

Then, AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix; and CA is defined and is a 7×4 matrix. The products AC, CB, and BA are all undefined.

In general, if $A = [a_{ij}]$ is an $m\times r$ matrix and $B = [b_{ij}]$ is an $r\times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r}\\ a_{21} & a_{22} & \cdots & a_{2r}\\ \vdots & \vdots & & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{ir}\\ \vdots & \vdots & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n}\\ \vdots & \vdots & & \vdots & & \vdots\\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$\boxed{(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

Partitioned Matrices

A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3×4 matrix A—the first is a partition of A into four ***submatrices*** A_{11}, A_{12}, A_{21}, and A_{22}; the second is a partition of A into its row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; and the third is a partition of A into its column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$:

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Matrix Multiplication by Columns and by Rows

Partitioning has many uses, one of which is for finding particular rows or columns of a matrix product AB without computing the entire product. Specifically, the following formulas, whose proofs are left as exercises, show how individual column vectors of AB can be obtained by partitioning B into column vectors and how individual row vectors of AB can be obtained by partitioning A into row vectors.

$$\boxed{AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]} \tag{6}$$

(*AB* computed column by column)

$$\boxed{AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}} \tag{7}$$

(*AB* computed row by row)

In words, these formulas state that

$$\boxed{j\text{th column vector of } AB = A[j\text{th column vector of } B]} \tag{8}$$

$$\boxed{i\text{th row vector of } AB = [i\text{th row vector of } A]B} \tag{9}$$

Historical Note

Gotthold Eisenstein (1823–1852)

The concept of matrix multiplication is due to the German mathematician Gotthold Eisenstein, who introduced the idea around 1844 to simplify the process of making substitutions in linear systems. The idea was then expanded on and formalized by Arthur Cayley (see p. 36) in his *Memoir on the Theory of Matrices* that was published in 1858. Eisenstein was a pupil of Gauss, who ranked him as the equal of Isaac Newton and Archimedes. However, Eisenstein, suffering from bad health his entire life, died at age 30, so his potential was never realized.

[Image: University of St Andrews/Wikipedia]

EXAMPLE 7 | Example 5 Revisited

If A and B are the matrices in Example 5, then from (8) the second column vector of AB can be obtained by the computation

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}}_{\text{Second column of } B} = \underbrace{\begin{bmatrix} 27 \\ -4 \end{bmatrix}}_{\text{Second column of } AB}$$

and from (9) the first row vector of AB can be obtained by the computation

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}}_{\text{First row of } A} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}}_{\text{First row of } AB}$$

Matrix Products as Linear Combinations

The following definition provides yet another way of thinking about matrix multiplication.

Definition 6

If $A_1, A_2, \dots, A_r$ are matrices of the same size, and if $c_1, c_2, \dots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a ***linear combination*** of $A_1, A_2, \dots, A_r$ with ***coefficients*** $c_1, c_2, \dots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \tag{10}$$

This proves the following theorem.

Theorem 1.3.1

If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

EXAMPLE 8 | Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

EXAMPLE 9 | Columns of a Product AB as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

It follows from Formula (6) and Theorem 1.3.1 that the jth column vector of AB can be expressed as a linear combination of the column vectors of A in which the coefficients in the linear combination are the entries from the jth column of B. The computations are as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Column-Row Expansion

Partitioning provides yet another way to view matrix multiplication. Specifically, suppose that an $m \times r$ matrix A is partitioned into its r column vectors $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_r$ (each of size $m \times 1$) and an $r \times n$ matrix B is partitioned into its r row vectors $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_r$ (each of size $1 \times n$). Each term in the sum

$$\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r$$

has size $m \times n$ so the sum itself is an $m \times n$ matrix. We leave it as an exercise for you to verify that the entry in row i and column j of the sum is given by the expression on the right side of Formula (5), from which it follows that

$$\boxed{AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r} \tag{11}$$

We call (11) the ***column-row expansion*** of AB.

EXAMPLE 10 | Column-Row Expansion

Find the column-row expansion of the product

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} \tag{12}$$

Solution The column vectors of A and the row vectors of B are, respectively,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \quad \mathbf{r}_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

Thus, it follows from (11) that the column-row expansion of AB is

$$\begin{aligned} AB &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix} \end{aligned} \tag{13}$$

As a check, we leave it for you to confirm that the product in (12) and the sum in (13) both yield

$$AB = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

Summarizing Matrix Multiplication

Putting it all together, we have given five different ways to compute a matrix product, each of which has its own use:

1. Entry by entry (Definition 5)
2. Row-column method (Formula (5))
3. Column by column (Formula (6))
4. Row by row (Formula (7))
5. Column-row expansion (Formula (11))

Matrix Form of a Linear System

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{matrix}$$

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times 1$ matrix on the left side of this equation can be written as a product to give

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$\boxed{A\mathbf{x} = \mathbf{b}}$$

The matrix A in this equation is called the ***coefficient matrix*** of the system. The augmented matrix for the system is obtained by adjoining $\mathbf{b}$ to A as the last column; thus the augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The vertical partition line in the augmented matrix $[A \mid \mathbf{b}]$ is optional, but is a useful way of visually separating the coefficient matrix A from the column vector $\mathbf{b}$.

Transpose of a Matrix

We conclude this section by defining two matrix operations that have no analogs in the arithmetic of real numbers.

Definition 7

If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

EXAMPLE 11 | Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Observe that not only are the columns of A^T the rows of A, but the rows of A^T are the columns of A. Thus the entry in row i and column j of A^T is the entry in row j and column i of A; that is,

$$\boxed{(A^T)_{ij} = (A)_{ji}} \tag{14}$$

Note the reversal of the subscripts.

In the special case where A is a square matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal. In (15) we see that A^T can also be obtained by "reflecting" A about its main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix} \tag{15}$$

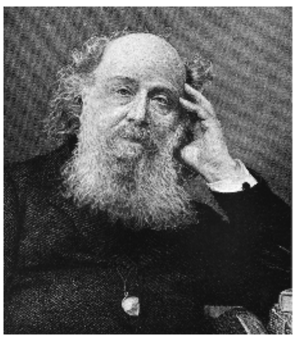
Interchange entries that are symmetrically positioned about the main diagonal.

Trace of a Matrix

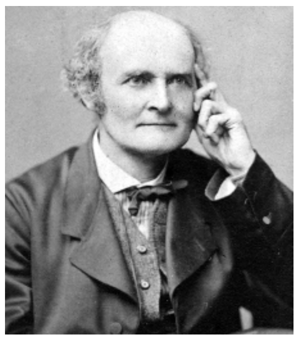
Definition 8

If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

Historical Note



James Sylvester (1814–1897)



Arthur Cayley (1821–1895)

The term *matrix* was first used by the English mathematician James Sylvester, who defined the term in 1850 to be an "oblong arrangement of terms." Sylvester communicated his work on matrices to a fellow English mathematician and lawyer named Arthur Cayley, who then introduced some of the basic operations on matrices in a book entitled *Memoir on the Theory of Matrices* that was published in 1858. As a matter of interest, Sylvester, who was Jewish, did not get his college degree because he refused to sign a required oath to the Church of England. He was appointed to a chair at the University of Virginia in the United States but resigned after swatting a student with a stick because he was reading a newspaper in class. Sylvester, thinking he had killed the student, fled back to England on the first available ship. Fortunately, the student was not dead, just in shock!

[Images: © Bettmann/CORBIS (Sylvester); Wikipedia Commons (Cayley)]

EXAMPLE 12 | Trace

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\boxed{\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}} \qquad \boxed{\operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11}$$

In the exercises you will have some practice working with the transpose and trace operations.

Exercise Set 1.3

In Exercises 1–2, suppose that A, B, C, D, and E are matrices with the following sizes:

A	B	C	D	E
(4×5)	(4×5)	(5×2)	(4×2)	(5×4)

In each part, determine whether the given matrix expression is defined. For those that are defined, give the size of the resulting matrix.

1. **a.** BA **b.** AB^T **c.** $AC + D$
d. $E(AC)$ **e.** $A - 3E^T$ **f.** $E(5B + A)$

2. **a.** CD^T **b.** DC **c.** $BC - 3D$
d. $D^T(BE)$ **e.** $B^TD + ED$ **f.** $BA^T + D$

In Exercises 3–6, use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

3. **a.** $D + E$ **b.** $D - E$ **c.** $5A$
d. $-7C$ **e.** $2B - C$ **f.** $4E - 2D$
g. $-3(D + 2E)$ **h.** $A - A$ **i.** $\operatorname{tr}(D)$
j. $\operatorname{tr}(D - 3E)$ **k.** $4\operatorname{tr}(7B)$ **l.** $\operatorname{tr}(A)$

4. **a.** $2A^T + C$ **b.** $D^T - E^T$ **c.** $(D - E)^T$
d. $B^T + 5C^T$ **e.** $\frac{1}{2}C^T - \frac{1}{4}A$ **f.** $B - B^T$
g. $2E^T - 3D^T$ **h.** $(2E^T - 3D^T)^T$ **i.** $(CD)E$
j. $C(BA)$ **k.** $\operatorname{tr}(DE^T)$ **l.** $\operatorname{tr}(BC)$

5. **a.** AB **b.** BA **c.** $(3E)D$
d. $(AB)C$ **e.** $A(BC)$ **f.** CC^T
g. $(DA)^T$ **h.** $(C^TB)A^T$ **i.** $\operatorname{tr}(DD^T)$
j. $\operatorname{tr}(4E^T - D)$ **k.** $\operatorname{tr}(C^TA^T + 2E^T)$ **l.** $\operatorname{tr}((EC^T)^TA)$

6. **a.** $(2D^T - E)A$ **b.** $(4B)C + 2B$
c. $(-AC)^T + 5D^T$ **d.** $(BA^T - 2C)^T$
e. $B^T(CC^T - A^TA)$ **f.** $D^TE^T - (ED)^T$

In Exercises 7–8, use the following matrices and either the row method or the column method, as appropriate, to find the indicated row or column.

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

7. **a.** the first row of AB **b.** the third row of AB
c. the second column of AB **d.** the first column of BA
e. the third row of AA **f.** the third column of AA

8. **a.** the first column of AB **b.** the third column of BB
c. the second row of BB **d.** the first column of AA
e. the third column of AB **f.** the first row of BA

In Exercises 9–10, use matrices A and B from Exercises 7–8.

9. **a.** Express each column vector of AA as a linear combination of the column vectors of A.
b. Express each column vector of BB as a linear combination of the column vectors of B.

10. **a.** Express each column vector of AB as a linear combination of the column vectors of A.

b. Express each column vector of BA as a linear combination of the column vectors of B.

In each part of Exercises 11–12, find matrices A, $\mathbf{x}$, and $\mathbf{b}$ that express the given linear system as a single matrix equation $A\mathbf{x} = \mathbf{b}$, and write out this matrix equation.

11. **a.**
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

b.
$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad\quad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

12. **a.**
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -3 \\ 2x_1 + x_2 \qquad &= 0 \\ -3x_2 + 4x_3 &= 1 \\ x_1 \qquad + x_3 &= 5 \end{aligned}$$

b.
$$\begin{aligned} 3x_1 + 3x_2 + 3x_3 &= -3 \\ -x_1 - 5x_2 - 2x_3 &= 3 \\ -4x_2 + x_3 &= 0 \end{aligned}$$

In each part of Exercises 13–14, express the matrix equation as a system of linear equations.

13. **a.** $\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 5 & -3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$

14. **a.** $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 3 & 7 \\ -2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 5 & 0 & 2 & -2 \\ 3 & 1 & 4 & 7 \\ -2 & 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In Exercises 15–16, find all values of k, if any, that satisfy the equation.

15. $\begin{bmatrix} k & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$

16. $\begin{bmatrix} 2 & 2 & k \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$

In Exercises 17–20, use the column-row expansion of AB to express this product as a sum of matrix products.

17. $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

18. $A = \begin{bmatrix} 0 & -2 \\ 4 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 \\ -3 & 0 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

20. $A = \begin{bmatrix} 0 & 4 & 2 \\ 1 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & -1 \end{bmatrix}$

21. For the linear system in Example 5 of Section 1.2, express the general solution that we obtained in that example as a linear combination of column vectors that contain only numerical entries. [*Suggestion:* Rewrite the general solution as a single column vector, then write that column vector as a sum of column vectors each of which contains at most one parameter, and then factor out the parameters.]

22. Follow the directions of Exercise 21 for the linear system in Example 6 of Section 1.2.

In Exercises 23–24, solve the matrix equation for a, b, c, and d.

23. $\begin{bmatrix} a & 3 \\ -1 & a+b \end{bmatrix} = \begin{bmatrix} 4 & d-2c \\ d+2c & -2 \end{bmatrix}$

24. $\begin{bmatrix} a-b & b+a \\ 3d+c & 2d-c \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$

25. **a.** Show that if A has a row of zeros and B is any matrix for which AB is defined, then AB also has a row of zeros.

b. Find a similar result involving a column of zeros.

26. In each part, find a 6×6 matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possible by using letters rather than specific numbers for the nonzero entries.

a. $a_{ij} = 0 \quad \text{if} \quad i \neq j$ **b.** $a_{ij} = 0 \quad \text{if} \quad i > j$

c. $a_{ij} = 0 \quad \text{if} \quad i < j$ **d.** $a_{ij} = 0 \quad \text{if} \quad |i - j| > 1$

In Exercises 27–28, how many 3×3 matrices A can you find for which the equation is satisfied for all choices of x, y, and z?

27. $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$ **28.** $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ 0 \\ 0 \end{bmatrix}$

29. A matrix B is said to be a ***square root*** of a matrix A if $BB = A$.

a. Find two square roots of $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

b. How many different square roots can you find of

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}?$$

c. Do you think that every 2×2 matrix has at least one square root? Explain your reasoning.

30. Let 0 denote a 2×2 matrix, each of whose entries is zero.

a. Is there a 2×2 matrix A such that $A \neq 0$ and $AA = 0$? Justify your answer.

b. Is there a 2×2 matrix A such that $A \neq 0$ and $AA = A$? Justify your answer.

31. Establish Formula (11) by using Formula (5) to show that

$$(AB)_{ij} = (\mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_r\mathbf{r}_r)_{ij}$$

32. Find a 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

a. $a_{ij} = i + j$ **b.** $a_{ij} = i^{j-1}$

c. $a_{ij} = \begin{cases} 1 & \text{if} \quad |i-j| > 1 \\ -1 & \text{if} \quad |i-j| \leq 1 \end{cases}$

33. Suppose that type I items cost \$1 each, type II items cost \$2 each, and type III items cost \$3 each. Also, suppose that the accompanying table describes the number of items of each type purchased during the first four months of the year.

TABLE Ex-33

	Type I	Type II	Type III
Jan.	3	4	3
Feb.	5	6	0
Mar.	2	9	4
Apr.	1	1	7

What information is represented by the following product?

$$\begin{bmatrix} 3 & 4 & 3 \\ 5 & 6 & 0 \\ 2 & 9 & 4 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

34. The accompanying table shows a record of May and June unit sales for a clothing store. Let M denote the 4×3 matrix of May sales and J the 4×3 matrix of June sales.

a. What does the matrix $M + J$ represent?

b. What does the matrix $M - J$ represent?

c. Find a column vector $\mathbf{x}$ for which $M\mathbf{x}$ provides a list of the number of shirts, jeans, suits, and raincoats sold in May.

d. Find a row vector $\mathbf{y}$ for which $\mathbf{y}M$ provides a list of the number of small, medium, and large items sold in May.

e. Using the matrices $\mathbf{x}$ and $\mathbf{y}$ that you found in parts (c) and (d), what does $\mathbf{y}M\mathbf{x}$ represent?

TABLE Ex-34

May Sales

	Small	Medium	Large
Shirts	45	60	75
Jeans	30	30	40
Suits	12	65	45
Raincoats	15	40	35

June Sales

	Small	Medium	Large
Shirts	30	33	40
Jeans	21	23	25
Suits	9	12	11
Raincoats	8	10	9

Working with Proofs

35. Prove: If A and B are $n \times n$ matrices, then

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

36. a. Prove: If AB and BA are both defined, then AB and BA are square matrices.

b. Prove: If A is an $m \times n$ matrix and $A(BA)$ is defined, then B is an $n \times m$ matrix.

True-False Exercises

TF. In parts **(a)–(o)** determine whether the statement is true or false, and justify your answer.

a. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

b. An $m \times n$ matrix has m column vectors and n row vectors.

c. If A and B are 2×2 matrices, then $AB = BA$.

d. The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B.

e. For every matrix A, it is true that $(A^T)^T = A$.

f. If A and B are square matrices of the same order, then

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$$

g. If A and B are square matrices of the same order, then

$$(AB)^T = A^T B^T$$

h. For every square matrix A, it is true that $\mathrm{tr}(A^T) = \mathrm{tr}(A)$.

i. If A is a 6×4 matrix and B is an $m \times n$ matrix such that $B^T A^T$ is a 2×6 matrix, then $m = 4$ and $n = 2$.

j. If A is an $n \times n$ matrix and c is a scalar, then $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$.

k. If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$.

l. If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$.

m. If $AB + BA$ is defined, then A and B are square matrices of the same size.

n. If B has a column of zeros, then so does AB if this product is defined.

o. If B has a column of zeros, then so does BA if this product is defined.

Working with Technology

T1. a. Compute the product AB of the matrices in Example 5, and compare your answer to that in the text.

b. Use your technology utility to extract the columns of A and the rows of B, and then calculate the product AB by a column-row expansion.

T2. Suppose that a manufacturer uses Type I items at \$1.35 each, Type II items at \$2.15 each, and Type III items at \$3.95 each. Suppose also that the accompanying table describes the purchases of those items (in thousands of units) for the first quarter of the year. Find a matrix product, the computation of which produces a matrix that lists the manufacturer's expenditure in each month of the first quarter. Compute that product.

	Type I	Type II	Type III
Jan.	3.1	4.2	3.5
Feb.	5.1	6.8	0
Mar.	2.2	9.5	4.0
Apr.	1.0	1.0	7.4

1.4 Inverses; Algebraic Properties of Matrices

In this section we will discuss some of the algebraic properties of matrix operations. We will see that many of the basic rules of arithmetic for real numbers hold for matrices, but we will also see that some do not.

Properties of Matrix Addition and Scalar Multiplication

The following theorem lists the basic algebraic properties of the matrix operations.

Theorem 1.4.1

Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(*a*) $A + B = B + A$ **[Commutative law for matrix addition]**

(*b*) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(*c*) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(*d*) $A(B + C) = AB + AC$ **[Left distributive law]**

(*e*) $(B + C)A = BA + CA$ **[Right distributive law]**

(*f*) $A(B - C) = AB - AC$

(*g*) $(B - C)A = BA - CA$

(*h*) $a(B + C) = aB + aC$

(*i*) $a(B - C) = aB - aC$

(*j*) $(a + b)C = aC + bC$

(*k*) $(a - b)C = aC - bC$

(*l*) $a(bC) = (ab)C$

(*m*) $a(BC) = (aB)C = B(aC)$

To prove any of the equalities in this theorem one must show that the matrix on the left side has the same size as that on the right and that the corresponding entries on the two sides are the same. Most of the proofs follow the same pattern, so we will prove part (*d*) as a sample. The proof of the associative law for multiplication is more complicated than the rest and is outlined in the exercises.

***Proof* (d)** We must show that $A(B+C)$ and $AB+AC$ have the same size and that corresponding entries are equal. To form $A(B+C)$, the matrices B and C must have the same size, say $m \times n$, and the matrix A must then have m columns, so its size must be of the form $r \times m$. This makes $A(B+C)$ an $r \times n$ matrix. It follows that $AB+AC$ is also an $r \times n$ matrix and, consequently, $A(B+C)$ and $AB+AC$ have the same size.

Suppose that $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. We want to show that corresponding entries of $A(B+C)$ and $AB+AC$ are equal; that is,

$$\big(A(B+C)\big)_{ij} = (AB+AC)_{ij}$$

for all values of i and j. But from the definitions of matrix addition and matrix multiplication, we have

$$\begin{aligned}\big(A(B+C)\big)_{ij} &= a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) + \cdots + a_{im}(b_{mj}+c_{mj}) \\ &= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{im}c_{mj}) \\ &= (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij} \quad \blacksquare\end{aligned}$$

There are three basic ways to prove that two matrices of the same size are equal—prove that corresponding entries are the same, prove that corresponding row vectors are the same, or prove that corresponding column vectors are the same.

Remark Although the operations of matrix addition and matrix multiplication were defined for pairs of matrices, associative laws (*b*) and (*c*) enable us to denote sums and products of three matrices as $A+B+C$ and ABC without inserting any parentheses. This is justified by the fact that no matter how parentheses are inserted, the associative laws guarantee that the same end result will be obtained. In general, *given any sum or any product of matrices, pairs of parentheses can be inserted or deleted anywhere within the expression without affecting the end result.*

EXAMPLE 1 | Associativity of Matrix Multiplication

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1(*c*).

Properties of Matrix Multiplication

Do not let Theorem 1.4.1 lull you into believing that *all* laws of real arithmetic carry over to matrix arithmetic. For example, you know that in real arithmetic it is always true that

$ab = ba$, which is called the *commutative law for multiplication*. In matrix arithmetic, however, the equality of AB and BA can fail for three possible reasons:

1. AB may be defined and BA may not (for example, if A is 2×3 and B is 3×4).
2. AB and BA may both be defined, but they may have different sizes (for example, if A is 2×3 and B is 3×2).
3. AB and BA may both be defined and have the same size, but the two products may be different (as illustrated in the next example).

EXAMPLE 2 | Order Matters in Matrix Multiplication

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$.

Because, as this example shows, it is *not* generally true that $AB = BA$, we say that ***matrix multiplication is not commutative***. This does not preclude the possibility of equality in certain cases—it is just not true in general. In those special cases where there is equality we say that A and B ***commute***.

Zero Matrices

A matrix whose entries are all zero is called a ***zero matrix***. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

We will denote a zero matrix by 0 unless it is important to specify its size, in which case we will denote the $m \times n$ zero matrix by $0_{m \times n}$.

It should be evident that if A and 0 are matrices with the same size, then

$$A + 0 = 0 + A = A$$

Thus, 0 plays the same role in this matrix equation that the number 0 plays in the numerical equation $a + 0 = 0 + a = a$.

The following theorem lists the basic properties of zero matrices. Since the results should be self-evident, we will omit the formal proofs.

Theorem 1.4.2

Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(*a*) $A + 0 = 0 + A = A$

(*b*) $A - 0 = A$

(*c*) $A - A = A + (-A) = 0$

(*d*) $0A = 0$

(*e*) If $cA = 0$, then $c = 0$ or $A = 0$.

Since we know that the commutative law of real arithmetic is not valid in matrix arithmetic, it should not be surprising that there are other rules that fail as well. For example, consider the following two laws of real arithmetic:

- If $ab = ac$ and $a \neq 0$, then $b = c$. **[The cancellation law]**
- If $ab = 0$, then at least one of the factors on the left is 0.

The next two examples show that these laws are not true in matrix arithmetic.

EXAMPLE 3 | Failure of the Cancellation Law

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).

EXAMPLE 4 | A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An identity matrix is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

To explain the role of identity matrices in matrix arithmetic, let us consider the effect of multiplying a general 2×3 matrix A on each side by an identity matrix. Multiplying on the right by the 3×3 identity matrix yields

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and multiplying on the left by the 2×2 identity matrix yields

$$I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

The same result holds in general; that is, if A is any $m \times n$ matrix, then

$$\boxed{AI_n = A \quad \text{and} \quad I_mA = A}$$

Thus, the identity matrices play the same role in matrix arithmetic that the number 1 plays in the numerical equation $a \cdot 1 = 1 \cdot a = a$.

