1

Structural Stability

1.1 Introduction

Structures fail mainly either due to material failure or because of buckling or structural instability. Material failures are governed by the material strength that may be the ultimate strength or the yield point strength of the material. The failure due to structural instability depends on the structural geometry, size, and its stiffness. It does not depend on the strength of the material. It is important to understand the failure due to structural instability, because using a higher strength material will not prevent this type of failure. More and more structures are failing because of stability problems because of the present trend to use high strength materials and large structures. The increase in size increases the slenderness ratio of the members of a structure, and these members reach their stability limit before their material strength. A look at different design codes makes it clear that in many situations the maximum force a system can support is governed by structural instability than by material strength.

An interesting question to ask is, if the material strength is not exceeded, then why does the member fail?. The answer may be that all systems take the path of least resistance when they deform, a basic law of nature. For slender members, it is easier to bend than to shorten under a compressive force resulting in the buckling of the member before it fails by exceeding its material strength. For short members it is easier to shorten than to bend under a compressive force. In practice, there is always a tendency of a slender member to bend sideways even if the intended force is an axial compression. This tendency is due to small accidental eccentricity, unintended lateral disturbing force, imperfections, or other irregularities in the member. For small compressive forces the internal resistance of a member to bending exceeds external action forcing it to bend. As the external forces increase, a limiting load is reached where their overturning effect to bend exceeds the internal resistance to bending of the member. As a result, more and more bending of the system called buckling occurs. The maximum compressive force at which the member can remain in equilibrium in the straight configuration without bending is called the buckling load. A system is called stable if small disturbances cause small deformations of the system configuration. Displaced shape equilibrium and the energy methods are the two most commonly used procedures to solve the buckling loads problem and to study the stability of equilibrium.

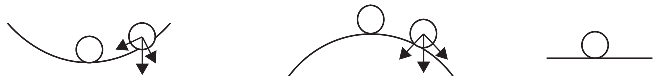

Figure 1.1 Types of equilibrium: (a) stable; (b) unstable; (c) neutral.

1.2 General Concepts

Concepts of stability can be explained by considering the equilibrium of a ball resting on three different surfaces [1] shown in Figure 1.1. The ball on the concave surface in Figure 1.1a is in stable equilibrium because any small displacement will increase the potential energy of the ball. The component of the self-weight parallel to the sliding surface will bring the ball back to its original equilibrium position. In Figure 1.1b, the ball rests on a convex surface, a small displacement from its equilibrium position will decrease the potential energy of the ball. The parallel component of the self-weight will slide the ball further from its initial configuration, and the equilibrium is unstable. If the ball is displaced on the flat surface, the potential energy of the ball remains the same, and the ball assumes a new equilibrium position. Thus, potential energy, Π, is a minimum for stable equilibrium, whereas it is a maximum for the unstable equilibrium position, and the potential energy remains the same for the position of neutral equilibrium. Energy methods are based on these concepts for solving the structural stability problems. If $\Delta\Pi > 0$, the displaced configuration is stable, whereas for $\Delta\Pi < 0$, the displaced shape is in unstable equilibrium, the transition $\Delta\Pi = 0$, which is the position of neutral equilibrium gives critical load at which the system becomes unstable by energy method.

Also, since we are studying the state of equilibrium in the slightly displaced position of the body, the equilibrium equations are written based on the displaced shape of the body in the displaced shape equilibrium method. Both methods can be used to formulate the equilibrium equations and calculate the critical loads. However, the displaced equilibrium approach does not give the nature of equilibrium when the critical load is reached. To answer that question, the second variation of potential energy $\delta^2\Pi$ is to be considered. The potential energy may be expanded into a Taylor series about the equilibrium state and written as

$$\Delta\Pi = \delta\Pi + \delta^2\Pi + \delta^3\Pi + \text{-----} \tag{1.1a}$$

where

$$\delta\Pi = \sum_{i=1}^{n} \frac{\partial \Pi}{\partial q_i}\delta q_i \tag{1.1b}$$

$$\delta^2\Pi = \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \Pi}{\partial q_i \partial q_j}\delta q_i \delta q_j \tag{1.1c}$$

$$\delta^3\Pi = \frac{1}{3!}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^3 \Pi}{\partial q_i \partial q_j \partial q_k}\delta q_i \delta q_j \delta q_k \tag{1.1d}$$

$\delta\Pi$, $\delta^2\Pi$, and $\delta^3\Pi$ are called the first, second and third derivatives respectively of the potential energy Π. The critical load P_{cr} is obtained from the conditions of equilibrium given by $\delta\Pi = 0$

for any δq_i, or $\frac{\partial \prod}{\partial q_i} = 0$ for each i [2]. The equilibrium state is stable if $\Delta\Pi > 0$. Therefore, the equilibrium state is stable for $\delta^2\Pi > 0$, and is unstable for $\delta^2\Pi < 0$.

Because energy is quadratic, it can also be written as

$$2\prod = \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} q_i q_j = \boldsymbol{q}^T \boldsymbol{K} \boldsymbol{q} \tag{1.1e}$$

where

$\boldsymbol{q}$ = column vector of the generalized displacements
$\boldsymbol{q}^T$ = transpose of the column vector
$\boldsymbol{K}$ = square matrix (n × n) with elements K_{ij}

For elastic structures, matrix $\boldsymbol{K}$ represents the stiffness matrix of the structure with regard to its generalized displacements, and Π is the potential energy. The stiffness elements are given by

$$K_{ij} = \frac{\partial^2 \prod}{\partial q_i \partial q_j} = \frac{\partial^2 \prod}{\partial q_j \partial q_i} = K_{ji} \tag{1.1f}$$

That shows the stiffness matrix is symmetric. The second variation of the potential energy from Eq. (1.1c) is

$$2\delta^2\prod = \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} \delta q_i \delta q_j \tag{1.1g}$$

For $\delta^2\Pi > 0$, the matrix with elements K_{ij} will be positive definite. A real symmetric matrix is positive definite if and only if all its principal minors are positive, that is,

$$D_1 = K_{11} > 0, \quad D_2 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} > 0, \text{------------} D_n = \begin{bmatrix} K_{11} & - & - & K_{1n} \\ - & - & - & - \\ - & - & - & - \\ K_{n1} & - & - & K_{nn} \end{bmatrix} > 0 \tag{1.1h}$$

or

$$D_1 = K_{11} > 0, \quad |D_2| = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} > 0, \text{-----------} D_n = \begin{vmatrix} K_{11} & - & - & K_{1n} \\ - & - & - & - \\ - & - & - & - \\ K_{n1} & - & - & K_{nn} \end{vmatrix} > 0 \tag{1.1i}$$

When systems are subjected to compressive forces three types of instabilities can occur: (i) bifurcation of equilibrium; (ii) maximum or limit load instabilities; and (iii) Finite disturbance instability.

1.2.1 Bifurcation of Equilibrium

Equilibrium paths are shown as load displacement plots in Figure 1.2. The equilibrium path starting from the unloaded configuration is called the fundamental or primary path. At a certain load the equilibrium path can continue to be the fundamental path or it could change to an alternate configuration if there is a small lateral perturbation. This alternate path is called