As the next theorem shows, identity matrices arise naturally as reduced row echelon forms of *square* matrices.

Theorem 1.4.3

If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

Proof Suppose that the reduced row echelon form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

Either the last row in this matrix consists entirely of zeros or it does not. If not, the matrix contains no zero rows, and consequently each of the n rows has a leading entry of 1. Since these leading 1's occur progressively farther to the right as we move down the matrix, each of these 1's must occur on the main diagonal. Since the other entries in the same column as one of these 1's are zero, R must be I_n. Thus, either R has a row of zeros or $R = I_n$. ■

Inverse of a Matrix

In real arithmetic every nonzero number a has a reciprocal $a^{-1}(= 1/a)$ with the property

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

The number a^{-1} is sometimes called the *multiplicative inverse* of a. Our next objective is to develop an analog of this result for matrix arithmetic. For this purpose we make the following definition.

Definition 1

If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B exists, then A is said to be ***singular***.

The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when $AB = BA = I$ we say that A and B are *inverses of one another*.

EXAMPLE 5 | An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

EXAMPLE 6 | A Class of Singular Matrices

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that

$$AB = BA = I$$

For this purpose let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{0}$ be the column vectors of A. Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \qquad \textbf{[Formula (6) of Section 1.3]}$$

The column of zeros shows that $BA \neq I$ and hence that A is singular.

As in Example 6, we will frequently denote a zero matrix with one row or one column by a boldface zero.

Properties of Inverses

It is reasonable to ask whether an invertible matrix can have more than one inverse. The next theorem shows that the answer is no—*an invertible matrix has exactly one inverse.*

Theorem 1.4.4

If B and C are both inverses of the matrix A, then $B = C$.

Proof Since B is an inverse of A, we have $BA = I$. Multiplying both sides on the right by C gives $(BA)C = IC = C$. But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$. ■

As a consequence of this important result, we can now speak of "the" inverse of an invertible matrix. If A is invertible, then its inverse will be denoted by the symbol A^{-1}. Thus,

$$\boxed{AA^{-1} = I \quad \text{and} \quad A^{-1}A = I} \tag{1}$$

The inverse of A plays much the same role in matrix arithmetic that the reciprocal a^{-1} plays in the numerical relationships $aa^{-1} = 1$ and $a^{-1}a = 1$.

Warning The symbol A^{-1} should not be interpreted as $1/A$. Division by matrices is not a defined operation.

In the next section we will develop a method for computing the inverse of an invertible matrix of any size. For now we give the following theorem that specifies conditions under which a 2×2 matrix is invertible and provides a simple formula for its inverse.

Theorem 1.4.5

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

The quantity $ad - bc$ in Theorem 1.4.5 is called the ***determinant*** of the 2×2 matrix A and is denoted by

$$\det(A) = ad - bc$$

or alternatively by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We will omit the proof, because we will study a more general version of this theorem later. For now, you should at least confirm the validity of Formula (2) by showing that $AA^{-1} = A^{-1}A = I$.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

FIGURE 1.4.1

Remark **Figure 1.4.1** illustrates that the determinant of a 2×2 matrix A is the product of the entries on its main diagonal minus the product of the entries *off* its main diagonal.

Historical Note

The formula for A^{-1} given in Theorem 1.4.5 first appeared (in a more general form) in Arthur Cayley's 1858 *Memoir on the Theory of Matrices*. The more general result that Cayley discovered will be studied later.

EXAMPLE 7 | Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$(a)\ A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (*a*) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (*b*) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

EXAMPLE 8 | Solution of a Linear System by Matrix Inversion

A problem that arises in many applications is to solve a pair of equations of the form

$$\begin{aligned} u &= ax + by \\ v &= cx + dy \end{aligned}$$

for x and y in terms of u and v. One approach is to treat this as a linear system of two equations in the unknowns x and y and use Gauss–Jordan elimination to solve for x and y. However, because the coefficients of the unknowns are *literal* rather than *numerical*, that procedure is a little clumsy. As an alternative approach, let us replace the two equations by the single matrix equation

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

which we can rewrite as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If we assume that the 2×2 matrix is invertible (i.e., $ad - bc \neq 0$), then we can multiply through on the left by the inverse and rewrite the equation as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Using Theorem 1.4.5, we can rewrite this equation as

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

from which we obtain

$$x = \frac{du - bv}{ad - bc}, \quad y = \frac{av - cu}{ad - bc}$$

The next theorem is concerned with inverses of matrix products.

Theorem 1.4.6

If A and B are invertible matrices with the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof We can establish the invertibility and obtain the stated formula at the same time by showing that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$. ■

Although we will not prove it, this result can be extended to three or more factors:

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

EXAMPLE 9 | The Inverse of a Product

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

If a product of matrices is singular, then at least one of the factors must be singular. Why?

We leave it for you to show that

$$AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

and also that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6.

Powers of a Matrix

If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$\boxed{A^0 = I \quad \text{and} \quad A^n = AA\cdots A \quad [n \text{ factors}]}$$

and if A is invertible, then we define the negative integer powers of A to be

$$\boxed{A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1} \quad [n \text{ factors}]}$$

Because these definitions parallel those for real numbers, the usual laws of nonnegative exponents hold; for example,

$$\boxed{A^rA^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}}$$

In addition, we have the following properties of negative exponents.

Theorem 1.4.7

If A is invertible and n is a nonnegative integer, then:

(*a*) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(*b*) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

We will prove part (c) and leave the proofs of parts (a) and (b) as exercises.

Proof (c) Properties (m) and (l) of Theorem 1.4.1 imply that

$$(kA)(k^{-1}A^{-1}) = k^{-1}(kA)A^{-1} = (k^{-1}k)AA^{-1} = (1)I = I$$

and similarly, $(k^{-1}A^{-1})(kA) = I$. Thus, kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}$. ■

EXAMPLE 10 | Properties of Exponents

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(*b*),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

EXAMPLE 11 | The Square of a Matrix Sum

In real arithmetic, where we have a commutative law for multiplication, we can write

$$(a + b)^2 = a^2 + ab + ba + b^2 = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

However, in matrix arithmetic, where we have no commutative law for multiplication, the best we can do is to write

$$(A + B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B *commute* (i.e., $AB = BA$) that we can go a step further and write

$$(A + B)^2 = A^2 + 2AB + B^2$$

Matrix Polynomials

If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m \tag{3}$$

where I is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting A for x and replacing the constant term a_0 by the matrix a_0I. An expression of form (3) is called a ***matrix polynomial in A***.

EXAMPLE 12 | A Matrix Polynomial

Find $p(A)$ for

$$p(x) = x^2 - 2x - 5 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 5I \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$.

Remark It follows from the fact that $A^rA^s = A^{r+s} = A^{s+r} = A^sA^r$ that powers of a square matrix commute, and since a matrix polynomial in A is built up from powers of A, any two matrix polynomials in A also commute; that is, for any polynomials p_1 and p_2 we have

$$p_1(A)p_2(A) = p_2(A)p_1(A) \tag{4}$$

Properties of the Transpose

The following theorem lists the main properties of the transpose.

Theorem 1.4.8

If the sizes of the matrices are such that the stated operations can be performed, then:

(*a*) $(A^T)^T = A$

(*b*) $(A+B)^T = A^T + B^T$

(*c*) $(A-B)^T = A^T - B^T$

(*d*) $(kA)^T = kA^T$

(*e*) $(AB)^T = B^TA^T$

If you keep in mind that transposing a matrix interchanges its rows and columns, then you should have little trouble visualizing the results in parts (*a*)–(*d*). For example, part (*a*) states the obvious fact that interchanging rows and columns twice leaves a matrix unchanged; and part (*b*) states that adding two matrices and then interchanging the rows and columns produces the same result as interchanging the rows and columns before adding. We will omit the formal proofs. Part (*e*) is less obvious, but for brevity we will omit its proof as well. The result in that part can be extended to three or more factors and restated as:

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

The following theorem establishes a relationship between the inverse of a matrix and the inverse of its transpose.

Theorem 1.4.9

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof We can establish the invertibility and obtain the formula at the same time by showing that

$$A^T(A^{-1})^T = (A^{-1})^TA^T = I$$

But from part (*e*) of Theorem 1.4.8 and the fact that $I^T = I$, we have

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$
$$(A^{-1})^TA^T = (AA^{-1})^T = I^T = I$$

which completes the proof. ■

EXAMPLE 13 | Inverse of a Transpose

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Since A is invertible, its determinant $ad - bc$ is nonzero. But the determinant of A^T is also $ad - bc$ (verify), so A^T is also invertible. It follows from Theorem 1.4.5 that

$$(A^T)^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{c}{ad-bc} \\ -\dfrac{b}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

which is the same matrix that results if A^{-1} is transposed (verify). Thus,

$$(A^T)^{-1} = (A^{-1})^T$$

as guaranteed by Theorem 1.4.9.

Exercise Set 1.4

*In Exercises **1–2**, verify that the following matrices and scalars satisfy the stated properties of Theorem 1.4.1.*

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & -4 \end{bmatrix},$$

$$C = \begin{bmatrix} 4 & 1 \\ -3 & -2 \end{bmatrix}, \quad a = 4, \quad b = -7$$

1. **a.** The associative law for matrix addition.

b. The associative law for matrix multiplication.

c. The left distributive law.

d. $(a + b)C = aC + bC$

2. **a.** $a(BC) = (aB)C = B(aC)$

b. $A(B - C) = AB - AC$ **c.** $(B + C)A = BA + CA$

d. $a(bC) = (ab)C$

*In Exercises **3–4**, verify that the matrices and scalars in Exercise 1 satisfy the stated properties.*

3. **a.** $(A^T)^T = A$ **b.** $(AB)^T = B^TA^T$

4. **a.** $(A + B)^T = A^T + B^T$ **b.** $(aC)^T = aC^T$

*In Exercises **5–8**, use Theorem 1.4.5 to compute the inverse of the matrix.*

5. $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$ **6.** $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

7. $C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ **8.** $D = \begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$

9. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

*In Exercises **11–14**, verify that the equations are valid for the matrices in Exercises 5–8.*

11. $(A^T)^{-1} = (A^{-1})^T$ **12.** $(A^{-1})^{-1} = A$

13. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ **14.** $(ABC)^T = C^TB^TA^T$

*In Exercises **15–18**, use the given information to find A.*

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ **16.** $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$ **18.** $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

*In Exercises **19–20**, compute the following using the given matrix A.*

a. A^3 **b.** A^{-3} **c.** $A^2 - 2A + I$

19. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ **20.** $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

*In Exercises **21–22**, compute $p(A)$ for the given matrix A and the following polynomials.*

a. $p(x) = x - 2$

b. $p(x) = 2x^2 - x + 1$

c. $p(x) = x^3 - 2x + 1$

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ **22.** $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

In Exercises 23–24, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

23. Find all values of $a, b, c,$ and d (if any) for which the matrices A and B commute.

24. Find all values of $a, b, c,$ and d (if any) for which the matrices A and C commute.

In Exercises 25–28, use the method of Example 8 to find the unique solution of the given linear system.

25. $\begin{aligned} 3x_1 - 2x_2 &= -1 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$ **26.** $\begin{aligned} -x_1 + 5x_2 &= 4 \\ -x_1 - 3x_2 &= 1 \end{aligned}$

27. $\begin{aligned} 6x_1 + x_2 &= 0 \\ 4x_1 - 3x_2 &= -2 \end{aligned}$ **28.** $\begin{aligned} 2x_1 - 2x_2 &= 4 \\ x_1 + 4x_2 &= 4 \end{aligned}$

If a polynomial $p(x)$ can be factored as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A)$$

In Exercises 29–30, verify this statement for the stated matrix A and polynomials

$$p(x) = x^2 - 9, \quad p_1(x) = x + 3, \quad p_2(x) = x - 3$$

29. The matrix A in Exercise 21.

30. An arbitrary square matrix A.

31. a. Give an example of two 2×2 matrices such that

$$(A + B)(A - B) \neq A^2 - B^2$$

b. State a valid formula for multiplying out

$$(A + B)(A - B)$$

c. What condition can you impose on A and B that will allow you to write $(A + B)(A - B) = A^2 - B^2$?

32. The numerical equation $a^2 = 1$ has exactly two solutions. Find at least eight solutions of the matrix equation $A^2 = I_3$. [*Hint:* Look for solutions in which all entries off the main diagonal are zero.]

33. a. Show that if a square matrix A satisfies the equation $A^2 + 2A + I = 0$, then A must be invertible. What is the inverse?

b. Show that if $p(x)$ is a polynomial with a nonzero constant term, and if A is a square matrix for which $p(A) = 0$, then A is invertible.

34. Is it possible for A^3 to be an identity matrix without A being invertible? Explain.

35. Can a matrix with a row of zeros or a column of zeros have an inverse? Explain.

36. Can a matrix with two identical rows or two identical columns have an inverse? Explain.

In Exercises 37–38, determine whether A is invertible, and if so, find the inverse. [Hint: *Solve $AX = I$ for X by equating corresponding entries on the two sides.*]

37. $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **38.** $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

In Exercises 39–40, simplify the expression assuming that A, B, C, and D are invertible.

39. $(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

40. $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

41. Show that if R is a $1 \times n$ matrix and C is an $n \times 1$ matrix, then $RC = \operatorname{tr}(CR)$.

42. If A is a square matrix and n is a positive integer, is it true that $(A^n)^T = (A^T)^n$? Justify your answer.

43. a. Show that if A is invertible and $AB = AC$, then $B = C$.

b. Explain why part (a) and Example 3 do not contradict one another.

44. Show that if A is invertible and k is any nonzero scalar, then $(kA)^n = k^n A^n$ for all integer values of n.

45. a. Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

b. What does the result in part (a) tell you about the matrix $A^{-1} + B^{-1}$?

46. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

a. Show that if A is idempotent, then so is $I - A$.

b. Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

47. Show that if A is a square matrix such that $A^k = 0$ for some positive integer k, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

48. Show that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfies the equation

$$A^2 - (a + d)A + (ad - bc)I = 0$$

49. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$C^T B^{-1} A^2 B A C^{-1} D A^{-2} B^T C^{-2} = C^T$$

50. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^T DBA^T C = AB^T$$

Working with Proofs

In Exercises 51–58, prove the stated result.

51. Theorem 1.4.1(a) **52.** Theorem 1.4.1(b)

53. Theorem 1.4.1(f) **54.** Theorem 1.4.1(c)

55. Theorem 1.4.2(c)

56. Theorem 1.4.2(b)

57. Theorem 1.4.8(d)

58. Theorem 1.4.8(e)

True-False Exercises

TF. In parts **(a)–(k)** determine whether the statement is true or false, and justify your answer.

a. Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

b. For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

c. For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

d. If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

e. If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^TB^T$.

f. The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

g. If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

h. If A is an invertible matrix, then so is A^T.

i. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

j. A square matrix containing a row or column of zeros cannot be invertible.

k. The sum of two invertible matrices of the same size must be invertible.

Working with Technology

T1. Let A be the matrix

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{7} & 0 \end{bmatrix}$$

Discuss the behavior of A^k as k increases indefinitely, that is, as $k \to \infty$.

T2. In each part use your technology utility to make a conjecture about the form of A^n for positive integer powers of n.

a. $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ **b.** $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

T3. The ***Fibonacci sequence*** (named for the Italian mathematician Leonardo Fibonacci 1170–1250) is

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144, \ldots$$

the terms of which are commonly denoted as

$$F_0,\ F_1,\ F_2,\ F_3, \ldots,\ F_n, \ldots$$

After the initial terms $F_0 = 0$ and $F_1 = 1$, each term is the sum of the previous two; that is,

$$F_n = F_{n-1} + F_{n-2}$$

Confirm that if

$$Q = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_0 \end{bmatrix}$$

1.5 Elementary Matrices and a Method for Finding A^{-1}

In this section we will develop an algorithm for finding the inverse of a matrix, and we will discuss some of the basic properties of invertible matrices.

Elementary Matrices

In Section 1.1 we defined three elementary row operations on a matrix A:

1. Multiply a row by a nonzero constant c.
2. Interchange two rows.
3. Add a constant c times one row to another.

It should be evident that if we let B be the matrix that results from A by performing one of the operations in this list, then the matrix A can be recovered from B by performing the corresponding operation in the following list:

1. Multiply the same row by $1/c$.
2. Interchange the same two rows.
3. If B resulted by adding c times row r_i of A to row r_j, then add $-c$ times r_j to r_i.

It follows that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A. Accordingly, we make the following definition.

Definition 1

Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

Our next goal is to show how matrix multiplication can be used to carry out an elementary row operation.

Definition 2

A matrix E is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

EXAMPLE 1 | Elementary Matrices and Row Operations

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

The following theorem, whose proof is left as an exercise, shows that when a matrix A is multiplied on the *left* by an elementary matrix E, the effect is to perform an elementary row operation on A.

Theorem 1.5.1 will be a useful tool for developing new results about matrices, but as a practical matter it is usually preferable to perform row operations directly.

Theorem 1.5.1

Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

EXAMPLE 2 | Using Elementary Matrices

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the matrix that results when we add 3 times the first row of A to the third row.

We know from the discussion at the beginning of this section that if E is an elementary matrix that results from performing an elementary row operation on an identity matrix I, then there is a second elementary row operation, which when applied to E produces I back again. **Table 1** lists these operations. The operations on the right side of the table are called the ***inverse operations*** of the corresponding operations on the left.

TABLE 1

Row Operation on I That Produces E	Row Operation on E That Reproduces I
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

EXAMPLE 3 | Row Operations and Inverse Row Operations

In each of the following, an elementary row operation is applied to the 2×2 identity matrix to obtain an elementary matrix E, then E is restored to the identity matrix by applying the inverse row operation.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7.

Multiply the second row by $\frac{1}{7}$.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Interchange the first and second rows. Interchange the first and second rows.

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 5\\0 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Add 5 times the second row to the first. Add -5 times the second row to the first.

The next theorem is a key result about invertibility of elementary matrices. It will be a building block for many results that follow.

Theorem 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Proof If E is an elementary matrix, then E results by performing some row operation on I. Let E_0 be the matrix that results when the inverse of that operation is performed on I. Applying Theorem 1.5.1 and using the fact that inverse row operations cancel the effect of each other, it follows that

$$E_0E = I \quad \text{and} \quad EE_0 = I$$

Thus, the elementary matrix E_0 is the inverse of E. ■

Equivalence Theorem

One of our objectives as we progress through this text is to show how seemingly diverse ideas in linear algebra are related. The following theorem, which relates results we have obtained about invertibility of matrices, homogeneous linear systems, reduced row echelon forms, and elementary matrices, is our first step in that direction. As we study new topics, more statements will be added to this theorem.

Theorem 1.5.3

Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(*a*) A is invertible.

(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.

(*d*) A is expressible as a product of elementary matrices.

Proof We will prove the equivalence by establishing the chain of implications: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

The following figure illustrates that the sequence of implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$$

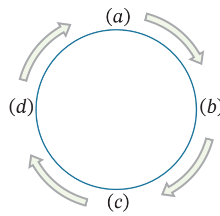
implies that

$$(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$$

and hence that

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$$

(see Appendix A).



(a) ⇒ (b) Assume A is invertible and let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = \mathbf{0}$. Multiplying both sides of this equation by A^{-1} gives

$$(A^{-1}A)\mathbf{x}_0 = A^{-1}\mathbf{0}$$

from which it follows that $\mathbf{x}_0 = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) ⇒ (c) Let $A\mathbf{x} = \mathbf{0}$ be the matrix form of the system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned} \tag{1}$$

and assume that the system has only the trivial solution. If we solve by Gauss–Jordan elimination, then the system of equations corresponding to the reduced row echelon form of the augmented matrix will be

$$\begin{aligned}
x_1 \qquad\qquad &= 0 \\
x_2 \qquad &= 0 \\
\ddots \quad & \\
x_n &= 0
\end{aligned} \tag{2}$$

Thus, the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

for (1) can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

for (2) by a sequence of elementary row operations. If we disregard the last column (all zeros) in each of these matrices, we can conclude that the reduced row echelon form of A is I_n.

(c) ⇒ (d) Assume that the reduced row echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations. By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2E_1A = I_n \tag{3}$$

By Theorem 1.5.2, $E_1, E_2, \ldots, E_k$ are invertible. Multiplying both sides of Equation (3) on the left successively by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ we obtain

$$A = E_1^{-1}E_2^{-1} \cdots E_k^{-1}I_n = E_1^{-1}E_2^{-1} \cdots E_k^{-1} \tag{4}$$

By Theorem 1.5.2, this equation expresses A as a product of elementary matrices.

(d) ⇒ (a) If A is a product of elementary matrices, then from Theorems 1.4.6 and 1.5.2, the matrix A is a product of invertible matrices and hence is invertible. ■

A Method for Inverting Matrices

As a first application of Theorem 1.5.3, we will develop a procedure (or algorithm) that can be used to tell whether a given matrix is invertible, and if so, produce its inverse. To derive this algorithm, assume for the moment, that A is an invertible $n \times n$ matrix. In Equation (3), the elementary matrices execute a sequence of row operations that reduce A to I_n. If we multiply both sides of this equation on the right by A^{-1} and simplify, we obtain

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

But this equation tells us that *the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}*. Thus, we have established the following result.

Inversion Algorithm To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

A simple method for carrying out this procedure is given in the following example.

EXAMPLE 4 | Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$ ← We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$ ← We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$ ← We multiplied the third row by -1.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Often it will not be known in advance if a given $n \times n$ matrix A is invertible. However, if it is not, then by parts (a) and (c) of Theorem 1.5.3 it will be impossible to reduce A to I_n by elementary row operations. This will be signaled by a row of zeros appearing on the *left side* of the partition at some stage of the inversion algorithm. If this occurs, then you can stop the computations and conclude that A is not invertible.

EXAMPLE 5 | Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

EXAMPLE 6 | Analyzing Homogeneous Systems

Use Theorem 1.5.3 to determine whether the given homogeneous system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Solution From parts (a) and (b) of Theorem 1.5.3 a homogeneous linear system has only the trivial solution if and only if its coefficient matrix is invertible. From Examples 4 and 5 the coefficient matrix of system (a) is invertible and that of system (b) is not. Thus, system (a) has only the trivial solution while system (b) has nontrivial solutions.

Exercise Set 1.5

In Exercises 1–2, determine whether the given matrix is elementary.

1. a. $\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. a. $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

In Exercises 3–4, find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

3. a. $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. a. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

In Exercises 5–6 an elementary matrix E and a matrix A are given. Identify the row operation corresponding to E and verify that the product EA results from applying the row operation to A.

5. a. $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

6. a. $E = \begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b. $E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

In Exercises 7–8, use the following matrices and find an elementary matrix E that satisfies the stated equation.

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 2 & -7 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 1 & 5 \\ -6 & 21 & 3 \\ 3 & 4 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

7. a. $EA = B$ **b.** $EB = A$

c. $EA = C$ **d.** $EC = A$

8. a. $EB = D$ **b.** $ED = B$

c. $EB = F$ **d.** $EF = B$

In Exercises 9–10, first use Theorem 1.4.5 and then use the inversion algorithm to find A^{-1}, if it exists.

9. a. $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

10. a. $A = \begin{bmatrix} 1 & -5 \\ 3 & -16 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$

In Exercises 11–12, use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

11. a. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ **b.** $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

12. a. $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$ **b.** $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

In Exercises 13–18, use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

13. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **14.** $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$ **18.** $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

In Exercises 19–20, find the inverse of each of the following 4×4 matrices, where k_1, k_2, k_3, k_4, and k are all nonzero.

19. a. $\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}$ **b.** $\begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

20. a. $\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}$

In Exercises 21–22, find all values of c, if any, for which the given matrix is invertible.

21. $\begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$ **22.** $\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$

In Exercises 23–26, express the matrix and its inverse as products of elementary matrices.

23. $\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$ **24.** $\begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix}$

25. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ **26.** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

In Exercises 27–28, show that the matrices A and B are row equivalent by finding a sequence of elementary row operations that produces B from A, and then use that result to find a matrix C such that $CA = B$.

27. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{bmatrix}$

28. $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 9 & 4 \\ -5 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

29. Show that if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$$

is an elementary matrix, then at least one entry in the third row must be zero.

30. Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

Working with Proofs

31. Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

32. Prove that if A is an invertible matrix and B is row equivalent to A, then B is also invertible.

33. Prove that if B is obtained from A by performing a sequence of elementary row operations, then there is a second sequence of elementary row operations, which when applied to B recovers A.

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

a. The product of two elementary matrices of the same size must be an elementary matrix.

b. Every elementary matrix is invertible.

c. If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent.

d. If A is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = 0$ has infinitely many solutions.

e. If A is an $n \times n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible.

f. If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

g. An expression of an invertible matrix A as a product of elementary matrices is unique.

Working with Technology

T1. It can be proved that if the partitioned matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is invertible, then its inverse is

$$\begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

provided that all of the inverses on the right side exist. Use this result to find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

1.6 More on Linear Systems and Invertible Matrices

In this section we will show how the inverse of a matrix can be used to solve a linear system, and we will develop some more results about invertible matrices.

Number of Solutions of a Linear System

In Section 1.1 we made the statement (based on Figures 1.1.1 and 1.1.2) that every linear system either has no solutions, has exactly one solution, or has infinitely many solutions. We are now in a position to prove this fundamental result.

Theorem 1.6.1

A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Proof If $A\mathbf{x} = \mathbf{b}$ is a system of linear equations, exactly one of the following is true: (a) the system has no solutions, (b) the system has exactly one solution, or (c) the system has more than one solution. The proof will be complete if we can show that the system has infinitely many solutions in case (c).

Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, the matrix $\mathbf{x}_0$ is nonzero; moreover,

$$A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

If we now let k be any scalar, then

$$\begin{aligned} A(\mathbf{x}_1 + k\mathbf{x}_0) &= A\mathbf{x}_1 + A(k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) \\ &= \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b} \end{aligned}$$

But this says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. ■

Solving Linear Systems by Matrix Inversion

Thus far we have studied two *procedures* for solving linear systems—Gauss–Jordan elimination and Gaussian elimination. The following theorem provides an actual *formula* for the solution of a linear system of n equations in n unknowns in the case where the coefficient matrix is invertible.

Theorem 1.6.2

If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof Since $A(A^{-1}\mathbf{b}) = \mathbf{b}$, it follows that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is the only solution, we will assume that $\mathbf{x}_0$ is an arbitrary solution and then show that $\mathbf{x}_0$ must be the solution $A^{-1}\mathbf{b}$.

If $\mathbf{x}_0$ is any solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_0 = \mathbf{b}$. Multiplying both sides of this equation by A^{-1}, we obtain $\mathbf{x}_0 = A^{-1}\mathbf{b}$. ■

EXAMPLE 1 | **Solution of a Linear System Using A^{-1}**

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2, the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1, x_2 = -1, x_3 = 2$.

Keep in mind that the method of Example 1 applies only when the system has as many equations as unknowns and the coefficient matrix is invertible.

Linear Systems with a Common Coefficient Matrix

Frequently, one is concerned with solving a sequence of systems

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3, \ldots, \quad A\mathbf{x} = \mathbf{b}_k$$

each of which has the same square coefficient matrix A. If A is invertible, then the solutions

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1, \quad \mathbf{x}_2 = A^{-1}\mathbf{b}_2, \quad \mathbf{x}_3 = A^{-1}\mathbf{b}_3, \ldots, \quad \mathbf{x}_k = A^{-1}\mathbf{b}_k$$

can be obtained with one matrix inversion and k matrix multiplications. An efficient way to do this is to form the partitioned matrix

$$[A \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_k] \tag{1}$$

in which the coefficient matrix A is "augmented" by all k of the matrices $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$, and then reduce (1) to reduced row echelon form by Gauss–Jordan elimination. In this way we can solve all k systems at once. This method has the added advantage that it applies even when A is not invertible.

EXAMPLE 2 | **Solving Two Linear Systems at Once**

Solve the systems

$$(a)\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

Solution The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced row echelon form yields (verify)

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

It follows from the last two columns that the solution of system (a) is $x_1 = 1, x_2 = 0, x_3 = 1$ and the solution of system (b) is $x_1 = 2, x_2 = 1, x_3 = -1$.

Properties of Invertible Matrices

Up to now, to show that an $n \times n$ matrix A is invertible, it has been necessary to find an $n \times n$ matrix B such that

$$AB = I \quad \text{and} \quad BA = I$$

The next theorem shows that if we can produce an $n \times n$ matrix B satisfying *either* condition, then the other condition will hold automatically.

Theorem 1.6.3

Let A be a square matrix.

(a) If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

We will prove part (a) and leave part (b) as an exercise.

Proof (a) Assume that $BA = I$. If we can show that A is invertible, the proof can be completed by multiplying $BA = I$ on both sides by A^{-1} to obtain

$$BAA^{-1} = IA^{-1} \quad \text{or} \quad BI = IA^{-1} \quad \text{or} \quad B = A^{-1}$$

To show that A is invertible, it suffices to show that the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (see Theorem 1.5.3). Let $\mathbf{x}_0$ be any solution of this system. If we multiply both sides of $A\mathbf{x}_0 = \mathbf{0}$ on the left by B, we obtain $BA\mathbf{x}_0 = B\mathbf{0}$ or $I\mathbf{x}_0 = \mathbf{0}$ or $\mathbf{x}_0 = \mathbf{0}$. Thus, the system of equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. ■

Equivalence Theorem

We are now in a position to add two more statements to the four given in Theorem 1.5.3.

Theorem 1.6.4

Equivalent Statements

If A is an $n \times n$ matrix, then the following are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.
(*d*) A is expressible as a product of elementary matrices.
(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Proof Since we proved in Theorem 1.5.3 that (*a*), (*b*), (*c*), and (*d*) are equivalent, it will be sufficient to prove that $(a) \Rightarrow (f) \Rightarrow (e) \Rightarrow (a)$.

$(a) \Rightarrow (f)$ This was already proved in Theorem 1.6.2.

$(f) \Rightarrow (e)$ This is almost self-evident, for if $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

$(e) \Rightarrow (a)$ If the system $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$, then, in particular, this is so for the systems

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be solutions of the respective systems, and let us form an $n \times n$ matrix C having these solutions as columns. Thus C has the form

$$C = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n]$$

As discussed in Section 1.3, the successive columns of the product AC will be

$$A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n$$

[see Formula (8) of Section 1.3]. Thus,

$$AC = [A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

By part (*b*) of Theorem 1.6.3, it follows that $C = A^{-1}$. Thus, A is invertible. ■

It follows from the equivalency of parts (*e*) and (*f*) that if you can show that $A\mathbf{x} = \mathbf{b}$ has *at least one* solution for every $n \times 1$ matrix $\mathbf{b}$, then you can conclude that it has *exactly one* solution for every $n \times 1$ matrix $\mathbf{b}$.

We know from earlier work that invertible matrix factors produce an invertible product. Conversely, the following theorem shows that if the product of square matrices is invertible, then the factors themselves must be invertible.

Theorem 1.6.5

Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

Proof We will show first that B is invertible by showing that the homogeneous system $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. If we assume that $\mathbf{x}_0$ is any solution of this system, then

$$(AB)\mathbf{x}_0 = A(B\mathbf{x}_0) = A\mathbf{0} = \mathbf{0}$$

so $\mathbf{x}_0 = \mathbf{0}$ by parts (a) and (b) of Theorem 1.6.4 applied to the invertible matrix AB. Thus, $B\mathbf{x} = \mathbf{0}$ has only the trivial solution, which implies that B is invertible. But this in turn implies that A is invertible since A can be expressed as

$$A = A(BB^{-1}) = (AB)B^{-1}$$

which is a product of two invertible matrices. This completes the proof. ■

In our later work the following fundamental problem will occur frequently in various contexts.

A Fundamental Problem Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices $\mathbf{b}$ such that the system of equations $A\mathbf{x} = \mathbf{b}$ is consistent.

If A is an invertible matrix, Theorem 1.6.2 completely solves this problem by asserting that for *every* $m \times 1$ matrix $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not square, or if A is square but not invertible, then Theorem 1.6.2 does not apply. In these cases $\mathbf{b}$ must usually satisfy certain conditions in order for $A\mathbf{x} = \mathbf{b}$ to be consistent. The following example illustrates how the methods of Section 1.2 can be used to determine such conditions.

EXAMPLE 3 | Determining Consistency by Elimination

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \phantom{{}+x_2} + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

which can be reduced to row echelon form as follows:

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \longleftarrow$$

-1 times the first row was added to the second and -2 times the first row was added to the third.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix} \longleftarrow$$

The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix} \longleftarrow$$

The second row was added to the third.

It is now evident from the third row in the matrix that the system has a solution if and only if b_1, b_2, and b_3 satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_1 + b_2$$

To express this condition another way, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a matrix of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

where b_1 and b_2 are arbitrary.

EXAMPLE 4 | Determining Consistency by Elimination

What conditions must b_1, b_2, and b_3 satisfy in order for the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \qquad\quad + 8x_3 &= b_3 \end{aligned}$$

to be consistent?

Solution The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{bmatrix}$$

Reducing this to reduced row echelon form yields (verify)

$$\begin{bmatrix} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{bmatrix} \tag{2}$$

In this case there are no restrictions on b_1, b_2, and b_3, so the system has the unique solution

$$x_1 = -40b_1 + 16b_2 + 9b_3, \quad x_2 = 13b_1 - 5b_2 - 3b_3, \quad x_3 = 5b_1 - 2b_2 - b_3 \tag{3}$$

for all values of b_1, b_2, and b_3.

What does the result in Example 4 tell you about the coefficient matrix of the system?

Exercise Set 1.6

In Exercises 1–8, solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

1. $\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}$

2. $\begin{aligned} 4x_1 - 3x_2 &= -3 \\ 2x_1 - 5x_2 &= 9 \end{aligned}$

3. $\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$

4. $\begin{aligned} 5x_1 + 3x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 + x_3 &= 5 \end{aligned}$

5. $\begin{aligned} x + y + z &= 5 \\ x + y - 4z &= 10 \\ -4x + y + z &= 0 \end{aligned}$

6. $\begin{aligned} - x - 2y - 3z &= 0 \\ w + x + 4y + 4z &= 7 \\ w + 3x + 7y + 9z &= 4 \\ -w - 2x - 4y - 6z &= 6 \end{aligned}$

7. $\begin{aligned} 3x_1 + 5x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$

8. $\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 5x_3 &= b_2 \\ 3x_1 + 5x_2 + 8x_3 &= b_3 \end{aligned}$

In Exercises 9–12, solve the linear systems. Using the given values for the b's solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

9. $\begin{aligned} x_1 - 5x_2 &= b_1 \\ 3x_1 + 2x_2 &= b_2 \end{aligned}$

i. $b_1 = 1, \; b_2 = 4$ **ii.** $b_1 = -2, \; b_2 = 5$

10. $\begin{aligned} -x_1 + 4x_2 + x_3 &= b_1 \\ x_1 + 9x_2 - 2x_3 &= b_2 \\ 6x_1 + 4x_2 - 8x_3 &= b_3 \end{aligned}$

i. $b_1 = 0, \; b_2 = 1, \; b_3 = 0$ **ii.** $b_1 = -3, \; b_2 = 4, \; b_3 = -5$

11. $$\begin{aligned} 4x_1 - 7x_2 &= b_1 \\ x_1 + 2x_2 &= b_2 \end{aligned}$$

i. $b_1 = 0,\quad b_2 = 1$ **ii.** $b_1 = -4,\quad b_2 = 6$
iii. $b_1 = -1,\quad b_2 = 3$ **iv.** $b_1 = -5,\quad b_2 = 1$

12. $$\begin{aligned} x_1 + 3x_2 + 5x_3 &= b_1 \\ -x_1 - 2x_2 \qquad &= b_2 \\ 2x_1 + 5x_2 + 4x_3 &= b_3 \end{aligned}$$

i. $b_1 = 1,\quad b_2 = 0,\quad b_3 = -1$
ii. $b_1 = 0,\quad b_2 = 1,\quad b_3 = 1$
iii. $b_1 = -1,\quad b_2 = -1,\quad b_3 = 0$

In Exercises ***13–17****, determine conditions on the* b_i*'s, if any, in order to guarantee that the linear system is consistent.*

13. $$\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$$

14. $$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

15. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

16. $$\begin{aligned} x_1 - 2x_2 - x_3 &= b_1 \\ -4x_1 + 5x_2 + 2x_3 &= b_2 \\ -4x_1 + 7x_2 + 4x_3 &= b_3 \end{aligned}$$

17. $$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\ -2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\ -3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\ 4x_1 - 3x_2 + x_3 + 3x_4 &= b_4 \end{aligned}$$

18. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a. Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

b. Solve $A\mathbf{x} = 4\mathbf{x}$.

In Exercises ***19–20****, solve the matrix equation for* X.

19. $$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

20. $$\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & -4 \end{bmatrix} X = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$$

Working with Proofs

21. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Prove that if k is any positive integer, then the system $A^k\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

22. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n \times n$ matrix. Prove that $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if $(QA)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

23. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations, and let $\mathbf{x}_1$ be a fixed solution. Prove that every solution to the system can be written in the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$, where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$. Prove also that every matrix of this form is a solution.

24. Use part (*a*) of Theorem 1.6.3 to prove part (*b*).

True-False Exercises

TF. In parts **(a)–(g)** determine whether the statement is true or false, and justify your answer.

a. It is impossible for a system of linear equations to have exactly two solutions.

b. If A is a square matrix, and if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x} = \mathbf{c}$ also must have a unique solution.

c. If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

d. If A and B are row equivalent matrices, then the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.

e. Let A be an $n \times n$ matrix and S is an $n \times n$ invertible matrix. If $\mathbf{x}$ is a solution to the system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the system $A\mathbf{y} = S\mathbf{b}$.

f. Let A be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

g. Let A and B be $n \times n$ matrices. If A or B (or both) are not invertible, then neither is AB.

Working with Technology

T1. Colors in print media, on computer monitors, and on television screens are implemented using what are called "color models." For example, in the RGB model, colors are created by mixing percentages of red (R), green (G), and blue (B), and in the YIQ model (used in TV broadcasting), colors are created by mixing percentages of luminescence (Y) with percentages of a chrominance factor (I) and a chrominance factor (Q). The conversion from the RGB model to the YIQ model is accomplished by the matrix equation

$$\begin{bmatrix} \text{Y} \\ \text{I} \\ \text{Q} \end{bmatrix} = \begin{bmatrix} .299 & .587 & .114 \\ .596 & -.275 & -.321 \\ .212 & -.523 & .311 \end{bmatrix} \begin{bmatrix} \text{R} \\ \text{G} \\ \text{B} \end{bmatrix}$$

What matrix would you use to convert the YIQ model to the RGB model?

T2. Let

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 5 & 1 \\ 0 & 3 & -1 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix},\ B_2 = \begin{bmatrix} 11 \\ 5 \\ 3 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

Solve the linear systems $A\mathbf{x} = B_1, A\mathbf{x} = B_2, A\mathbf{x} = B_3$ using the method of Example 2.

1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will play an important role in our subsequent work.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{1}$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of (1) is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \tag{2}$$

We leave it for you to confirm that $DD^{-1} = D^{-1}D = I_m$.

Powers of diagonal matrices are easy to compute; we also leave it for you to verify that if D is the diagonal matrix (1) and k is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \tag{3}$$

EXAMPLE 1 | Inverses and Powers of Diagonal Matrices

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

Matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

In words, *to multiply a matrix A on the left by a diagonal matrix D, multiply successive rows of A by the successive diagonal entries of D, and to multiply A on the right by D, multiply successive columns of A by the successive diagonal entries of D.*

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

EXAMPLE 2 | Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

Remark Observe that diagonal matrices are both upper triangular and lower triangular since they have zeros below and above the main diagonal. Observe also that a *square* matrix in row echelon form is upper triangular since it has zeros below the main diagonal.

Properties of Triangular Matrices

Example 2 illustrates the following four facts about triangular matrices that we will state without formal proof:

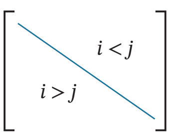


FIGURE 1.7.1

- A square matrix $A = [a_{ij}]$ is upper triangular if and only if all entries below the main diagonal are zero; that is, $a_{ij} = 0$ if $i > j$ (**Figure 1.7.1**).
- A square matrix $A = [a_{ij}]$ is lower triangular if and only if all entries above the main diagonal are zero; that is, $a_{ij} = 0$ if $i < j$ (Figure 1.7.1).
- A square matrix $A = [a_{ij}]$ is upper triangular if and only if the ith row starts with at least $i - 1$ zeros for every i.
- A square matrix $A = [a_{ij}]$ is lower triangular if and only if the jth column starts with at least $j - 1$ zeros for every j.

The following theorem lists some of the basic properties of triangular matrices.

Theorem 1.7.1

(*a*) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(*b*) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

(*c*) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(*d*) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Part (*a*) is evident from the fact that transposing a square matrix can be accomplished by reflecting the entries about the main diagonal; we omit the formal proof. We will prove (*b*), but we will defer the proofs of (*c*) and (*d*) to the next chapter, where we will have the tools to prove those results more efficiently.

***Proof* (b)** We will prove the result for lower triangular matrices; the proof for upper triangular matrices is similar. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular $n \times n$ matrices, and let $C = [c_{ij}]$ be the product $C = AB$. We can prove that C is lower triangular by showing that $c_{ij} = 0$ for $i < j$. But from the definition of matrix multiplication,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

If we assume that $i < j$, then the terms in this expression can be grouped as follows:

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}}_{\substack{\text{Terms in which the row}\\ \text{number of } b \text{ is less than}\\ \text{the column number of } b}} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}_{\substack{\text{Terms in which the row}\\ \text{number of } a \text{ is less than}\\ \text{the column number of } a}}$$

In the first grouping all of the b factors are zero since B is lower triangular, and in the second grouping all of the a factors are zero since A is lower triangular. Thus, $c_{ij} = 0$, which is what we wanted to prove. ■

EXAMPLE 3 | Computations with Triangular Matrices

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from part (*c*) of Theorem 1.7.1 that the matrix A is invertible but the matrix B is not. Moreover, the theorem also tells us that A^{-1}, AB, and BA must be upper triangular. We leave it for you to confirm these three statements by showing that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Remark Observe that in this example the diagonal entries of AB and BA are the same and are the products of the corresponding diagonal entries of A and B. Also observe that the diagonal entries of A^{-1} are the reciprocals of the diagonal entries of A. In the exercises we ask you to show that this happens whenever upper or lower triangular matrices are multiplied or inverted.

Symmetric Matrices

Definition 1

A square matrix A is said to be ***symmetric*** if $A = A^T$.

It is easy to recognize a symmetric matrix by inspection: The entries on the main diagonal have no restrictions, but mirror images of entries *across* the main diagonal must be equal. Here is a picture using the second matrix in Example 4:

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$$

EXAMPLE 4 | Symmetric Matrices

The following matrices are symmetric since each is equal to its own transpose (verify).

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

Remark It follows from Formula (14) of Section 1.3 that a square matrix A is symmetric if and only if

$$\boxed{(A)_{ij} = (A)_{ji}} \tag{4}$$

for all values of i and j.

The following theorem lists the main algebraic properties of symmetric matrices. The proofs are direct consequences of Theorem 1.4.8 and are omitted.

Theorem 1.7.2

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(*a*) A^T is symmetric.

(*b*) $A + B$ and $A - B$ are symmetric.

(*c*) kA is symmetric.

It is not true, in general, that the product of symmetric matrices is symmetric. To see why this is so, let A and B be symmetric matrices with the same size. Then it follows from part (*e*) of Theorem 1.4.8 and the symmetry of A and B that

$$(AB)^T = B^TA^T = BA$$

Thus, $(AB)^T = AB$ if and only if $AB = BA$, that is, if and only if A and B commute. In summary, we have the following result.

Theorem 1.7.3

The product of two symmetric matrices is symmetric if and only if the matrices commute.

EXAMPLE 5 | Products of Symmetric Matrices

The first of the following equations shows a product of symmetric matrices that *is not* symmetric, and the second shows a product of symmetric matrices that *is* symmetric. We conclude that the factors in the first equation do not commute, but those in the second equation do. We leave it for you to verify that this is so.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Invertibility of Symmetric Matrices

In general, a symmetric matrix need not be invertible. For example, a diagonal matrix with a zero on the main diagonal is symmetric but not invertible. However, the following theorem shows that if a symmetric matrix happens to be invertible, then its inverse must also be symmetric.

Theorem 1.7.4

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof Assume that A is symmetric and invertible. From Theorem 1.4.9 and the fact that $A = A^T$, we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

which proves that A^{-1} is symmetric. ■

Later in this text, we will obtain general conditions on A under which AA^T and A^TA are invertible. However, in the special case where A is *square*, we have the following result.

Theorem 1.7.5

If A is an invertible matrix, then AA^T and A^TA are also invertible.

Proof Since A is invertible, so is A^T by Theorem 1.4.9. Thus AA^T and A^TA are invertible, since they are the products of invertible matrices. ■

Products AA^T and A^TA are Symmetric

Matrix products of the form AA^T and A^TA arise in a variety of applications. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices—the matrix AA^T has size $m \times m$, and the matrix A^TA has size $n \times n$. Such products are always symmetric since

$$(AA^T)^T = (A^T)^T A^T = AA^T \quad \text{and} \quad (A^TA)^T = A^T(A^T)^T = A^TA$$

EXAMPLE 6 | The Product of a Matrix and Its Transpose Is Symmetric

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that A^TA and AA^T are symmetric as expected.

Exercise Set 1.7

In Exercises 1–2, classify the matrix as upper triangular, lower triangular, or diagonal, and decide by inspection whether the matrix is invertible. Recall that a diagonal matrix is both upper and lower triangular, so there may be more than one answer in some parts.

1. a. $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$

c. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & -2 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix}$

2. a. $\begin{bmatrix} 4 & 0 \\ 1 & 7 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & -2 \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 7 & 0 & 0 \end{bmatrix}$

In Exercises 3–6, find the product by inspection.

3. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & -5 \\ -3 & -1 & 0 \end{bmatrix}\begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} -3 & 2 & 0 & 4 & -4 \\ 1 & -5 & 3 & 0 & 3 \\ -6 & 2 & 2 & 2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & -2 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

In Exercises 7–10, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection.

7. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ **8.** $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

9. $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$ **10.** $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 11–12, compute the product by inspection.

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

In Exercises 13–14, compute the indicated quantity.

13. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39}$ **14.** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{1000}$

In Exercises 15–16, use what you have learned in this section about multiplying by diagonal matrices to compute the product by inspection.

15. a. $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$ **b.** $\begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

16. a. $\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ **b.** $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}$

In Exercises 17–18, create a symmetric matrix by substituting appropriate numbers for the $\times$'s.

17. a. $\begin{bmatrix} 2 & -1 \\ \times & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & \times & \times & \times \\ 3 & 1 & \times & \times \\ 7 & -8 & 0 & \times \\ 2 & -3 & 9 & 0 \end{bmatrix}$

18. a. $\begin{bmatrix} 0 & \times \\ 3 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 7 & -3 & 2 \\ \times & 4 & 5 & -7 \\ \times & \times & 1 & -6 \\ \times & \times & \times & 3 \end{bmatrix}$

In Exercises 19–22, determine by inspection whether the matrix is invertible.

19. $\begin{bmatrix} 0 & 6 & -1 \\ 0 & 7 & -4 \\ 0 & 0 & -2 \end{bmatrix}$ **20.** $\begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -5 & 0 & 0 \\ 4 & -3 & 4 & 0 \\ 1 & -2 & 1 & 3 \end{bmatrix}$ **22.** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ -4 & -6 & 0 & 0 \\ 0 & 3 & 8 & -5 \end{bmatrix}$

In Exercises 23–24, find the diagonal entries of AB by inspection.

23. $A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$

24. $A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 0 & 0 \\ -3 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 2 & 6 \end{bmatrix}$

In Exercises 25–26, find all values of the unknown constant(s) for which A is symmetric.

25. $A = \begin{bmatrix} 4 & -3 \\ a+5 & -1 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$

In Exercises 27–28, find all values of x for which A is invertible.

27. $A = \begin{bmatrix} x-1 & x^2 & x^4 \\ 0 & x+2 & x^3 \\ 0 & 0 & x-4 \end{bmatrix}$

28. $A = \begin{bmatrix} x-\frac{1}{2} & 0 & 0 \\ x & x-\frac{1}{3} & 0 \\ x^2 & x^3 & x+\frac{1}{4} \end{bmatrix}$

29. If A is an invertible upper triangular or lower triangular matrix, what can you say about the diagonal entries of A^{-1}?

30. Show that if A is a symmetric $n \times n$ matrix and B is any $n \times m$ matrix, then the following products are symmetric:

$$B^TB, \quad BB^T, \quad B^TAB$$

In Exercises 31–32, find a diagonal matrix A that satisfies the given condition.

31. $A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **32.** $A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

33. Verify Theorem 1.7.1(b) for the matrix product AB and Theorem 1.7.1(d) for the matrix A, where

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

34. Let A be an $n \times n$ symmetric matrix.

a. Show that A^2 is symmetric.

b. Show that $2A^2 - 3A + I$ is symmetric.

35. Verify Theorem 1.7.4 for the given matrix A.

a. $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix}$

36. Find all 3×3 diagonal matrices A that satisfy $A^2 - 3A - 4I = 0$.

37. Let $A = [a_{ij}]$ be an $n \times n$ matrix. Determine whether A is symmetric.

a. $a_{ij} = i^2 + j^2$ **b.** $a_{ij} = i^2 - j^2$

c. $a_{ij} = 2i + 2j$ **d.** $a_{ij} = 2i^2 + 2j^3$

38. On the basis of your experience with Exercise 37, devise a general test that can be applied to a formula for a_{ij} to determine whether $A = [a_{ij}]$ is symmetric.

39. Find an upper triangular matrix that satisfies

$$A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$$

40. If the $n \times n$ matrix A can be expressed as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ can be expressed as $LU\mathbf{x} = \mathbf{b}$ and can be solved in two steps:

Step 1. Let $U\mathbf{x} = \mathbf{y}$, so that $LU\mathbf{x} = \mathbf{b}$ can be expressed as $L\mathbf{y} = \mathbf{b}$. Solve this system.

Step 2. Solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

In each part, use this two-step method to solve the given system.

a. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$

In the text we defined a matrix A to be symmetric if $A^T = A$. Analogously, a matrix A is said to be **skew-symmetric** *if $A^T = -A$. Exercises 41–45 are concerned with matrices of this type.*

41. Fill in the missing entries (marked with $\times$) so the matrix A is skew-symmetric.

a. $A = \begin{bmatrix} \times & \times & 4 \\ 0 & \times & \times \\ \times & -1 & \times \end{bmatrix}$ **b.** $A = \begin{bmatrix} \times & 0 & \times \\ \times & \times & -4 \\ 8 & \times & \times \end{bmatrix}$

42. Find all values of a, b, c, and d for which A is skew-symmetric.

$$A = \begin{bmatrix} 0 & 2a-3b+c & 3a-5b+5c \\ -2 & 0 & 5a-8b+6c \\ -3 & -5 & d \end{bmatrix}$$

43. We showed in the text that the product of symmetric matrices is symmetric if and only if the matrices commute. Is the product of commuting skew-symmetric matrices skew-symmetric? Explain.

Working with Proofs

44. Prove that every square matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix. [*Hint:* Note the identity $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.]

45. Prove the following facts about skew-symmetric matrices.

a. If A is an invertible skew-symmetric matrix, then A^{-1} is skew-symmetric.

b. If A and B are skew-symmetric matrices, then so are A^T, $A + B$, $A - B$, and kA for any scalar k.

46. Prove: If the matrices A and B are both upper triangular or both lower triangular, then the diagonal entries of both AB and BA are the products of the diagonal entries of A and B.

47. Prove: If $A^TA = A$, then A is symmetric and $A = A^2$.

True-False Exercises

TF. In parts (a)–(m) determine whether the statement is true or false, and justify your answer.

a. The transpose of a diagonal matrix is a diagonal matrix.

b. The transpose of an upper triangular matrix is an upper triangular matrix.

c. The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

d. All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

e. All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

f. The inverse of an invertible lower triangular matrix is an upper triangular matrix.

g. A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

h. The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

i. A matrix that is both symmetric and upper triangular must be a diagonal matrix.

j. If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.

k. If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.

l. If A^2 is a symmetric matrix, then A is a symmetric matrix.

m. If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

Working with Technology

T1. Starting with the formula stated in Exercise T1 of Section 1.5, derive a formula for the inverse of the "block diagonal" matrix

$$\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

in which D_1 and D_2 are invertible, and use your result to compute the inverse of the matrix

$$M = \begin{bmatrix} 1.24 & 2.37 & 0 & 0 \\ 3.08 & -1.01 & 0 & 0 \\ 0 & 0 & 2.76 & 4.92 \\ 0 & 0 & 3.23 & 5.54 \end{bmatrix}$$

1.8 Introduction to Linear Transformations

Up to now we have treated matrices simply as rectangular arrays of numbers and have been concerned primarily with developing algebraic properties of those arrays. In this section we will view matrices in a completely different way. Here we will be concerned with how matrices can be used to transform or "map" one vector into another by matrix multiplication. This will be the foundation for much of our work in subsequent sections.

Recall that in Section 1.1 we defined an "ordered n-tuple" to be a sequence of n real numbers, and we observed that a solution of a linear system in n unknowns, say

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

can be expressed as the ordered n-tuple

$$(s_1, s_2, \dots, s_n) \tag{1}$$

Recall also that if $n = 2$, then the n-tuple is called an "ordered pair," and if $n = 3$, it is called an "ordered triple." For two ordered n-tuples to be regarded as the same, they must list the same numbers in the same order. Thus, for example, $(1, 2)$ and $(2, 1)$ are different ordered pairs.

The set of all ordered n-tuples of real numbers is denoted by the symbol R^n. The elements of R^n are called ***vectors*** and are denoted in boldface type, such as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$. When convenient, ordered n-tuples can be denoted in matrix notation as column vectors. For example, the matrix

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \tag{2}$$

can be used as an alternative to (1). We call (1) the ***comma-delimited form*** of a vector and (2) the ***column-vector form***. For each $i = 1, 2, \dots, n$, let $\mathbf{e}_i$ denote the vector in R^n with a 1 in the ith position and zeros elsewhere. In column form these vectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We call the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ the ***standard basis vectors*** for R^n. For example, the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are the standard basis vectors for R^3.

The vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in R^n are termed "basis vectors" because all other vectors in R^n are expressible in exactly one way as a linear combination of them. For example, if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then we can express $\mathbf{x}$ as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

The term "vector" is used in various ways in mathematics, physics, engineering, and other applications. The idea of viewing n-tuples as vectors will be discussed in more detail in Chapter 3, at which point we will also explain how this idea relates to a more familiar notion of a vector.

Functions and Transformations

Recall that a ***function*** is a rule that associates with each element of a set A one and only one element in a set B. If f associates the element b with the element a, then we write

$$b = f(a)$$

and we say that b is the ***image*** of a under f or that $f(a)$ is the ***value*** of f at a. The set A is called the ***domain*** of f and the set B the ***codomain*** of f (**Figure 1.8.1**). The subset of the codomain that consists of all images of elements in the domain is called the ***range*** of f.

In many applications the domain and codomain of a function are sets of real numbers, but in this text we will be concerned with functions for which the domain is R^n and the codomain is R^m for some positive integers m and n. In this setting it is common to use italicized capital letters for functions, the letter T being typical.

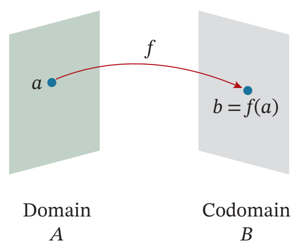


FIGURE 1.8.1

Definition 1

If T is a function with domain R^n and codomain R^m, then we say that T is a ***transformation*** from R^n to R^m or that T ***maps*** from R^n to R^m, which we denote by writing

$$T : R^n \to R^m$$

In the special case where $m = n$, a transformation is sometimes called an ***operator*** on R^n.

Matrix Transformations

In this section we will be concerned with the class of transformations from R^n to R^m that arise from linear systems. Specifically, suppose that we have the system of linear equations

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\;\;\vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{3}$$

which we can write in matrix notation as

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{4}$$

or more briefly as

$$\mathbf{w} = A\mathbf{x} \tag{5}$$

Up to now we have been viewing (5) as a compact way of writing system (3). Another way to view this formula is as a transformation that maps a vector $\mathbf{x}$ in R^n into a vector $\mathbf{w}$ in R^m by multiplying $\mathbf{x}$ on the left by A. We call this a ***matrix transformation*** (or ***matrix operator*** in the special case where $m = n$). We denote it by

$$T_A : R^n \to R^m$$

(see **Figure 1.8.2**). This notation is useful when it is important to make the domain and codomain clear. The subscript on T_A serves as a reminder that the transformation results from multiplying vectors in R^n by the matrix A. In situations where specifying the domain and codomain is not essential, we will express (5) as

$$\mathbf{w} = T_A(\mathbf{x}) \tag{6}$$

We call the transformation T_A ***multiplication by A***. On occasion we will find it convenient to express (6) in the schematic form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w} \tag{7}$$

which is read "T_A maps $\mathbf{x}$ into $\mathbf{w}$."

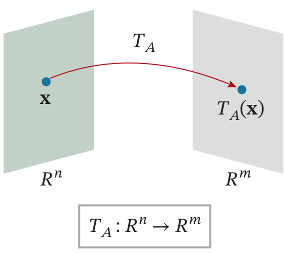


FIGURE 1.8.2

EXAMPLE 1 | A Matrix Transformation from R^4 to R^3

The transformation from R^4 to R^3 defined by the equations

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned} \tag{8}$$

can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from which we see that the transformation can be interpreted as multiplication by

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \tag{9}$$

Although the image under the transformation T_A of any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

in R^4 could be computed directly from the defining equations in (8), we will find it preferable to use the matrix in (9). For example, if

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

then it follows from (9) that

$$T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

EXAMPLE 2 | Zero Transformations

If 0 is the $m \times n$ zero matrix, then

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

so multiplication by zero maps every vector in R^n into the zero vector in R^m. We call T_0 the ***zero transformation*** from R^n to R^m.

EXAMPLE 3 | Identity Operators

If I is the $n \times n$ identity matrix, then

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in R^n to itself. We call T_I the ***identity operator*** on R^n.

Properties of Matrix Transformations

The following theorem lists four basic properties of matrix transformations that follow from properties of matrix multiplication.

Theorem 1.8.1

For every matrix A the matrix transformation $T_A : R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

(*a*) $T_A(\mathbf{0}) = \mathbf{0}$

(*b*) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(*c*) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

(*d*) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

Proof All four parts are restatements from the transformation viewpoint of the following properties of matrix arithmetic given in Theorem 1.4.1:

$$A\mathbf{0} = \mathbf{0}, \quad A(k\mathbf{u}) = k(A\mathbf{u}), \quad A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} \; \blacksquare$$

It follows from parts (*b*) and (*c*) of Theorem 1.8.1 that a matrix transformation maps a linear combination of vectors in R^n into the corresponding linear combination of vectors in R^m in the sense that

$$\boxed{T_A(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T_A(\mathbf{u}_1) + k_2T_A(\mathbf{u}_2) + \cdots + k_rT_A(\mathbf{u}_r)} \tag{10}$$

Matrix transformations are not the only kinds of transformations. For example, if

$$\begin{aligned} w_1 &= x_1^2 + x_2^2 \\ w_2 &= x_1x_2 \end{aligned} \tag{11}$$

then there are no constants a, b, c, and d for which

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 \\ x_1x_2 \end{bmatrix}$$

so that the equations in (11) do not define a matrix transformation from R^2 to R^2.

This leads us to the following two questions.

Question 1. Are there algebraic properties of a transformation $T: R^n \to R^m$ that can be used to determine whether T is a matrix transformation?

Question 2. If we discover that a transformation $T: R^n \to R^m$ is a matrix transformation, how can we find a matrix A for which $T = T_A$?

The following theorem and its proof will provide the answers.

Theorem 1.8.2

$T : R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

Proof If T is a matrix transformation, then properties (i) and (ii) follow respectively from parts (c) and (b) of Theorem 1.8.1.

Conversely, assume that properties (i) and (ii) hold. We must show that there exists an $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n. Recall that the derivation of Formula (10) used only the additivity and homogeneity properties of T_A. Since we are assuming that T has those properties, it must be true that

$$T(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T(\mathbf{u}_1) + k_2T(\mathbf{u}_2) + \cdots + k_rT(\mathbf{u}_r) \tag{12}$$

for all scalars $k_1, k_2, \ldots, k_r$ and all vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ in R^n. Let A be the matrix

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)] \tag{13}$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n. It follows from Theorem 1.3.1 that $A\mathbf{x}$ is a linear combination of the columns of A in which the successive coefficients are the entries $x_1, x_2, \ldots, x_n$ of $\mathbf{x}$. That is,

$$A\mathbf{x} = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Using Formula (10) we can rewrite this as

$$A\mathbf{x} = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) = T(\mathbf{x})$$

which completes the proof. ■

The two properties listed in Theorem 1.8.2 are called ***linearity conditions***, and a transformation that satisfies these conditions is called a ***linear transformation***. Using this terminology Theorem 1.8.2 can be restated as follows.

Theorem 1.8.3

Every linear transformation from R^n to R^m is a matrix transformation and conversely every matrix transformation from R^n to R^m is a linear transformation.

Briefly stated, this theorem tells us that for transformations from R^n to R^m the terms "linear transformation" and "matrix transformation" are synonymous.

Depending on whether n-tuples and m-tuples are regarded as vectors or points, the geometric effect of a matrix transformation $T_A\colon R^n \to R^m$ is to map each vector (point) in R^n into a vector (point) in R^m (**Figure 1.8.3**).

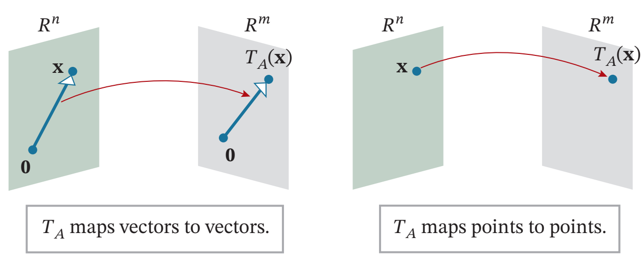


FIGURE 1.8.3

The following theorem states that if two matrix transformations from R^n to R^m have the same image for each point of R^n, then the matrices themselves must be the same.

Theorem 1.8.4

If $T_A\colon R^n \to R^m$ and $T_B\colon R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

Proof To say that $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector in R^n is the same as saying that

$$A\mathbf{x} = B\mathbf{x}$$

for every vector $\mathbf{x}$ in R^n. This will be true, in particular, if $\mathbf{x}$ is any of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n; that is,

$$A\mathbf{e}_j = B\mathbf{e}_j \quad (j = 1, 2, \ldots, n) \tag{14}$$

Since every entry of $\mathbf{e}_j$ is 0 except for the jth, which is 1, it follows from Theorem 1.3.1 that $A\mathbf{e}_j$ is the jth column of A and $B\mathbf{e}_j$ is the jth column of B. Thus, (14) implies that corresponding columns of A and B are the same, and hence that $A = B$. ■

Theorem 1.8.4 is significant because it tells us that there is a *one-to-one correspondence* between $m \times n$ matrices and matrix transformations from R^n to R^m in the sense that every $m \times n$ matrix A produces exactly one matrix transformation (multiplication by A) and every matrix transformation from R^n to R^m arises from exactly one $m \times n$ matrix; we call that matrix the ***standard matrix*** for the transformation.

A Procedure for Finding Standard Matrices

In the course of proving Theorem 1.8.2 we showed in Formula (13) that if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n (in column form), then the standard matrix for a linear transformation $T: R^n \to R^m$ is given by the formula

$$\boxed{A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)]} \tag{15}$$

This formula reveals a key property of linear transformations from R^n to R^m, namely, that they are completely determined by their actions on the standard basis vectors for R^n. It also suggests the following procedure that can be used to find the standard matrix for such transformations.

Finding the Standard Matrix for a Matrix Transformation

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

EXAMPLE 4 | Finding a Standard Matrix

Find the standard matrix A for the linear transformation $T: R^2 \to R^3$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix} \tag{16}$$

Solution We leave it for you to verify that

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, it follows from Formulas (15) and (16) that the standard matrix is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

As a check, observe that

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

which shows that multiplication by A produces the same result as the transformation T (see Equation (16)).

EXAMPLE 5 | Computing with Standard Matrices

For the linear transformation in Example 4, use the standard matrix A obtained in that example to find

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Solution The transformation is multiplication by A, so

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}$$

Although we could have obtained the result in Example 5 by substituting values for the variables in (13), the method used in that example is preferable for large-scale problems in that matrix multiplication is better suited for computer computations.

For transformation problems posed in comma-delimited form, a good procedure is to rewrite the problem in column-vector form and use the methods previously illustrated.

EXAMPLE 6 | Finding a Standard Matrix

Rewrite the transformation $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$ in column-vector form and find its standard matrix.

Solution

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is

$$\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$$

EXAMPLE 7

Find the standard matrix A for the linear transformation $T : R^2 \to R^2$ for which

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -6 \end{bmatrix} \tag{17}$$

Solution Our objective is to find the images of the standard basis vectors and then use Formula (15) to obtain the standard matrix. To start, we will rewrite the standard basis vectors as linear combinations of

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

This leads to the vector equations

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 2 \\ -1 \end{bmatrix} \tag{18}$$

which we can rewrite as

$$\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As these systems have the same coefficient matrix, we can solve both at once using the method in Example 2 of Section 1.6. We leave it for you to do this and to show that

$$c_1 = 1, \; c_2 = 1, \; k_1 = 2, \; k_2 = 1$$

Substituting these values in (18) and using the linearity properties of T, we obtain

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -10 \\ 10 \end{bmatrix} + \begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Thus, it follows from Formula (15) that the standard matrix for T is

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

You can check this result using multiplication by A to verify (17).

Remark This section is but a first step in the study of linear transformations, which is one of the major themes in this text. We will delve deeper into this topic in Chapter 4, at which point we will have more background and a richer source of examples to work with.

There are many ways to transform the vector spaces R^2 and R^3, some of the most important of which can be accomplished by matrix transformations. For example, rotations about the origin, reflections about lines and planes through the origin, and projections onto lines and planes through the origin can all be accomplished using a matrix operator with an appropriate 2×2 or 3×3 matrix.

Reflection Operators

Some of the most basic matrix operators on R^2 and R^3 are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called ***reflection operators***. **Table 1** shows the standard matrices for the reflections about the coordinate axes and the line $y = x$ in R^2, and **Table 2** shows the standard matrices for the reflections about the coordinate planes in R^3. In each case the standard matrix was obtained by finding the images of the standard basis vectors, converting those images to column vectors, and then using those column vectors as successive columns of the standard matrix.

TABLE 1

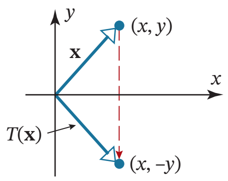
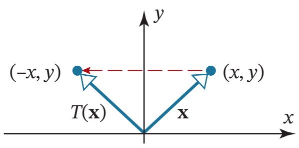
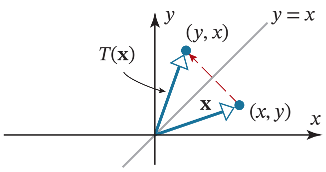
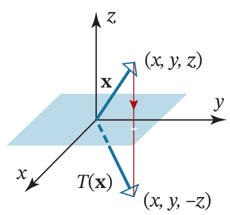
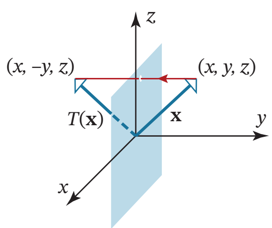
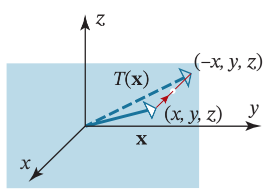
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x,y)=(x,-y)$		$T(\mathbf{e}_1)=T(1,0)=(1,0)$ $T(\mathbf{e}_2)=T(0,1)=(0,-1)$	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Reflection about the y-axis $T(x,y)=(-x,y)$		$T(\mathbf{e}_1)=T(1,0)=(-1,0)$ $T(\mathbf{e}_2)=T(0,1)=(0,1)$	$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$
Reflection about the line $y=x$ $T(x,y)=(y,x)$		$T(\mathbf{e}_1)=T(1,0)=(0,1)$ $T(\mathbf{e}_2)=T(0,1)=(1,0)$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

TABLE 2

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Reflection about the xy-plane $T(x,y,z)=(x,y,-z)$		$T(\mathbf{e}_1)=T(1,0,0)=(1,0,0)$ $T(\mathbf{e}_2)=T(0,1,0)=(0,1,0)$ $T(\mathbf{e}_3)=T(0,0,1)=(0,0,-1)$	$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$
Reflection about the xz-plane $T(x,y,z)=(x,-y,z)$		$T(\mathbf{e}_1)=T(1,0,0)=(1,0,0)$ $T(\mathbf{e}_2)=T(0,1,0)=(0,-1,0)$ $T(\mathbf{e}_3)=T(0,0,1)=(0,0,1)$	$\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$
Reflection about the yz-plane $T(x,y,z)=(-x,y,z)$		$T(\mathbf{e}_1)=T(1,0,0)=(-1,0,0)$ $T(\mathbf{e}_2)=T(0,1,0)=(0,1,0)$ $T(\mathbf{e}_3)=T(0,0,1)=(0,0,1)$	$\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Projection Operators

Matrix operators on R^2 and R^3 that map each point into its orthogonal projection onto a fixed line or plane through the origin are called ***projection operators*** (or more precisely, ***orthogonal projection*** operators). **Table 3** shows the standard matrices for the orthogonal projections onto the coordinate axes in R^2, and **Table 4** shows the standard matrices for the orthogonal projections onto the coordinate planes in R^3.

TABLE 3

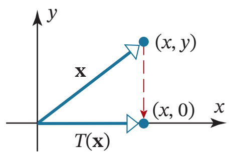
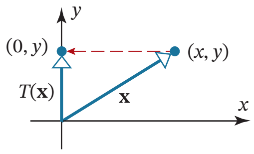
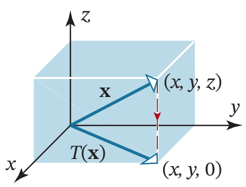
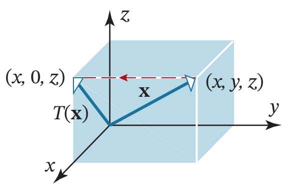
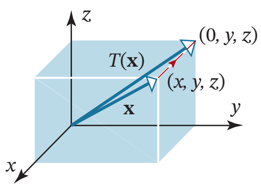
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

TABLE 4

Operator	Illustration	Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	Standard Matrix
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix multiplication is really not needed to accomplish the reflections and projections in these tables, as the results are evident geometrically. For example, although the computation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$$

shows that the orthogonal projection of (x, y, z) onto the xz-plane is $(x, 0, z)$, that result is evident from the illustration in Table 4. However, in the next section and subsequently we will study more complicated matrix transformations in which the end results are not evident and matrix multiplication is essential.

Rotation Operators

Matrix operators on R^2 that move points along arcs of circles centered at the origin are called ***rotation operators***. Let us consider how to find the standard matrix for the rotation operator $T: R^2 \to R^2$ that moves points *counterclockwise* about the origin through a positive angle θ. **Figure 1.8.4** shows a typical vector $\mathbf{x}$ in R^2 and its image $T(\mathbf{x})$ under such a rotation. As illustrated in **Figure 1.8.5**, the images of the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ under a rotation through an angle θ are

$$T(\mathbf{e}_1) = T(1, 0) = (\cos\theta, \sin\theta) \quad \text{and} \quad T(\mathbf{e}_2) = T(0, 1) = (-\sin\theta, \cos\theta)$$

so it follows from Formula (15) that the standard matrix for T is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

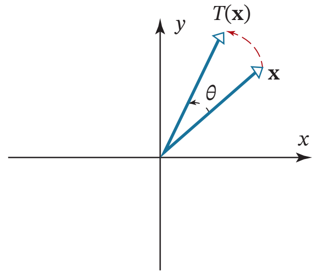


FIGURE 1.8.4

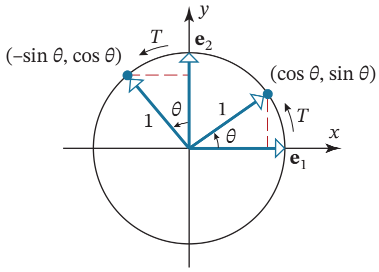


FIGURE 1.8.5

In keeping with common usage we will denote this matrix as

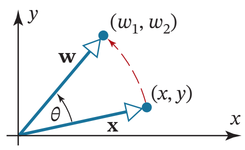
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{19}$$

and call it the ***rotation matrix*** for R^2. These ideas are summarized in **Table 5**.

In the plane, counterclockwise angles are positive and clockwise angles are negative. The rotation matrix for a *clockwise* rotation of $-\theta$ radians can be obtained by replacing θ by $-\theta$ in (19). After simplification this yields

$$R_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

TABLE 5

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Counterclockwise rotation about the origin through an angle θ		$T(\mathbf{e}_1) = T(1, 0) = (\cos\theta, \sin\theta)$ $T(\mathbf{e}_2) = T(0, 1) = (-\sin\theta, \cos\theta)$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

EXAMPLE 8 | A Rotation Matrix

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ radians $(= 30^\circ)$ about the origin.

Solution It follows from (19) with $\theta = \pi/6$ that

$$R_{\pi/6}\mathbf{x} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix} \approx \begin{bmatrix} 0.37 \\ 1.37 \end{bmatrix}$$

or in comma-delimited notation, $R_{\pi/6}(1, 1) \approx (0.37, 1.37)$.

Concluding Remark

Rotations in R^3 are substantially more complicated than those in R^2 and will be considered later in this text.

Exercise Set 1.8

In Exercises 1–2, find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$.

1. a. A has size 3×2. b. A has size 2×3.
   c. A has size 3×3. d. A has size 1×6.

2. a. A has size 4×5. b. A has size 5×4.
   c. A has size 4×4. d. A has size 3×1.

In Exercises 3–4, find the domain and codomain of the transformation defined by the equations.

3. a. $\begin{aligned} w_1 &= 4x_1 + 5x_2 \\ w_2 &= x_1 - 8x_2 \end{aligned}$ b. $\begin{aligned} w_1 &= 5x_1 - 7x_2 \\ w_2 &= 6x_1 + x_2 \\ w_3 &= 2x_1 + 3x_2 \end{aligned}$

4. a. $\begin{aligned} w_1 &= x_1 - 4x_2 + 8x_3 \\ w_2 &= -x_1 + 4x_2 + 2x_3 \\ w_3 &= -3x_1 + 2x_2 - 5x_3 \end{aligned}$ b. $\begin{aligned} w_1 &= 2x_1 + 7x_2 - 4x_3 \\ w_2 &= 4x_1 - 3x_2 + 2x_3 \end{aligned}$

In Exercises 5–6, find the domain and codomain of the transformation defined by the matrix product.

5. a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

6. a. $\begin{bmatrix} 6 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 1 & -6 \\ 3 & 7 & -4 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

In Exercises 7–8, find the domain and codomain of the transformation T defined by the formula.

7. a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
   b. $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$

8. a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$
   b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

In Exercises 9–10, find the domain and codomain of the transformation T defined by the formula.

9. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 \\ x_1 - x_2 \\ 3x_2 \end{bmatrix}$ 10. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 - x_3 \\ 0 \end{bmatrix}$

In Exercises 11–12, find the standard matrix for the transformation defined by the equations.

11. a. $\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$ b. $\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= - x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$

12. a. $\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$ b. $\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$

13. Find the standard matrix for the transformation T defined by the formula.
   a. $T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2)$
   b. $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$
   c. $T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$
   d. $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$

14. Find the standard matrix for the operator T defined by the formula.
   a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
   b. $T(x_1, x_2) = (x_1, x_2)$
   c. $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$
   d. $T(x_1, x_2, x_3) = (4x_1, 7x_2, -8x_3)$

15. Find the standard matrix for the operator $T: R^3 \to R^3$ defined by
$$\begin{aligned} w_1 &= 3x_1 + 5x_2 - x_3 \\ w_2 &= 4x_1 - x_2 + x_3 \\ w_3 &= 3x_1 + 2x_2 - x_3 \end{aligned}$$
and then compute $T(-1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

16. Find the standard matrix for the transformation $T: R^4 \to R^2$ defined by
$$\begin{aligned} w_1 &= 2x_1 + 3x_2 - 5x_3 - x_4 \\ w_2 &= x_1 - 5x_2 + 2x_3 - 3x_4 \end{aligned}$$
and then compute $T(1, -1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

In Exercises 17–18, find the standard matrix for the transformation and use it to compute $T(\mathbf{x})$. Check your result by substituting directly in the formula for T.

17. a. $T(x_1, x_2) = (-x_1 + x_2, x_2)$; $\mathbf{x} = (-1, 4)$
   b. $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + x_3, 0)$; $\mathbf{x} = (2, 1, -3)$

18. a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$; $\mathbf{x} = (-2, 2)$
   b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$; $\mathbf{x} = (1, 0, 5)$

In Exercises 19–20, find $T_A(\mathbf{x})$, and express your answer in matrix form.

19. a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$
   b. $A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

20. a. $A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
   b. $A = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

In Exercises 21–22, use Theorem 1.8.2 to show that T is a matrix transformation.

21. a. $T(x,y) = (2x+y, x-y)$

b. $T(x_1,x_2,x_3) = (x_1, x_3, x_1+x_2)$

22. a. $T(x,y,z) = (x+y, y+z, x)$

b. $T(x_1,x_2) = (x_2,x_1)$

In Exercises 23–24, use Theorem 1.8.2 to show that T is not a matrix transformation.

23. a. $T(x,y) = (x^2, y)$

b. $T(x,y,z) = (x,y,xz)$

24. a. $T(x,y) = (x, y+1)$

b. $T(x_1,x_2,x_3) = (x_1, x_2, \sqrt{x_3})$

25. A function of the form $f(x) = mx+b$ is commonly called a "linear function" because the graph of $y = mx+b$ is a line. Is f a matrix transformation on R?

26. Show that $T(x,y) = (0,0)$ defines a matrix operator on R^2 but $T(x,y) = (1,1)$ does not.

In Exercises 27–28, the images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

27. $T(\mathbf{e}_1) = \begin{bmatrix}1\\3\\0\end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix}0\\0\\1\end{bmatrix},\ T(\mathbf{e}_3) = \begin{bmatrix}4\\-3\\-1\end{bmatrix};\quad \mathbf{x} = \begin{bmatrix}2\\1\\0\end{bmatrix}$

28. $T(\mathbf{e}_1) = \begin{bmatrix}2\\1\\3\end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix}-3\\-1\\0\end{bmatrix},\ T(\mathbf{e}_3) = \begin{bmatrix}1\\0\\2\end{bmatrix};\quad \mathbf{x} = \begin{bmatrix}3\\2\\1\end{bmatrix}$

29. Use matrix multiplication to find the reflection of $(-1,2)$ about the

a. x-axis. **b.** y-axis. **c.** line $y = x$.

30. Use matrix multiplication to find the reflection of (a,b) about the

a. x-axis. **b.** y-axis. **c.** line $y = x$.

31. Use matrix multiplication to find the reflection of $(2,-5,3)$ about the

a. xy-plane. **b.** xz-plane. **c.** yz-plane.

32. Use matrix multiplication to find the reflection of (a,b,c) about the

a. xy-plane. **b.** xz-plane. **c.** yz-plane.

33. Use matrix multiplication to find the orthogonal projection of $(2,-5)$ onto the

a. x-axis. **b.** y-axis.

34. Use matrix multiplication to find the orthogonal projection of (a,b) onto the

a. x-axis. **b.** y-axis.

35. Use matrix multiplication to find the orthogonal projection of $(-2,1,3)$ onto the

a. xy-plane. **b.** xz-plane. **c.** yz-plane.

36. Use matrix multiplication to find the orthogonal projection of (a,b,c) onto the

a. xy-plane. **b.** xz-plane. **c.** yz-plane.

37. Use matrix multiplication to find the image of the vector $(3,-4)$ when it is rotated about the origin through an angle of

a. $\theta = 30°$. **b.** $\theta = -60°$.

c. $\theta = 45°$. **d.** $\theta = 90°$.

38. Use matrix multiplication to find the image of the nonzero vector $\mathbf{v} = (v_1, v_2)$ when it is rotated about the origin through

a. a positive angle α. **b.** a negative angle $-\alpha$.

39. Let $T: R^2 \to R^2$ be a linear operator for which the images of the standard basis vectors for R^2 are $T(\mathbf{e}_1) = (a,b)$ and $T(\mathbf{e}_2) = (c,d)$. Find $T(1,1)$.

40. Let $T_A: R^2 \to R^2$ be multiplication by

$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis vectors for R^2. Find the following vectors by inspection.

a. $T_A(k\mathbf{e}_1)$ **b.** $T_A(k\mathbf{e}_1 + l\mathbf{e}_2)$

41. Let $T_A: R^3 \to R^3$ be multiplication by

$$A = \begin{bmatrix}-1 & 3 & 0\\2 & 1 & 2\\4 & 5 & -3\end{bmatrix}$$

and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for R^3. Find the following vectors by inspection.

a. $T_A(\mathbf{e}_1)$, $T_A(\mathbf{e}_2)$, and $T_A(\mathbf{e}_3)$

b. $T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ **c.** $T_A(7\mathbf{e}_3)$

42. For each orthogonal projection operator in Table 4 use the standard matrix to compute $T(1,2,3)$, and convince yourself that your result makes sense geometrically.

43. For each reflection operator in Table 2 use the standard matrix to compute $T(1,2,3)$, and convince yourself that your result makes sense geometrically.

44. If multiplication by A rotates a vector $\mathbf{x}$ in the xy-plane through an angle θ, what is the effect of multiplying $\mathbf{x}$ by A^T? Explain your reasoning.

45. Find the standard matrix A for the linear transformation $T: R^2 \to R^2$ for which

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\end{bmatrix},\ T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}-2\\5\end{bmatrix}$$

46. Find the standard matrix A for the linear transformation $T: R^3 \to R^3$ for which

$$T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right) = \begin{bmatrix}2\\-3\\10\end{bmatrix},\ T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\3\\8\end{bmatrix},\ T\left(\begin{bmatrix}-3\\-1\\2\end{bmatrix}\right) = \begin{bmatrix}-5\\-11\\7\end{bmatrix}$$

47. Let $\mathbf{x}_0$ be a nonzero column vector in R^2, and suppose that $T: R^2 \to R^2$ is the transformation defined by the formula $T(\mathbf{x}) = \mathbf{x}_0 + R_\theta\mathbf{x}$, where R_θ is the standard matrix of the rotation of R^2 about the origin through the angle θ. Give a geometric description of this transformation. Is it a matrix transformation? Explain.

48. In each part of the accompanying figure, find the standard matrix for the pictured operator.

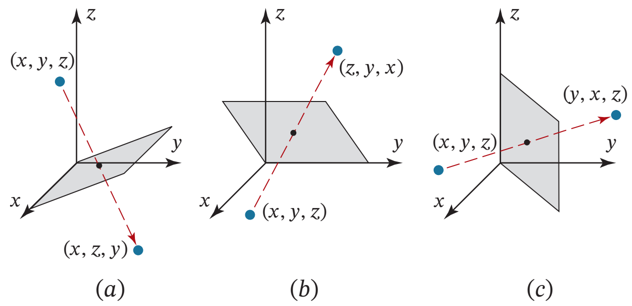


FIGURE Ex-48

49. In a sentence, describe the geometric effect of multiplying a vector $\mathbf{x}$ by the matrix

$$A = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Working with Proofs

50. a. Prove: If $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$; that is, T maps the zero vector in R^n into the zero vector in R^m.

b. The converse of this is not true. Find an example of a mapping $T: R^n \to R^m$ for which $T(\mathbf{0}) = \mathbf{0}$ but which is not a matrix transformation.

True-False Exercises

TF. In parts **(a)–(g)** determine whether the statement is true or false, and justify your answer.

a. If A is a 2×3 matrix, then the domain of the transformation T_A is R^2.

b. If A is an $m \times n$ matrix, then the codomain of the transformation T_A is R^n.

c. There is at least one linear transformation $T: R^n \to R^m$ for which $T(2\mathbf{x}) = 4T(\mathbf{x})$ for some vector $\mathbf{x}$ in R^n.

d. There are linear transformations from R^n to R^m that are not matrix transformations.

e. If $T_A: R^n \to R^n$ and if $T_A(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^n, then A is the $n \times n$ zero matrix.

f. There is only one matrix transformation $T: R^n \to R^m$ such that $T(-\mathbf{x}) = -T(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n.

g. If $\mathbf{b}$ is a nonzero vector in R^n, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a matrix operator on R^n.

1.9 Compositions of Matrix Transformations

In this section we will discuss the analogs of matrix multiplication and matrix inversion for matrix transformations, and we illustrate those ideas with familiar geometric operations such as rotations, reflections, and projections in the plane. One of the by-products of our work on compositions will be an explanation of why matrix multiplication was defined in such an unusual way.

Compositions of Matrix Transformations

Simply stated, the "composition" of matrix transformations is the process of first applying a matrix transformation to a vector and then applying another matrix transformation to the image vector. For example, suppose that T_A is a matrix transformation from R^n to R^k and T_B is a matrix transformation from R^k to R^m. If $\mathbf{x}$ is a vector in R^n, then T_A maps this vector into a vector $T_A(\mathbf{x})$ in R^k, and T_B, in turn, maps that vector into the vector $T_B(T_A(\mathbf{x}))$ in R^m. This process creates a transformation directly from R^n to R^m that we call the ***composition of T_B with T_A*** and which we denote by the symbol

$$T_B \circ T_A$$

which is read "T_B circle T_A." As illustrated in **Figure 1.9.1**, the transformation T_A in the formula is performed first; that is,

$$\boxed{(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))} \tag{1}$$

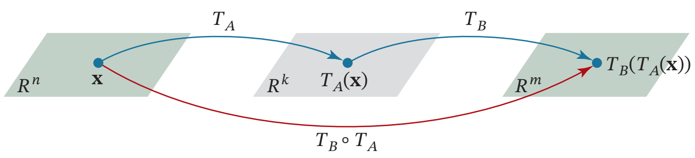


FIGURE 1.9.1

In the introduction to this section we promised to explain why matrix multiplication was defined in such an unusual way. The following theorem does that by showing that our definition of matrix multiplication is precisely what is required to ensure that the composition of two matrix transformations has the same effect as the transformation that results when the underlying matrices are multiplied.

Theorem 1.9.1

If $T_A: R^n \to R^k$ and $T_B: R^k \to R^m$ are matrix transformations, then $T_B \circ T_A$ is also a matrix transformation and

$$T_B \circ T_A = T_{BA} \tag{2}$$

Proof First we will show that $T_B \circ T_A$ is a linear transformation, thereby establishing that it is a matrix transformation by Theorem 1.8.3. Then we will show that the standard matrix for this transformation is BA to complete the proof.

To prove that $T_B \circ T_A$ is linear we must show that it has the additivity and homogeneity properties stated in Theorem 1.8.2. For this purpose, let $\mathbf{x}$ and $\mathbf{y}$ be vectors in R^n and observe that

$$\begin{aligned}(T_B \circ T_A)(\mathbf{x}+\mathbf{y}) &= T_B(T_A(\mathbf{x}+\mathbf{y}))\\ &= T_B(T_A(\mathbf{x}) + T_A(\mathbf{y})) && [\textbf{because } T_A \textbf{ is linear}]\\ &= T_B(T_A(\mathbf{x})) + T_B(T_A(\mathbf{y})) && [\textbf{because } T_B \textbf{ is linear}]\\ &= (T_B \circ T_A)(\mathbf{x}) + (T_B \circ T_A)(\mathbf{y})\end{aligned}$$

which proves additivity. Moreover,

$$\begin{aligned}(T_B \circ T_A)(k\mathbf{x}) &= T_B(T_A(k\mathbf{x}))\\ &= T_B(kT_A(\mathbf{x})) && [\textbf{because } T_A \textbf{ is linear}]\\ &= kT_B(T_A(\mathbf{x})) && [\textbf{because } T_B \textbf{ is linear}]\\ &= k(T_B \circ T_A)(\mathbf{x})\end{aligned}$$

which proves homogeneity and establishes that $T_B \circ T_A$ is a matrix transformation. Thus, there is an $m \times n$ matrix C such that

$$T_B \circ T_A = T_C \tag{3}$$

To find the appropriate matrix C that satisfies equation (3), observe that

$$T_C(\mathbf{x}) = (T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = T_B(A\mathbf{x}) = B(A\mathbf{x}) = (BA)\mathbf{x} = T_{BA}(\mathbf{x})$$

It now follows from Theorem 1.8.4 that $C = BA$. ■

EXAMPLE 1 | The Standard Matrix for a Composition

Let $T_1: R^3 \to R^2$ and $T_2: R^2 \to R^3$ be the linear transformations given by

$$T_1(x, y, z) = (x + 2y, x + 2z - y)$$

and

$$T_2(x, y) = (3x + y, x, x - 2y)$$

Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

Solution The standard basis vectors for R^3 are $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$. From which it follows that

$$T_1(\mathbf{e}_1) = (1,\ 1), \quad T_1(\mathbf{e}_2) = (2, -1) \quad \text{and} \quad T_1(\mathbf{e}_3) = (0, 2)$$

Thus

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

is the standard matrix for T_1. Similarly, the standard basis vectors for R^2 are $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$, so

$$T_2(\mathbf{e}_1) = (3, 1, 1) \quad \text{and} \quad T_2(\mathbf{e}_2) = (1, 0, 2)$$

Thus

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix}$$

is the standard matrix for T_2. Applying equation (3), the standard matrix for $T_2 \circ T_1$ is

$$BA = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 2 \\ 1 & 2 & 0 \\ -1 & 4 & -4 \end{bmatrix}$$

and the standard matrix for $T_1 \circ T_2$ is

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$$

Commutativity of Matrix Transformations

Since it is *not* generally true that $AB = BA$, it is also *not* generally true that $T_{AB} = T_{BA}$, so in general

$$T_A \circ T_B \neq T_B \circ T_A$$

Thus, *composition of matrix transformations is not commutative.* In those special cases where equality holds, we say that T_A and T_B ***commute***. Note, for example, that the linear transformations in Example 1 do not commute, since $AB \neq BA$.

EXAMPLE 2 | Composition Is Not Commutative

Let $T_A\colon R^2 \to R^2$ be the reflection about the line $y = x$, and let $T_B\colon R^2 \to R^2$ be the orthogonal projection onto the y-axis. Figure 1.9.2 illustrates graphically that $T_A \circ T_B$ and $T_B \circ T_A$ have different effects on a vector $\mathbf{x}$. This same conclusion can be reached by showing that the standard matrices for T_A and T_B do not commute:

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so $AB \neq BA$.

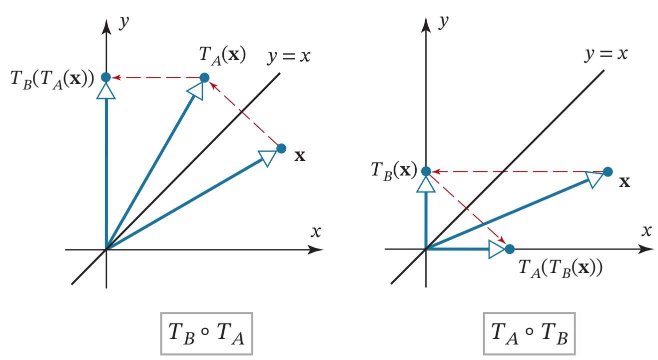


FIGURE 1.9.2

EXAMPLE 3 | Composition of Rotations Is Commutative

It is evident geometrically that the effect of rotating a vector about the origin through an angle θ_1 and then rotating the resulting vector through an angle θ_2 has the same effect as first rotating through the angle θ_2 and then rotating through the angle θ_1 since in both cases the original vector has been rotated through a total angle of $\theta = \theta_1 + \theta_2 = \theta_2 + \theta_1$. This suggests that the matrix transformations $T_{A_1}: R^2 \to R^2$ and $T_{A_2}: R^2 \to R^2$ that rotate vectors about the origin through the angles θ_1 and θ_2, respectively, should commute; that is

$$T_{A_1} \circ T_{A_2} = T_{A_2} \circ T_{A_1}$$

or equivalently

$$T_{A_1 A_2} = T_{A_2 A_1}$$

To verify that this is so, we need only show that $A_1 A_2 = A_2 A_1$. But from Table 5 of Section 1.8 we know that

$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

so (with the help of some basic trigonometric identities) it follows that

$$\begin{aligned} A_1 A_2 &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & -(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2) \\ \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 & -\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) \end{bmatrix} \\ &= A_2 A_1 \end{aligned}$$

Using the notation R_θ for a rotation of R^2 about the origin through an angle θ, the computation in Example 3 shows that

$$R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$$

EXAMPLE 4 | Composition of Two Reflections

Let $T_1: R^2 \to R^2$ be the reflection about the y-axis, and let $T_2: R^2 \to R^2$ be the reflection about the x-axis. In this case $T_1 \circ T_2$ and $T_2 \circ T_1$ are the same; both map every vector $\mathbf{x} = (x, y)$ into its negative $-\mathbf{x} = (-x, -y)$ (as evidenced by the following computation and Figure 1.9.3):

$$(T_1 \circ T_2)(x, y) = T_1(x, -y) = (-x, -y)$$
$$(T_2 \circ T_1)(x, y) = T_2(-x, y) = (-x, -y)$$

The equality of $T_1 \circ T_2$ and $T_2 \circ T_1$ can also be deduced by showing that the standard matrices for T_1 and T_2 commute. For this purpose let the standard matrices for these transformations be A_1 and A_2, respectively. Then it follows from Table 1 of Section 1.8 that

$$A_1 A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$A_2 A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We see from **Figure 1.9.3** that the composition $T_1 T_2(\mathbf{x}) = T_2 T_1(\mathbf{x})$ has the net effect of rotating the vector $\mathbf{x}$ through an angle of $\pi/2$ $(=180°)$, thereby reflecting that vector through the origin into the vector $-\mathbf{x}$. We call the linear operator $T(\mathbf{x}) = -\mathbf{x}$ the ***reflection about the origin.***

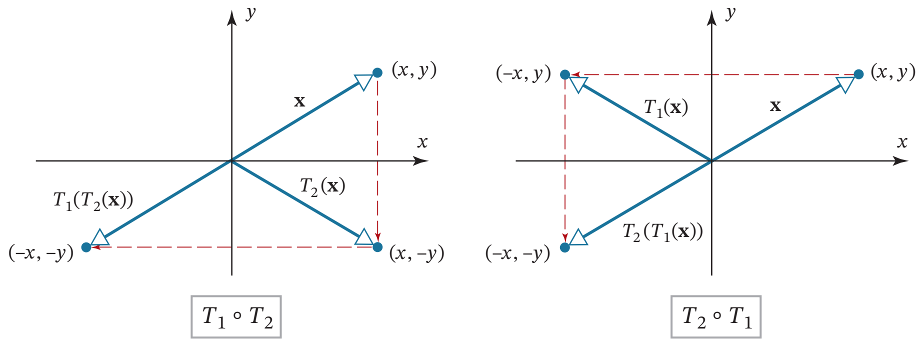


FIGURE 1.9.3

Compositions can be defined for any finite succession of matrix transformations whose domains and ranges have the appropriate dimensions. For example, to extend Formula (3) to three factors, consider the matrix transformations

$$T_A : R^n \to R^k, \quad T_B : R^k \to R^l, T_C : R^l \to R^m$$

We define the composition $(T_C \circ T_B \circ T_A) : R^n \to R^m$ by

$$(T_C \circ T_B \circ T_A)(\mathbf{x}) = T_C(T_B(T_A(\mathbf{x})))$$

As above, it can be shown that this is a matrix transformation whose standard matrix is CBA and that

$$\boxed{T_C \circ T_B \circ T_A = T_{CBA}} \tag{4}$$

EXAMPLE 5 | Composition of Three Matrix Transformations

Find the image of a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

under the matrix transformation that first rotates $\mathbf{x}$ about the origin through an angle of $\pi/6$, then reflects the resulting vector about the line $y = x$, and then projects that vector orthogonally onto the y-axis.

Solution Let A, B, and C be the standard matrices for the rotation, the reflection, and the orthogonal projection, respectively. Then from Tables 1, 3, and 5 of Section 1.8 these matrices are

$$A = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The three transformations in the stated succession can be viewed as the composition

$$T_C \circ T_B \circ T_A = T_{CBA}$$

whose standard matrix is

$$\begin{aligned} CBA &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ \cos(\pi/6) & -\sin(\pi/6) \end{bmatrix} \end{aligned}$$

Thus, the image of the vector $\mathbf{x}$ expressed as a column vector is

$$\begin{bmatrix} 0 & 0 \\ \cos(\pi/6) & -\sin(\pi/6) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ (\sqrt{3}/2)x - (1/2)y \end{bmatrix}$$

Invertibility of Matrix Operators

If $T_A: R^n \to R^n$ is a matrix operator whose standard matrix A is invertible, then we say that T_A is ***invertible***, and we define the ***inverse*** of T_A as

$$\boxed{T_A^{-1} = T_{A^{-1}}} \tag{5}$$

or restated in words, *the inverse of multiplication by A is multiplication by the inverse of A.* Thus, by definition, the standard matrix for T_A^{-1} is A^{-1}, from which it follows that

$$T_A^{-1} \circ T_A = T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_I$$

It follows from this that for any vector $\mathbf{x}$ in R^n

$$\left(T_A^{-1} \circ T_A\right)(\mathbf{x}) = T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

and similarly that $\left(T_A \circ T_A^{-1}\right)(\mathbf{x}) = \mathbf{x}$. Thus, when T_A and T_A^{-1} are composed in either order they cancel out the effect of one another (**Figure 1.9.4**).

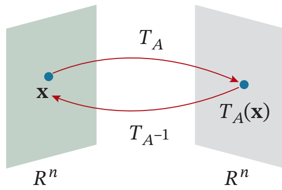


FIGURE 1.9.4

EXAMPLE 6 | Inverse of a Rotation Operator

Let $T: R^2 \to R^2$ be the operator that rotates each vector in R^2 through the angle θ, so the standard matrix for T is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

It is evident geometrically that to undo the effect of T, one must rotate each vector in R^2 through the angle $-\theta$. But this is precisely what T^{-1} does, since it follows from (5) and Theorem 1.4.5 that the standard matrix for this transformation is

$$R_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = R_{-\theta}$$

EXAMPLE 7 | Inverse Transformations from Linear Equations

Consider the operator $T: R^2 \to R^2$ defined by the equations

$$\begin{aligned} w_1 &= 2x_1 + x_2 \\ w_2 &= 3x_1 + 4x_2 \end{aligned}$$

Find $T^{-1}(w_1, w_2)$.

Solution The matrix form of these equations is

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so the standard matrix for T is

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

This matrix is invertible, and the standard matrix for T^{-1} is

$$A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Thus

$$A^{-1}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}w_1 - \frac{1}{5}w_2 \\ -\frac{3}{5}w_1 + \frac{2}{5}w_2 \end{bmatrix}$$

from which we conclude that

$$T^{-1}(w_1, w_2) = \left(\tfrac{4}{5}w_1 - \tfrac{1}{5}w_2, -\tfrac{3}{5}w_1 + \tfrac{2}{5}w_2\right)$$

Since not every matrix has an inverse, it should not be surprising that the same is true for matrix transformations. As a simple example, consider a transformation $T: R^2 \to R^2$ that projects a vector $\mathbf{x}$ orthogonally onto either the x-axis or the y-axis. You can see in Table 3 of Section 1.8 that the standard matrices for these transformations are not invertible, so in neither case does an invertible matrix A exist to satisfy Equation (5).

Exercise Set 1.9

*In Exercises **1–4**, determine whether the operators T_1 and T_2 commute; that is, whether $T_1 \circ T_2 = T_2 \circ T_1$.*

1. **a.** $T_1: R^2 \to R^2$ is the reflection about the line $y = x$, and $T_2: R^2 \to R^2$ is the orthogonal projection onto the x-axis.

b. $T_1: R^2 \to R^2$ is the reflection about the x-axis, and $T_2: R^2 \to R^2$ is the reflection about the line $y = x$.

2. **a.** $T_1: R^2 \to R^2$ is the orthogonal projection onto the x-axis, and $T_2: R^2 \to R^2$ is the orthogonal projection onto the y-axis.

b. $T_1: R^2 \to R^2$ is the rotation about the origin through an angle of $\pi/4$, and $T_2: R^2 \to R^2$ is the reflection about the y-axis.

3. $T_1: R^3 \to R^3$ is the reflection about the xy-plane and $T_2: R^3 \to R^3$ is the orthogonal projection onto the yz-plane.

4. $T_1: R^3 \to R^3$ is the reflection about the xy-plane and $T_2: R^3 \to R^3$ is given by the formula $T(x, y, z) = (2x, 3y, z)$.

*In Exercises **5–6**, let T_A and T_B be the operators whose standard matrices are given. Find the standard matrices for $T_B \circ T_A$ and $T_A \circ T_B$.*

5. $A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 4 \\ -1 & 5 & 2 \\ 2 & -3 & 8 \end{bmatrix}$

7. Find the standard matrix for the stated composition in R^2.

a. A rotation of 90°, followed by a reflection about the line $y = x$.

b. An orthogonal projection onto the y-axis, followed by a 45° degree rotation about the origin.

c. A reflection about the x-axis, followed by a rotation about the origin of 60°.

8. Find the standard matrix for the stated composition in R^2.

a. A rotation about the origin of 60°, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.

b. An orthogonal projection onto the x-axis, followed by a rotation about the origin of 45°, followed by a reflection about the y-axis.

c. A rotation about the origin of 15°, followed by a rotation about the origin of 105°, followed by a rotation about the origin of 60°.

9. Find the standard matrix for the stated composition in R^3.

a. A reflection about the yz-plane, followed by an orthogonal projection onto the xz-plane.

b. A reflection about the xy-plane, followed by an orthogonal projection onto the xy-plane.

c. An orthogonal projection onto the xy-plane, followed by a reflection about the yz-plane.

10. Find the standard matrix for the stated composition in R^3.

a. A reflection about the xy-plane, followed by an orthogonal projection onto the xz-plane, followed by the transformation that sends each vector $\mathbf{x}$ to the vector $-\mathbf{x}$.

b. A reflection about the xy-plane, followed by a reflection about the xz-plane, followed by an orthogonal projection onto the yz-plane.

c. An orthogonal projection onto the yz-plane, followed by the transformation that maps each vector $\mathbf{x}$ to the vector $2\mathbf{x}$, followed by a reflection about the xy-plane.

11. Let $T_1(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ and $T_2(x_1, x_2) = (3x_1, 2x_1 + 4x_2)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2))$.

12. Let $T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

13. Let $T_1(x_1, x_2) = (x_1 - x_2, 2x_2 - x_1, 3x_1)$ and $T_2(x_1, x_2, x_3) = (4x_2, x_1 + 2x_2)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2))$.

14. Let $T_1(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + 3x_3, x_2 - x_4)$ and $T_2(x_1, x_2) = (-x_1, 0, x_1 + x_2, 3x_2)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2, x_3, x_4))$.

15. Let $T_1: R^2 \to R^4$ and $T_2: R^4 \to R^3$ be given by:
$T_1(x, y) = (y, x, x + y, x - y)$
$T_2(x, y, z, w) = (x + w, y + w, z + w)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$.

c. Explain why $T_1 \circ T_2$ is not defined.

d. Use the matrix found in part (b) to find a formula for $(T_2 \circ T_1)(x, y)$.

16. Let $T_1: R^2 \to R^3$ and $T_2: R^3 \to R^4$ be given by:
$T_1(x, y) = (x + 2y, 0, 2x + y)$
$T_2(x, y, z) = (3z, x - y, 3z, y - x)$.

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$.

c. Explain why $T_1 \circ T_2$ is not defined.

d. Use the matrix found in part (b) to find a formula for $(T_2 \circ T_1)(x, y)$.

In Exercises 17–18, express the equations in matrix form, and then use Theorem 1.5.3(c) to determine whether the operator defined by the equations is invertible.

17. a. $\begin{aligned} w_1 &= 8x_1 + 4x_2 \\ w_2 &= 2x_1 + x_2 \end{aligned}$ **b.** $\begin{aligned} w_1 &= -x_1 + 3x_2 + 2x_3 \\ w_2 &= 2x_1 + 4x_3 \\ w_3 &= x_1 + 3x_2 + 6x_3 \end{aligned}$

18. a. $\begin{aligned} w_1 &= 2x_1 - 3x_2 \\ w_2 &= 5x_1 + x_2 \end{aligned}$ **b.** $\begin{aligned} w_1 &= x_1 + 2x_2 + 3x_3 \\ w_2 &= 2x_1 + 5x_2 + 3x_3 \\ w_3 &= x_1 + 8x_3 \end{aligned}$

19. Determine whether the matrix operator $T: R^2 \to R^2$ defined by the equations is invertible; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2)$.

a. $\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$ **b.** $\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$

20. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is invertible; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a. $\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$ **b.** $\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= - 2x_2 + 5x_3 \end{aligned}$

In Exercises 21–22, determine whether the matrix operator is invertible. If so, describe in words the effect of its inverse.

21. a. Reflection about the x-axis in R^2.

b. A 60° rotation about the origin in R^2.

c. Orthogonal projection onto the x-axis in R^2.

22. a. Reflection about the line $y = x$.

b. Orthogonal projection onto the y-axis.

c. Reflection about the origin.

In Exercises 23–24, determine whether T_A is invertible. If so, compute $T_A^{-1}(\mathbf{x})$.

23. a. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **b.** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

24. a. $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

25. Let $T_A: R^2 \to R^2$ be multiplication by

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

a. What is the geometric effect of applying this transformation to a vector $\mathbf{x}$ in R^2?

b. Express the operator T_A as a composition of two linear operators on R^2.

26. Let $T_A: R^2 \to R^2$ be multiplication by

$$A = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

a. What is the geometric effect of applying this transformation to a vector $\mathbf{x}$ in R^2?

b. Express the operator T_A as a composition of two linear operators on R^2.

Working with Proofs

27. Prove that the matrix transformations T_A and T_B commute if and only if the matrices A and B commute.

28. Let T_A and T_B be matrix operators on R^n. Prove that $T_A \circ T_B$ is invertible if and only if both T_A and T_B are invertible.

29. Prove that the matrix operator T_A on R^n is invertible if and only if for every $\mathbf{b}$ in R^n there exists a unique vector $\mathbf{x}$ in R^n such that $T_A(\mathbf{x}) = \mathbf{b}$.

True-False Exercises

TF. In parts **(a)–(g)** determine whether the statement is true or false, and justify your answer.

a. If T_A and T_B are matrix operators on R^n, then $T_A(T_B(\mathbf{x})) = T_B(T_A(\mathbf{x}))$ for every vector $\mathbf{x}$ in R^n.

b. If T_A and T_B are matrix operators on R^n and $\mathbf{x}$ is a vector in R^n, then $T_B \circ T_A(\mathbf{x}) = BA\mathbf{x}$

c. A composition of two rotation operators about the origin of R^2 is another rotation about the origin.

d. A composition of two reflection operators in R^2 is another reflection operator.

e. The inverse transformation for a reflection in R^2 about the line $y = x$ is the reflection about the line $y = x$.

f. The inverse transformation for a 90° rotation about the origin in R^2 is a 90° rotation about the origin.

g. The inverse transformation for a reflection about the origin in R^2 is a reflection about the origin.

Working with Technology

T1. a. Find the standard matrix for the linear operator on R^2 that performs a counterclockwise rotation of 47° about the origin, followed by a reflection about the y-axis, followed by a counterclockwise rotation of 33° about the origin.

b. Find the image of the point $(1, 1)$ under the operator in part (a).

1.10 Applications of Linear Systems

In this section we will discuss some brief applications of linear systems. These are but a small sample of the wide variety of real-world problems to which our study of linear systems is applicable.

Network Analysis

The concept of a *network* appears in a variety of applications. Loosely stated, a ***network*** is a set of ***branches*** through which something "flows." For example, the branches might be electrical wires through which electricity flows, pipes through which water or oil flows, traffic lanes through which vehicular traffic flows, or economic linkages through which money flows, to name a few possibilities.

In most networks, the branches meet at points, called ***nodes*** or ***junctions***, where the flow divides. For example, in an electrical network, nodes occur where three or more wires join, in a traffic network they occur at street intersections, and in a financial network they occur at banking centers where incoming money is distributed to individuals or other institutions.

In the study of networks, there is generally some numerical measure of the rate at which the medium flows through a branch. For example, the flow rate of electricity is often measured in amperes, the flow rate of water or oil in gallons per minute, the flow rate of traffic in vehicles per hour, and the flow rate of European currency in millions of Euros per day. We will restrict our attention to networks in which there is ***flow conservation*** at each node, by which we mean that *the rate of flow into any node is equal to the rate of flow out of that node.* This ensures that the flow medium does not build up at the nodes and block the free movement of the medium through the network.

A common problem in network analysis is to use known flow rates in certain branches to find the flow rates in all of the branches. Here is an example.

EXAMPLE 1 | Network Analysis Using Linear Systems

Figure 1.10.1 shows a network with four nodes in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

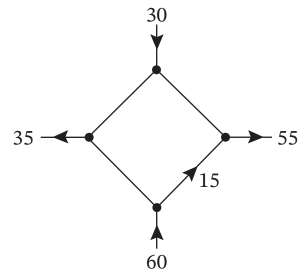


FIGURE 1.10.1

Solution As illustrated in **Figure 1.10.2**, we have assigned arbitrary directions to the unknown flow rates x_1, x_2, and x_3. We need not be concerned if some of the directions are incorrect, since an incorrect direction will be signaled by a negative value for the flow rate when we solve for the unknowns.

It follows from the conservation of flow at node A that

$$x_1 + x_2 = 30$$

Similarly, at the other nodes we have

$$\begin{aligned} x_2 + x_3 &= 35 \quad (\text{node } B) \\ x_3 + 15 &= 60 \quad (\text{node } C) \\ x_1 + 15 &= 55 \quad (\text{node } D) \end{aligned}$$

These four conditions produce the linear system

$$\begin{aligned} x_1 + x_2 &= 30 \\ x_2 + x_3 &= 35 \\ x_3 &= 45 \\ x_1 &= 40 \end{aligned}$$

which we can now try to solve for the unknown flow rates. In this particular case the system is sufficiently simple that it can be solved by inspection (work from the bottom up). We leave it for you to confirm that the solution is

$$x_1 = 40, \quad x_2 = -10, \quad x_3 = 45$$

The fact that x_2 is negative tells us that the direction assigned to that flow in Figure 1.10.2 is incorrect; that is, the flow in that branch is *into* node A.

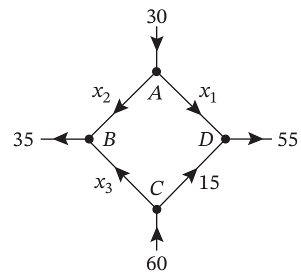


FIGURE 1.10.2

EXAMPLE 2 | Design of Traffic Patterns

The network in **Figure 1.10.3*a*** shows a proposed plan for the traffic flow around a new park that will house the Liberty Bell in Philadelphia, Pennsylvania. The plan calls for a computerized traffic light at the north exit on Fifth Street, and the diagram indicates the average number of vehicles per hour that are expected to flow in and out of the streets that border the complex. All streets are one-way.

(*a*) How many vehicles per hour should the traffic light let through to ensure that the average number of vehicles per hour flowing into the complex is the same as the average number of vehicles flowing out?

(*b*) Assuming that the traffic light has been set to balance the total flow in and out of the complex, what can you say about the average number of vehicles per hour that will flow along the streets that border the complex?

Solution (*a*) If, as indicated in **Figure 1.10.3*b***, we let x denote the number of vehicles per hour that the traffic light must let through, then the total number of vehicles per hour that flow in and out of the complex will be

Flowing in: $500 + 400 + 600 + 200 = 1700$

Flowing out: $x + 700 + 400$

Equating the flows in and out shows that the traffic light should let $x = 600$ vehicles per hour pass through.

Solution (*b*) To avoid traffic congestion, the flow in must equal the flow out at each intersection. For this to happen, the following conditions must be satisfied:

Intersection	Flow In		Flow Out
A	$400 + 600$	$=$	$x_1 + x_2$
B	$x_2 + x_3$	$=$	$400 + x$
C	$500 + 200$	$=$	$x_3 + x_4$
D	$x_1 + x_4$	$=$	700

Thus, with $x = 600$, as computed in part (a), we obtain the following linear system:

$$\begin{aligned}
x_1 + x_2 &= 1000\\
x_2 + x_3 &= 1000\\
x_3 + x_4 &= 700\\
x_1 + x_4 &= 700
\end{aligned}$$

We leave it for you to show that the system has infinitely many solutions and that these are given by the parametric equations

$$x_1 = 700 - t, \quad x_2 = 300 + t, \quad x_3 = 700 - t, \quad x_4 = t \tag{1}$$

However, the parameter t is not completely arbitrary here, since there are physical constraints to be considered. For example, the average flow rates must be nonnegative since we have assumed the streets to be one-way, and a negative flow rate would indicate a flow in the wrong direction. This being the case, we see from (1) that t can be any real number that satisfies $0 \le t \le 700$, which implies that the average flow rates along the streets will fall in the ranges

$$0 \le x_1 \le 700, \quad 300 \le x_2 \le 1000, \quad 0 \le x_3 \le 700, \quad 0 \le x_4 \le 700$$

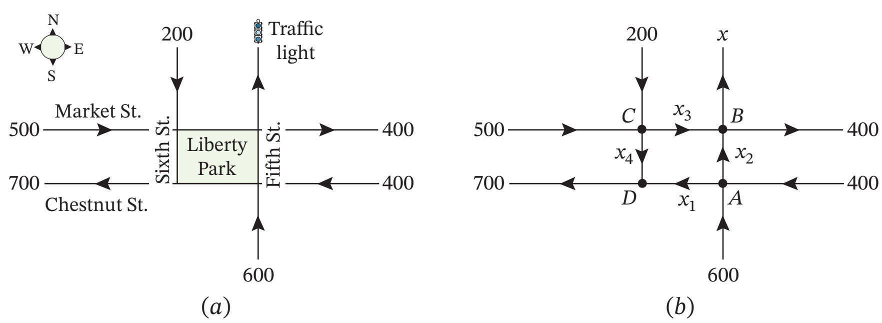


FIGURE 1.10.3

Electrical Circuits

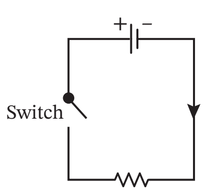


FIGURE 1.10.4

Next we will show how network analysis can be used to analyze electrical circuits consisting of batteries and resistors. A ***battery*** is a source of electric energy, and a ***resistor***, such as a lightbulb, is an element that dissipates electric energy. **Figure 1.10.4** shows a schematic diagram of a circuit with one battery (represented by the symbol ⊣⊢), one resistor (represented by the symbol –⩘⩘–), and a switch. The battery has a ***positive pole*** (+) and a ***negative pole*** (−). When the switch is closed, electrical current is considered to flow from the positive pole of the battery, through the resistor, and back to the negative pole (indicated by the arrowhead in the figure).

Electrical current, which is a flow of electrons through wires, behaves much like the flow of water through pipes. A battery acts like a pump that creates "electrical pressure" to increase the flow rate of electrons, and a resistor acts like a restriction in a pipe that reduces the flow rate of electrons. The technical term for electrical pressure is ***electrical potential***; it is commonly measured in ***volts*** (V). The degree to which a resistor reduces the electrical potential is called its ***resistance*** and is commonly measured in ***ohms*** (Ω). The rate of flow of electrons in a wire is called ***current*** and is commonly measured in ***amperes*** (also called ***amps***) (A). The precise effect of a resistor is given by the following law:

Ohm's Law If a current of I amperes passes through a resistor with a resistance of R ohms, then there is a resulting drop of E volts in electrical potential that is the product of the current and resistance; that is,

$$E = IR$$

A typical electrical network will have multiple batteries and resistors joined by some configuration of wires. A point at which three or more wires in a network are joined is called a ***node*** (or ***junction point***). A ***branch*** is a wire connecting two nodes, and a ***closed loop*** is a succession of connected branches that begin and end at the same node. For example, the electrical network in **Figure 1.10.5** has two nodes and three closed loops—two inner loops and one outer loop. As current flows through an electrical network, it undergoes increases and decreases in electrical potential, called ***voltage rises*** and ***voltage drops***, respectively. The behavior of the current at the nodes and around closed loops is governed by two fundamental laws:

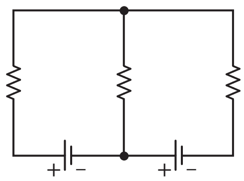


FIGURE 1.10.5

Kirchhoff's Current Law The sum of the currents flowing into any node is equal to the sum of the currents flowing out.

Kirchhoff's Voltage Law In one traversal of any closed loop, the sum of the voltage rises equals the sum of the voltage drops.

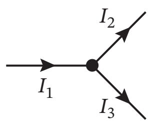
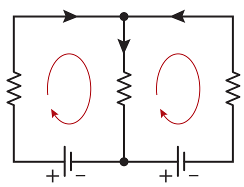


FIGURE 1.10.6

Kirchhoff's current law is a restatement of the principle of flow conservation at a node that was stated for general networks. Thus, for example, the currents at the top node in **Figure 1.10.6** satisfy the equation $I_1 = I_2 + I_3$.

In circuits with multiple loops and batteries there is usually no way to tell in advance which way the currents are flowing, so the usual procedure in circuit analysis is to assign *arbitrary* directions to the current flows in the branches and let the mathematical computations determine whether the assignments are correct. In addition to assigning directions to the current flows, Kirchhoff's voltage law requires a direction of travel for each closed loop. The choice is arbitrary, but for consistency we will always take this direction to be *clockwise* (**Figure 1.10.7**). We also make the following conventions:

- A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned to the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned to the loop.
- A voltage rise occurs at a battery if the direction assigned to the loop is from − to + through the battery, and a voltage drop occurs at a battery if the direction assigned to the loop is from + to − through the battery.

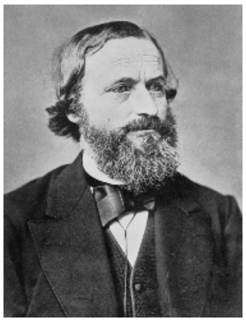


Clockwise closed-loop convention with arbitrary direction assignments to currents in the branches

FIGURE 1.10.7

If you follow these conventions when calculating currents, then those currents whose directions were assigned correctly will have positive values and those whose directions were assigned incorrectly will have negative values.

Historical Note



The German physicist Gustav Kirchhoff was a student of Gauss. His work on Kirchhoff's laws, announced in 1854, was a major advance in the calculation of currents, voltages, and resistances of electrical circuits. Kirchhoff was severely disabled and spent most of his life on crutches or in a wheelchair.

[Image: Courtesy of Library of Congress]

Gustav Kirchhoff
(1824–1887)

EXAMPLE 3 | A Circuit with One Closed Loop

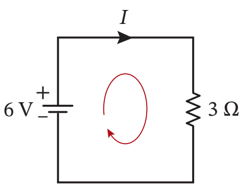


FIGURE 1.10.8

Determine the current I in the circuit shown in **Figure 1.10.8**.

Solution Since the direction assigned to the current through the resistor is the same as the direction of the loop, there is a voltage drop at the resistor. By Ohm's law this voltage drop is $E = IR = 3I$. Also, since the direction assigned to the loop is from $-$ to $+$ through the battery, there is a voltage rise of 6 volts at the battery. Thus, it follows from Kirchhoff's voltage law that

$$3I = 6$$

from which we conclude that the current is $I = 2$ A. Since I is positive, the direction assigned to the current flow is correct.

EXAMPLE 4 | A Circuit with Three Closed Loops

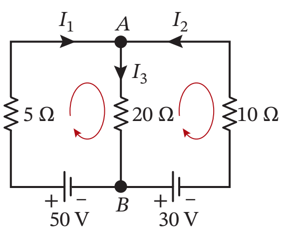


FIGURE 1.10.9

Determine the currents I_1, I_2, and I_3 in the circuit shown in **Figure 1.10.9**.

Solution Using the assigned directions for the currents, Kirchhoff's current law provides one equation for each node:

Node	Current In		Current Out
A	$I_1 + I_2$	$=$	I_3
B	I_3	$=$	$I_1 + I_2$

However, these equations are really the same, since both can be expressed as

$$I_1 + I_2 - I_3 = 0 \tag{2}$$

To find unique values for the currents we will need two more equations, which we will obtain from Kirchhoff's voltage law. We can see from the network diagram that there are three closed loops, a left inner loop containing the 50 V battery, a right inner loop containing the 30 V battery, and an outer loop that contains both batteries. Thus, Kirchhoff's voltage law will actually produce three equations. With a clockwise traversal of the loops, the voltage rises and drops in these loops are as follows:

	Voltage Rises	Voltage Drops
Left Inside Loop	50	$5I_1 + 20I_3$
Right Inside Loop	$30 + 10I_2 + 20I_3$	0
Outside Loop	$30 + 50 + 10I_2$	$5I_1$

These conditions can be rewritten as

$$\begin{aligned} 5I_1 \qquad\qquad + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \\ 5I_1 - 10I_2 \qquad\quad &= 80 \end{aligned} \tag{3}$$

However, the last equation is superfluous, since it is the difference of the first two. Thus, if we combine (2) and the first two equations in (3), we obtain the following linear system of three equations in the three unknown currents:

$$\begin{aligned} I_1 + \quad I_2 - \quad I_3 &= 0 \\ 5I_1 \qquad\qquad + 20I_3 &= 50 \\ 10I_2 + 20I_3 &= -30 \end{aligned}$$

We leave it for you to show that the solution of this system in amps is $I_1 = 6$, $I_2 = -5$, and $I_3 = 1$. The fact that I_2 is negative tells us that the direction of this current is opposite to that indicated in Figure 1.10.9.

Balancing Chemical Equations

Chemical compounds are represented by ***chemical formulas*** that describe the atomic makeup of their molecules. For example, water is composed of two hydrogen atoms and one oxygen atom, so its chemical formula is H_2O; and stable oxygen is composed of two oxygen atoms, so its chemical formula is O_2.

When chemical compounds are combined under the right conditions, the atoms in their molecules rearrange to form new compounds. For example, when methane burns, the methane (CH_4) and stable oxygen (O_2) react to form carbon dioxide (CO_2) and water (H_2O). This is indicated by the ***chemical equation***

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O \tag{4}$$

The molecules to the left of the arrow are called the ***reactants*** and those to the right the ***products***. In this equation the plus signs serve to separate the molecules and are not intended as algebraic operations. However, this equation does not tell the whole story, since it fails to account for the proportions of molecules required for a ***complete reaction*** (no reactants left over). For example, we can see from the right side of (4) that to produce one molecule of carbon dioxide and one molecule of water, one needs *three* oxygen atoms for each carbon atom. However, from the left side of (4) we see that one molecule of methane and one molecule of stable oxygen have only *two* oxygen atoms for each carbon atom. Thus, on the reactant side the ratio of methane to stable oxygen cannot be one-to-one in a complete reaction.

A chemical equation is said to be ***balanced*** if for each type of atom in the reaction, the same number of atoms appears on each side of the arrow. For example, the balanced version of Equation (4) is

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O \tag{5}$$

by which we mean that one methane molecule combines with two stable oxygen molecules to produce one carbon dioxide molecule and two water molecules. In theory, one could multiply this equation through by any positive integer. For example, multiplying through by 2 yields the balanced chemical equation

$$2CH_4 + 4O_2 \longrightarrow 2CO_2 + 4H_2O$$

However, the standard convention is to use the smallest positive integers that will balance the equation.

Equation (4) is sufficiently simple that it could have been balanced by trial and error, but for more complicated chemical equations we will need a systematic method. There are various methods that can be used, but we will give one that uses systems of linear equations. To illustrate the method let us reexamine Equation (4). To balance this equation we must find positive integers, x_1, x_2, x_3, and x_4 such that

$$x_1\,(CH_4) + x_2\,(O_2) \longrightarrow x_3\,(CO_2) + x_4\,(H_2O) \tag{6}$$

For each of the atoms in the equation, the number of atoms on the left must be equal to the number of atoms on the right. Expressing this in tabular form we have

	Left Side		Right Side
Carbon	x_1	$=$	x_3
Hydrogen	$4x_1$	$=$	$2x_4$
Oxygen	$2x_2$	$=$	$2x_3 + x_4$

from which we obtain the homogeneous linear system

$$\begin{aligned} x_1 \qquad\quad - x_3 \qquad\quad &= 0 \\ 4x_1 \qquad\qquad\qquad - 2x_4 &= 0 \\ 2x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

We leave it for you to show that the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

from which we conclude that the general solution of the system is

$$x_1 = t/2, \quad x_2 = t, \quad x_3 = t/2, \quad x_4 = t$$

where t is arbitrary. The smallest positive integer values for the unknowns occur when we let $t = 2$, so the equation can be balanced by letting $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 2$. This agrees with our earlier conclusions, since substituting these values into Equation (6) yields Equation (5).

EXAMPLE 5 | Balancing Chemical Equations Using Linear Systems

Balance the chemical equation

$$\begin{array}{ccccccc} HCl & + & Na_3PO_4 & \longrightarrow & H_3PO_4 & + & NaCl \\ \text{[hydrochloric acid]} & + & \text{[sodium phosphate]} & \longrightarrow & \text{[phosphoric acid]} & + & \text{[sodium chloride]} \end{array}$$

Solution Let x_1, x_2, x_3, and x_4 be positive integers that balance the equation

$$x_1\,(HCl) + x_2\,(Na_3PO_4) \longrightarrow x_3\,(H_3PO_4) + x_4\,(NaCl) \tag{7}$$

Equating the number of atoms of each type on the two sides yields

$$\begin{aligned} 1x_1 &= 3x_3 & &\textbf{Hydrogen (H)} \\ 1x_1 &= 1x_4 & &\textbf{Chlorine (Cl)} \\ 3x_2 &= 1x_4 & &\textbf{Sodium (Na)} \\ 1x_2 &= 1x_3 & &\textbf{Phosphorus (P)} \\ 4x_2 &= 4x_3 & &\textbf{Oxygen (O)} \end{aligned}$$

from which we obtain the homogeneous linear system

$$\begin{aligned} x_1 \qquad\quad - 3x_3 \qquad\quad &= 0 \\ x_1 \qquad\qquad\qquad\quad - x_4 &= 0 \\ 3x_2 \qquad\quad - x_4 &= 0 \\ x_2 - x_3 \qquad\quad &= 0 \\ 4x_2 - 4x_3 \qquad\quad &= 0 \end{aligned}$$

We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we conclude that the general solution of the system is

$$x_1 = t, \quad x_2 = t/3, \quad x_3 = t/3, \quad x_4 = t$$

where t is arbitrary. To obtain the smallest positive integers that balance the equation, we let $t = 3$, in which case we obtain $x_1 = 3, x_2 = 1, x_3 = 1$, and $x_4 = 3$. Substituting these values in (7) produces the balanced equation

$$3HCl + Na_3PO_4 \longrightarrow H_3PO_4 + 3NaCl$$

Polynomial Interpolation

An important problem in various applications is to find a polynomial whose graph passes through a specified set of points in the plane; this is called an ***interpolating polynomial*** for the points. The simplest example of such a problem is to find a linear polynomial

$$p(x) = ax + b \tag{8}$$

whose graph passes through two known distinct points, (x_1, y_1) and (x_2, y_2), in the xy-plane (**Figure 1.10.10**). You have probably encountered various methods in analytic geometry for finding the equation of a line through two points, but here we will give a method based on linear systems that can be adapted to general polynomial interpolation.

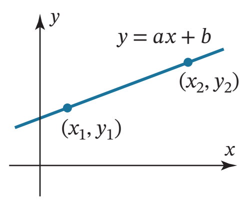


FIGURE 1.10.10

The graph of (8) is the line $y = ax + b$, and for this line to pass through the points (x_1, y_1) and (x_2, y_2), we must have

$$y_1 = ax_1 + b \quad \text{and} \quad y_2 = ax_2 + b$$

Therefore, the unknown coefficients a and b can be obtained by solving the linear system

$$\begin{aligned} ax_1 + b &= y_1 \\ ax_2 + b &= y_2 \end{aligned}$$

We don't need any fancy methods to solve this system—the value of a can be obtained by subtracting the equations to eliminate b, and then the value of a can be substituted into either equation to find b. We leave it as an exercise for you to find a and b and then show that they can be expressed in the form

$$a = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad b = \frac{y_1x_2 - y_2x_1}{x_2 - x_1} \tag{9}$$

provided $x_1 \neq x_2$. Thus, for example, the line $y = ax + b$ that passes through the points

$$(2, 1) \quad \text{and} \quad (5, 4)$$

can be obtained by taking $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (5, 4)$, in which case (9) yields

$$a = \frac{4-1}{5-2} = 1 \quad \text{and} \quad b = \frac{(1)(5)-(4)(2)}{5-2} = -1$$

Therefore, the equation of the line is

$$y = x - 1$$

(**Figure 1.10.11**).

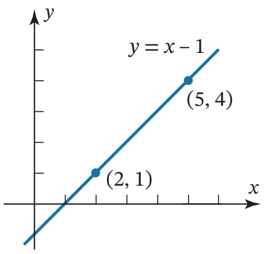


FIGURE 1.10.11

Now let us consider the more general problem of finding a polynomial whose graph passes through n points with distinct x-coordinates

$$(x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3), \ldots, \quad (x_n, y_n) \tag{10}$$

Since there are n conditions to be satisfied, intuition suggests that we should begin by looking for a polynomial of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{11}$$

since a polynomial of this form has n coefficients that are at our disposal to satisfy the n conditions. However, we want to allow for cases where the points may lie on a line or have some other configuration that would make it possible to use a polynomial whose degree is less than $n - 1$; thus, we allow for the possibility that a_{n-1} and other coefficients in (11) may be zero.

The following theorem, which we will not prove, is the basic result on polynomial interpolation.

Theorem 1.10.1

Polynomial Interpolation

Given any n points in the xy-plane that have distinct x-coordinates, there is a unique polynomial of degree $n - 1$ or less whose graph passes through those points.

Let us now consider how we might go about finding the interpolating polynomial (11) whose graph passes through the points in (10). Since the graph of this polynomial is the graph of the equation

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \tag{12}$$

it follows that the coordinates of the points must satisfy

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_1 & + & a_2 x_1^2 & + \cdots + & a_{n-1} x_1^{n-1} & = & y_1 \\ a_0 & + & a_1 x_2 & + & a_2 x_2^2 & + \cdots + & a_{n-1} x_2^{n-1} & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_0 & + & a_1 x_n & + & a_2 x_n^2 & + \cdots + & a_{n-1} x_n^{n-1} & = & y_n \end{array} \tag{13}$$

In these equations the values of x's and y's are assumed to be known, so we can view this as a linear system in the unknowns $a_0, a_1, \ldots, a_{n-1}$. From this point of view the augmented matrix for the system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{bmatrix} \tag{14}$$

and hence the interpolating polynomial can be found by reducing this matrix to reduced row echelon form, say by Gauss-Jordan elimination, as in the following example.

EXAMPLE 6 | Polynomial Interpolation by Gauss–Jordan Elimination

Find a cubic polynomial whose graph passes through the points

$$(1, 3), \quad (2, -2), \quad (3, -5), \quad (4, 0)$$

Solution Since there are four points, we will use an interpolating polynomial of degree $n = 3$. Denote this polynomial by

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

and denote the x- and y-coordinates of the given points by

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3, \quad x_4 = 4 \quad \text{and} \quad y_1 = 3, \quad y_2 = -2, \quad y_3 = -5, \quad y_4 = 0$$

Thus, it follows from (14) that the augmented matrix for the linear system in the unknowns a_0, a_1, a_2, and a_3 is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & y_1 \\ 1 & x_2 & x_2^2 & x_2^3 & y_2 \\ 1 & x_3 & x_3^2 & x_3^3 & y_3 \\ 1 & x_4 & x_4^2 & x_4^3 & y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{bmatrix}$$

We leave it for you to confirm that the reduced row echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

from which it follows that $a_0 = 4, a_1 = 3, a_2 = -5, a_3 = 1$. Thus, the interpolating polynomial is

$$p(x) = 4 + 3x - 5x^2 + x^3$$

The graph of this polynomial and the given points are shown in Figure 1.10.12.

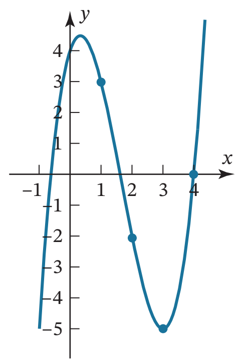


FIGURE 1.10.12

Remark Later we will give a more efficient method for finding interpolating polynomials that is better suited for problems in which the number of data points is large.

EXAMPLE 7 | Approximate Integration

CALCULUS REQUIRED

There is no way to evaluate the integral

$$\int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$$

directly since there is no way to express an antiderivative of the integrand in terms of elementary functions. This integral could be approximated by Simpson's rule or some comparable method, but an alternative approach is to approximate the integrand by an interpolating polynomial and integrate the approximating polynomial. For example, let us consider the five points

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1$$

that divide the interval [0, 1] into four equally spaced subintervals (Figure 1.10.13). The values of

$$f(x) = \sin\left(\frac{\pi x^2}{2}\right)$$

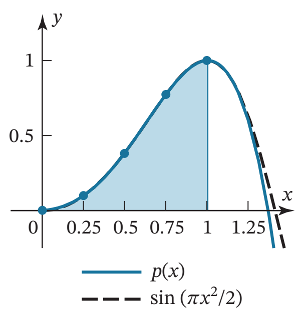


FIGURE 1.10.13

at these points approximately

$$f(0) = 0, \quad f(0.25) = 0.098017, \quad f(0.5) = 0.382683,$$
$$f(0.75) = 0.77301, \quad f(1) = 1$$

The interpolating polynomial is (verify)

$$p(x) = 0.098796x + 0.762356x^2 + 2.14429x^3 - 2.00544x^4 \tag{15}$$

and

$$\int_0^1 p(x)\,dx \approx 0.438501 \tag{16}$$

As shown in Figure 1.10.13, the graphs of f and p match very closely over the interval $[0, 1]$, so the approximation is quite good.

Exercise Set 1.10

1. The accompanying figure shows a network in which the flow rate and direction of flow in certain branches are known. Find the flow rates and directions of flow in the remaining branches.

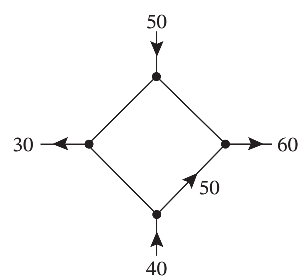


FIGURE Ex-1

2. The accompanying figure shows known flow rates of hydrocarbons into and out of a network of pipes at an oil refinery.

a. Set up a linear system whose solution provides the unknown flow rates.

b. Solve the system for the unknown flow rates.

c. Find the flow rates and directions of flow if $x_4 = 50$ and $x_6 = 0$.

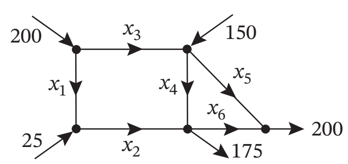


FIGURE Ex-2

3. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.

a. Set up a linear system whose solution provides the unknown flow rates.

b. Solve the system for the unknown flow rates.

c. If the flow along the road from A to B must be reduced for construction, what is the minimum flow that is required to keep traffic flowing on all roads?

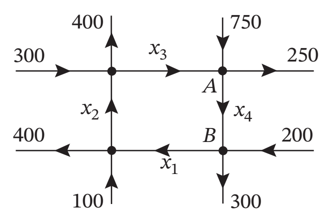


FIGURE Ex-3

4. The accompanying figure shows a network of one-way streets with traffic flowing in the directions indicated. The flow rates along the streets are measured as the average number of vehicles per hour.

a. Set up a linear system whose solution provides the unknown flow rates.

b. Solve the system for the unknown flow rates.

c. Is it possible to close the road from A to B for construction and keep traffic flowing on the other streets? Explain.

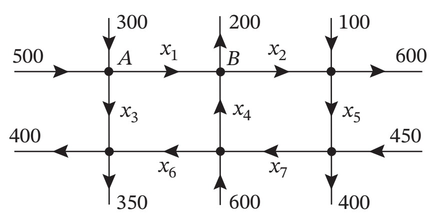
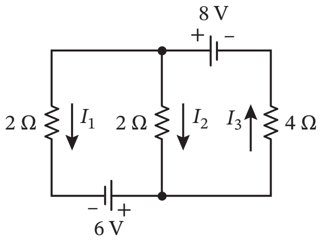


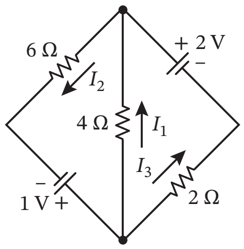
FIGURE Ex-4

In Exercises **5–8**, *analyze the given electrical circuits by finding the unknown currents.*

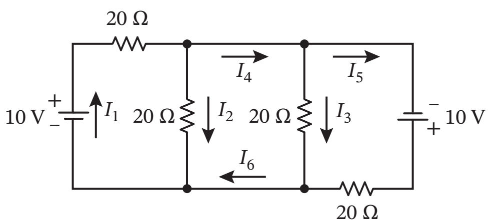
5.



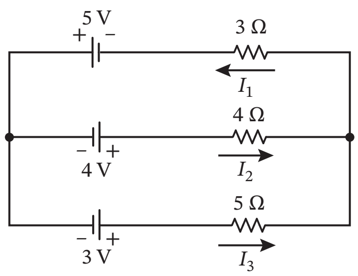
6.



7.



8.



In Exercises **9–12**, *write a balanced equation for the given chemical reaction.*

9. $C_3H_8 + O_2 \rightarrow CO_2 + H_2O$ [propane combustion]

10. $C_6H_{12}O_6 \rightarrow CO_2 + C_2H_5OH$ [fermentation of sugar]

11. $CH_3COF + H_2O \rightarrow CH_3COOH + HF$

12. $CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$ [photosynthesis]

13. Find the quadratic polynomial whose graph passes through the points $(1, 1)$, $(2, 2)$, and $(3, 5)$.

14. Find the quadratic polynomial whose graph passes through the points $(0, 0)$, $(-1, 1)$, and $(1, 1)$.

15. Find the cubic polynomial whose graph passes through the points $(-1, -1)$, $(0, 1)$, $(1, 3)$, $(4, -1)$.

16. The accompanying figure shows the graph of a cubic polynomial. Find the polynomial.

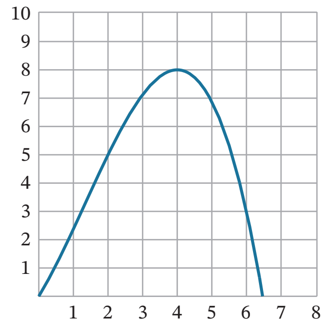


FIGURE Ex-16

17. a. Find an equation that represents the family of all second-degree polynomials that pass through the points $(0, 1)$ and $(1, 2)$. [*Hint:* The equation will involve one arbitrary parameter that produces the members of the family when varied.]

b. By hand, or with the help of a graphing utility, sketch four curves in the family.

18. In this section we have selected only a few applications of linear systems. Using the Internet as a search tool, try to find some more real-world applications of such systems. Select one that is of interest to you and write a paragraph about it.

True-False Exercises

TF. In parts **(a)–(e)** determine whether the statement is true or false, and justify your answer.

a. In any network, the sum of the flows out of a node must equal the sum of the flows into a node.

b. When a current passes through a resistor, there is an increase in the electrical potential in a circuit.

c. Kirchhoff's current law states that the sum of the currents flowing into a node equals the sum of the currents flowing out of the node.

d. A chemical equation is called balanced if the total number of atoms on each side of the equation is the same.

e. Given any n points in the xy-plane, there is a unique polynomial of degree $n - 1$ or less whose graph passes through those points.

Working with Technology

T1. The following table shows the lifting force on an aircraft wing measured in a wind tunnel at various wind velocities. Model the data with an interpolating polynomial of degree 5, and use that polynomial to estimate the lifting force at 2000 ft/s.

Velocity (100 ft/s)	1	2	4	8	16	32
Lifting Force (100 lb)	0	3.12	15.86	33.7	81.5	123.0

T2. **(*Calculus required*)** Use the method of Example 7 to approximate the integral

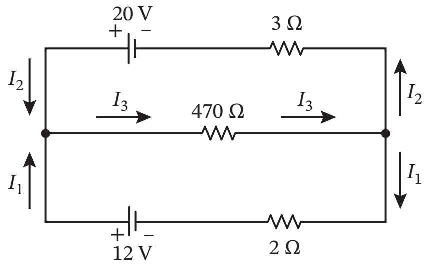
$$\int_0^1 e^{x^2}\,dx$$

by subdividing the interval of integration into five equal parts and using an interpolating polynomial to approximate the integrand. Compare your answer to that obtained using the numerical integration capability of your technology utility.

T3. Use the method of Example 5 to balance the chemical equation

$$Fe_2O_3 + Al \rightarrow Al_2O_3 + Fe$$
$$(Fe = \text{iron}, Al = \text{aluminum}, O = \text{oxygen})$$

T4. Determine the currents in the accompanying circuit.



1.11 Leontief Input-Output Models

In 1973 the economist Wassily Leontief was awarded the Nobel prize for his work on economic modeling in which he used matrix methods to study the relationships among different sectors in an economy. In this section we will discuss some of the ideas developed by Leontief.

Inputs and Outputs in an Economy

One way to analyze an economy is to divide it into ***sectors*** and study how the sectors interact with one another. For example, a simple economy might be divided into three sectors—manufacturing, agriculture, and utilities. Typically, a sector will produce certain ***outputs*** but will require ***inputs*** from the other sectors and itself. For example, the agricultural sector may produce wheat as an output but will require inputs of farm machinery from the manufacturing sector, electrical power from the utilities sector, and food from its own sector to feed its workers. Thus, we can imagine an economy to be a network in which inputs and outputs flow in and out of the sectors; the study of such flows is called ***input-output analysis***. Inputs and outputs are commonly measured in monetary units (dollars or millions of dollars, for example), but other units of measurement are also possible.

The flows between sectors of a real economy are not always obvious. For example, in World War II the United States had a demand for 50,000 new airplanes that required the construction of many new aluminum manufacturing plants. This produced an unexpectedly large demand for certain copper electrical components, which in turn produced a copper shortage. The problem was eventually resolved by using silver borrowed from Fort Knox as a copper substitute. In all likelihood modern input-output analysis would have anticipated the copper shortage.

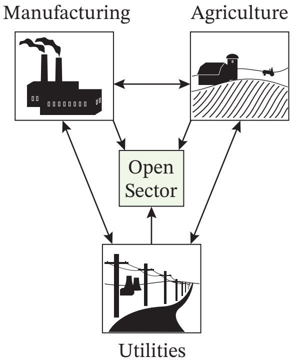


FIGURE 1.11.1

Most sectors of an economy will produce outputs, but there may exist sectors that consume outputs without producing anything themselves (the consumer market, for example). Those sectors that do not produce outputs are called ***open sectors***. Economies with no open sectors are called ***closed economies***, and economies with one or more open sectors are called ***open economies*** (**Figure 1.11.1**). In this section we will be concerned with economies with one open sector, and our primary goal will be to determine the output levels that are required for the productive sectors to sustain themselves and satisfy the demand of the open sector.

Leontief Model of an Open Economy

Let us consider a simple open economy with one open sector and three product-producing sectors: manufacturing, agriculture, and utilities. Assume that inputs and outputs are measured in dollars and that the inputs required by the productive sectors to produce one dollar's worth of output are in accordance with **Table 1**.

TABLE 1

		Input Required per Dollar Output		
		Manufacturing	**Agriculture**	**Utilities**
Provider	**Manufacturing**	\$ 0.50	\$ 0.10	\$ 0.10
	Agriculture	\$ 0.20	\$ 0.50	\$ 0.30
	Utilities	\$ 0.10	\$ 0.30	\$ 0.40

Usually, one would suppress the labeling and express this matrix as

$$C = \begin{bmatrix} 0.5 & 0.1 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.4 \end{bmatrix} \tag{1}$$

This is called the ***consumption matrix*** (or sometimes the ***technology matrix***) for the economy. The column vectors

$$\mathbf{c}_1 = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0.1 \\ 0.5 \\ 0.3 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.4 \end{bmatrix}$$

in C list the inputs required by the manufacturing, agricultural, and utilities sectors, respectively, to produce \$1.00 worth of output. These are called the ***consumption vectors*** of the sectors. For example, $\mathbf{c}_1$ tells us that to produce \$1.00 worth of output the manufacturing sector needs \$0.50 worth of manufacturing output, \$0.20 worth of agricultural output, and \$0.10 worth of utilities output.

What is the economic significance of the row sums of the consumption matrix?

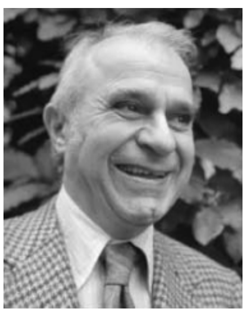
Continuing with the above example, suppose that the open sector wants the economy to supply it manufactured goods, agricultural products, and utilities with dollar values:

d_1 dollars of manufactured goods
d_2 dollars of agricultural products
d_3 dollars of utilities

The column vector $\mathbf{d}$ that has these numbers as successive components is called the ***outside demand vector***. Since the product-producing sectors consume some of their own output, the dollar value of their output must cover their own needs plus the outside demand. Suppose that the dollar values required to do this are

x_1 dollars of manufactured goods
x_2 dollars of agricultural products
x_3 dollars of utilities

Historical Note



Wassily Leontief (1906–1999)

It is somewhat ironic that it was the Russian-born Wassily Leontief who won the Nobel prize in 1973 for pioneering the modern methods for analyzing free-market economies. Leontief was a precocious student who entered the University of Leningrad at age 15. Bothered by the intellectual restrictions of the Soviet system, he was put in jail for anti-Communist activities, after which he headed for the University of Berlin, receiving his Ph.D. there in 1928. He came to the United States in 1931, where he held professorships at Harvard and then New York University.

[Image: © Bettmann/CORBIS]

The column vector $\mathbf{x}$ that has these numbers as successive components is called the ***production vector*** for the economy. For the economy with consumption matrix (1), that portion of the production vector $\mathbf{x}$ that will be consumed by the three productive sectors is

$$\underbrace{x_1\begin{bmatrix}0.5\\0.2\\0.1\end{bmatrix}}_{\boxed{\text{Fractions consumed by manufacturing}}} + \underbrace{x_2\begin{bmatrix}0.1\\0.5\\0.3\end{bmatrix}}_{\boxed{\text{Fractions consumed by agriculture}}} + \underbrace{x_3\begin{bmatrix}0.1\\0.3\\0.4\end{bmatrix}}_{\boxed{\text{Fractions consumed by utilities}}} = \begin{bmatrix}0.5 & 0.1 & 0.1\\0.2 & 0.5 & 0.3\\0.1 & 0.3 & 0.4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = C\mathbf{x}$$

The vector $C\mathbf{x}$ is called the ***intermediate demand vector*** for the economy. Once the intermediate demand is met, the portion of the production that is left to satisfy the outside demand is $\mathbf{x} - C\mathbf{x}$. Thus, if the outside demand vector is $\mathbf{d}$, then $\mathbf{x}$ must satisfy the equation

$$\underset{\boxed{\text{Amount produced}}}{\mathbf{x}} - \underset{\boxed{\text{Intermediate demand}}}{C\mathbf{x}} = \underset{\boxed{\text{Outside demand}}}{\mathbf{d}}$$

which we will find convenient to rewrite as

$$\boxed{(I - C)\mathbf{x} = \mathbf{d}} \tag{2}$$

The matrix $I - C$ is called the ***Leontief matrix*** and (2) is called the ***Leontief equation***.

EXAMPLE 1 | Satisfying Outside Demand

Consider the economy described in Table 1. Suppose that the open sector has a demand for \$7900 worth of manufacturing products, \$3950 worth of agricultural products, and \$1975 worth of utilities.

(*a*) Can the economy meet this demand?

(*b*) If so, find a production vector $\mathbf{x}$ that will meet it exactly.

Solution The consumption matrix, production vector, and outside demand vector are

$$C = \begin{bmatrix}0.5 & 0.1 & 0.1\\0.2 & 0.5 & 0.3\\0.1 & 0.3 & 0.4\end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix}7900\\3950\\1975\end{bmatrix} \tag{3}$$

To meet the outside demand, the vector $\mathbf{x}$ must satisfy the Leontief equation (2), so the problem reduces to solving the linear system

$$\underset{I-C}{\begin{bmatrix}0.5 & -0.1 & -0.1\\-0.2 & 0.5 & -0.3\\-0.1 & -0.3 & 0.6\end{bmatrix}}\underset{\mathbf{x}}{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}} = \underset{\mathbf{d}}{\begin{bmatrix}7900\\3950\\1975\end{bmatrix}} \tag{4}$$

(if consistent). We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & 27{,}500\\0 & 1 & 0 & 33{,}750\\0 & 0 & 1 & 24{,}750\end{array}\right]$$

This tells us that (4) is consistent, and the economy can satisfy the demand of the open sector exactly by producing \$27,500 worth of manufacturing output, \$33,750 worth of agricultural output, and \$24,750 worth of utilities output.

Productive Open Economies

In the preceding discussion we considered an open economy with three product-producing sectors; the same ideas apply to an open economy with n product-producing sectors. In this case, the consumption matrix, production vector, and outside demand vector have the form

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

where all entries are nonnegative and

$c_{ij} =$ the monetary value of the output of the ith sector that is needed by the jth sector to produce one unit of output

$x_i =$ the monetary value of the output of the ith sector

$d_i =$ the monetary value of the output of the ith sector that is required to meet the demand of the open sector

Remark Note that the jth column vector of C contains the monetary values that the jth sector requires of the other sectors to produce one monetary unit of output, and the ith row vector of C contains the monetary values required of the ith sector by the other sectors for each of them to produce one monetary unit of output.

As discussed in our example above, a production vector $\mathbf{x}$ that meets the demand $\mathbf{d}$ of the outside sector must satisfy the Leontief equation

$$(I - C)\mathbf{x} = \mathbf{d}$$

If the matrix $I - C$ is invertible, then this equation has the unique solution

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \tag{5}$$

for every demand vector $\mathbf{d}$. However, for $\mathbf{x}$ to be a valid production vector it must have nonnegative entries, so the problem of importance in economics is to determine conditions under which the Leontief equation has a solution with nonnegative entries.

It is evident from the form of (5) that if $I - C$ is invertible, and if $(I - C)^{-1}$ has nonnegative entries, then for every demand vector $\mathbf{d}$ the corresponding $\mathbf{x}$ will also have nonnegative entries, and hence will be a valid production vector for the economy. Economies for which $(I - C)^{-1}$ has nonnegative entries are said to be ***productive***. Such economies are desirable because demand can always be met by some level of production. The following theorem, whose proof can be found in many books on economics, gives conditions under which open economies are productive.

Theorem 1.11.1

If C is the consumption matrix for an open economy, and if all of the column sums are less than 1, then the matrix $I - C$ is invertible, the entries of $(I - C)^{-1}$ are nonnegative, and the economy is productive.

Remark The jth column sum of C represents the total dollar value of input that the jth sector requires to produce \$1 of output, so if the jth column sum is less than 1, then the jth sector requires less than \$1 of input to produce \$1 of output; in this case we say that the jth sector is ***profitable***. Thus, Theorem 1.11.1 states that if all product-producing sectors of an open economy are profitable, then the economy is productive. In the exercises we will ask you to show that an open economy is productive if all of the row sums of C are less than 1 (Exercise 11). Thus, an open economy is productive if *either* all of the column sums or all of the row sums of C are less than 1.

EXAMPLE 2 | An Open Economy Whose Sectors Are All Profitable

The column sums of the consumption matrix C in (1) are less than 1, so $(I - C)^{-1}$ exists and has nonnegative entries. Use a calculating utility to confirm this, and use this inverse to solve Equation (4) in Example 1.

Solution We leave it for you to show that

$$(I - C)^{-1} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix}$$

This matrix has nonnegative entries, and

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} \approx \begin{bmatrix} 2.65823 & 1.13924 & 1.01266 \\ 1.89873 & 3.67089 & 2.15190 \\ 1.39241 & 2.02532 & 2.91139 \end{bmatrix}\begin{bmatrix} 7900 \\ 3950 \\ 1975 \end{bmatrix} \approx \begin{bmatrix} 27{,}500 \\ 33{,}750 \\ 24{,}750 \end{bmatrix}$$

which is consistent with the solution in Example 1.

Exercise Set 1.11

1. An automobile mechanic (M) and a body shop (B) use each other's services. For each \$1.00 of business that M does, it uses \$0.50 of its own services and \$0.25 of B's services, and for each \$1.00 of business that B does it uses \$0.10 of its own services and \$0.25 of M's services.

   a. Construct a consumption matrix for this economy.

   b. How much must M and B each produce to provide customers with \$7000 worth of mechanical work and \$14,000 worth of body work?

2. A simple economy produces food (F) and housing (H). The production of \$1.00 worth of food requires \$0.30 worth of food and \$0.10 worth of housing, and the production of \$1.00 worth of housing requires \$0.20 worth of food and \$0.60 worth of housing.

   a. Construct a consumption matrix for this economy.

   b. What dollar value of food and housing must be produced for the economy to provide consumers \$130,000 worth of food and \$130,000 worth of housing?

3. Consider the open economy described by the accompanying table, where the input is in dollars needed for \$1.00 of output.

   a. Find the consumption matrix for the economy.

   b. Suppose that the open sector has a demand for \$1930 worth of housing, \$3860 worth of food, and \$5790 worth of utilities. Use row reduction to find a production vector that will meet this demand exactly.

TABLE Ex-3

Input Required per Dollar Output

Provider	Housing	Food	Utilities
Housing	\$ 0.10	\$ 0.60	\$ 0.40
Food	\$ 0.30	\$ 0.20	\$ 0.30
Utilities	\$ 0.40	\$ 0.10	\$ 0.20

4. A company produces Web design, software, and networking services. View the company as an open economy described by the accompanying table, where input is in dollars needed for \$1.00 of output.

   a. Find the consumption matrix for the company.

   b. Suppose that the customers (the open sector) have a demand for \$5400 worth of Web design, \$2700 worth of software, and \$900 worth of networking. Use row reduction to find a production vector that will meet this demand exactly.

TABLE Ex-4

Input Required per Dollar Output

Provider	Web Design	Software	Networking
Web Design	\$ 0.40	\$ 0.20	\$ 0.45
Software	\$ 0.30	\$ 0.35	\$ 0.30
Networking	\$ 0.15	\$ 0.10	\$ 0.20

In Exercises 5–6, use matrix inversion to find the production vector **x** *that meets the demand* **d** *for the consumption matrix* C.

5. $C = \begin{bmatrix} 0.1 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}$; $\mathbf{d} = \begin{bmatrix} 50 \\ 60 \end{bmatrix}$

6. $C = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 0.7 \end{bmatrix}$; $\mathbf{d} = \begin{bmatrix} 22 \\ 14 \end{bmatrix}$

7. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

a. Show that the economy can meet a demand of $d_1 = 2$ units from the first sector and $d_2 = 0$ units from the second sector, but it cannot meet a demand of $d_1 = 2$ units from the first sector and $d_2 = 1$ unit from the second sector.

b. Give both a mathematical and an economic explanation of the result in part (a).

8. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}$$

If the open sector demands the same dollar value from each product-producing sector, which such sector must produce the greatest dollar value to meet the demand?

9. Consider an open economy with consumption matrix

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & 0 \end{bmatrix}$$

Show that the Leontief equation $\mathbf{x} - C\mathbf{x} = \mathbf{d}$ has a unique solution for every demand vector **d** if $c_{21}c_{12} < 1 - c_{11}$.

Working with Proofs

10. a. Consider an open economy with a consumption matrix C whose column sums are less than 1, and let **x** be the production vector that satisfies an outside demand **d**; that is, $(I - C)^{-1}\mathbf{d} = \mathbf{x}$. Let $\mathbf{d}_j$ be the demand vector that is obtained by increasing the jth entry of **d** by 1 and leaving the other entries fixed. Prove that the production vector $\mathbf{x}_j$ that meets this demand is

$$\mathbf{x}_j = \mathbf{x} + j\text{th column vector of } (I - C)^{-1}$$

b. In words, what is the economic significance of the jth column vector of $(I - C)^{-1}$? [*Hint:* Look at $\mathbf{x}_j - \mathbf{x}$.]

11. Prove: If C is an $n \times n$ matrix whose entries are nonnegative and whose row sums are less than 1, then $I - C$ is invertible and has nonnegative entries. [*Hint:* $(A^T)^{-1} = (A^{-1})^T$ for any invertible matrix A.]

True-False Exercises

TF. In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

a. Sectors of an economy that produce outputs are called open sectors.

b. A closed economy is an economy that has no open sectors.

c. The rows of a consumption matrix represent the outputs in a sector of an economy.

d. If the column sums of the consumption matrix are all less than 1, then the Leontief matrix is invertible.

e. The Leontief equation relates the production vector for an economy to the outside demand vector.

Working with Technology

T1. The following table describes an open economy with three sectors in which the table entries are the dollar inputs required to produce one dollar of output. The outside demand during a 1-week period if \$50,000 of coal, \$75,000 of electricity, and \$1,250,000 of manufacturing. Determine whether the economy can meet the demand.

Input Required per Dollar Output

		Electricity	Coal	Manufacturing
Provider	Electricity	\$ 0.1	\$ 0.25	\$ 0.2
	Coal	\$ 0.3	\$ 0.4	\$ 0.5
	Manufacturing	\$ 0.1	\$ 0.15	\$ 0.1

Chapter 1 Supplementary Exercises

In Exercises 1–4 the given matrix represents an augmented matrix for a linear system. Write the corresponding set of linear equations for the system, and use Gaussian elimination to solve the linear system. Introduce free parameters as necessary.

1. $\begin{bmatrix} 3 & -1 & 0 & 4 & 1 \\ 2 & 0 & 3 & 3 & -1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 4 & -1 \\ -2 & -8 & 2 \\ 3 & 12 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -4 & 1 & 6 \\ -4 & 0 & 3 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 1 & -2 \\ -9 & -3 & 6 \\ 6 & 2 & 1 \end{bmatrix}$

5. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$x = \tfrac{3}{5}x' - \tfrac{4}{5}y'$$
$$y = \tfrac{4}{5}x' + \tfrac{3}{5}y'$$

6. Use Gauss–Jordan elimination to solve for x' and y' in terms of x and y.

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

7. Find positive integers that satisfy

$$\begin{aligned} x + y + z &= 9 \\ x + 5y + 10z &= 44 \end{aligned}$$

8. A box containing pennies, nickels, and dimes has 13 coins with a total value of 83 cents. How many coins of each type are in the box? Is the economy productive?

9. Let

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix}$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

a. a unique solution.

b. a one-parameter solution.

c. a two-parameter solution. **d.** no solution.

10. For which value(s) of a does the following system have zero solutions? One solution? Infinitely many solutions?

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ x_3 &= 2 \\ (a^2 - 4)x_3 &= a - 2 \end{aligned}$$

11. Find a matrix K such that $AKB = C$ given that

$$A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$$

12. How should the coefficients a, b, and c be chosen so that the system

$$\begin{aligned} ax + by - 3z &= -3 \\ -2x - by + cz &= -1 \\ ax + 3y - cz &= -3 \end{aligned}$$

has the solution $x = 1$, $y = -1$, and $z = 2$?

13. In each part, solve the matrix equation for X.

a. $X\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}$

b. $X\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}X - X\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$

14. Let A be a square matrix.

a. Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$.

b. Show that

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^n$$

if $A^{n+1} = 0$.

15. Find values of a, b, and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

16. ***(Calculus required)*** Find values of a, b, and c such that the graph of $p(x) = ax^2 + bx + c$ passes through the point $(-1, 0)$ and has a horizontal tangent at $(2, -9)$.

17. Let J_n be the $n \times n$ matrix each of whose entries is 1. Show that if $n > 1$, then

$$(I - J_n)^{-1} = I - \frac{1}{n-1}J_n$$

18. Show that if a square matrix A satisfies

$$A^3 + 4A^2 - 2A + 7I = 0$$

then so does A^T.

19. Prove: If B is invertible, then $AB^{-1} = B^{-1}A$ if and only if $AB = BA$.

20. Prove: If A is invertible, then $A + B$ and $I + BA^{-1}$ are both invertible or both not invertible.

21. Prove: If A is an $m \times n$ matrix and B is the $n \times 1$ matrix each of whose entries is $1/n$, then

$$AB = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_m \end{bmatrix}$$

where $\bar{r}_i$ is the average of the entries in the ith row of A.

22. ***(Calculus required)*** If the entries of the matrix

$$C = \begin{bmatrix} c_{11}(x) & c_{12}(x) & \cdots & c_{1n}(x) \\ c_{21}(x) & c_{22}(x) & \cdots & c_{2n}(x) \\ \vdots & \vdots & & \vdots \\ c_{m1}(x) & c_{m2}(x) & \cdots & c_{mn}(x) \end{bmatrix}$$

are differentiable functions of x, then we define

$$\frac{dC}{dx} = \begin{bmatrix} c'_{11}(x) & c'_{12}(x) & \cdots & c'_{1n}(x) \\ c'_{21}(x) & c'_{22}(x) & \cdots & c'_{2n}(x) \\ \vdots & \vdots & & \vdots \\ c'_{m1}(x) & c'_{m2}(x) & \cdots & c'_{mn}(x) \end{bmatrix}$$

Show that if the entries in A and B are differentiable functions of x and the sizes of the matrices are such that the stated operations can be performed, then

a. $\dfrac{d}{dx}(kA) = k\dfrac{dA}{dx}$

b. $\dfrac{d}{dx}(A + B) = \dfrac{dA}{dx} + \dfrac{dB}{dx}$

c. $\dfrac{d}{dx}(AB) = \dfrac{dA}{dx}B + A\dfrac{dB}{dx}$

23. ***(Calculus required)*** Use part (c) of Exercise 22 to show that

$$\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$$

State all the assumptions you make in obtaining this formula.

24. Assuming that the stated inverses exist, prove the following equalities.

a. $(C^{-1} + D^{-1})^{-1} = C(C + D)^{-1}D$

b. $(I + CD)^{-1}C = C(I + DC)^{-1}$

c. $(C + DD^T)^{-1}D = C^{-1}D(I + D^TC^{-1}D)^{-1}$

Partitioned matrices can be multiplied by the row-column rule just as if the matrix entries were numbers provided that the sizes of all matrices are such that the necessary operations can be performed. Thus, for example, if A is partitioned into a 2×2 matrix and B into a 2×1 matrix, then

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} \qquad (*)$$

provided that the sizes are such that AB, the two sums, and the four products are all defined.

25. Let A and B be the following partitioned matrices.

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 2 & 1 & 4 \\ 4 & 1 & 0 & 3 & -1 \\ \hline 0 & -3 & 4 & 2 & -2 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{cc} 3 & 0 \\ 2 & 1 \\ 4 & -1 \\ \hline 0 & 3 \\ 2 & 5 \end{array}\right] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

a. Confirm that the sizes of all matrices are such that the product AB can be obtained using Formula $(*)$.

b. Confirm that the result obtained using Formula $(*)$ agrees with that obtained using ordinary matrix multiplication.

26. Suppose that an invertible matrix A is partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Show that

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where

$$B_{11} = (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}, \qquad B_{12} = -B_{11}A_{12}A_{22}^{-1}$$
$$B_{21} = -A_{22}^{-1}A_{21}B_{11}, \qquad B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}$$

provided all the inverses in these formulas exist.

27. In the special case where matrix A_{21} in Exercise 26 is zero, the matrix A simplifies to

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

which is said to be in ***block upper triangular form***. Use the result of Exercise 26 to show that in this case

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

28. A linear system whose coefficient matrix has a pivot position in every row must be consistent. Explain why this must be so.

29. What can you say about the consistency or inconsistency of a linear system of three equations in five unknowns whose coefficient matrix has three pivot columns?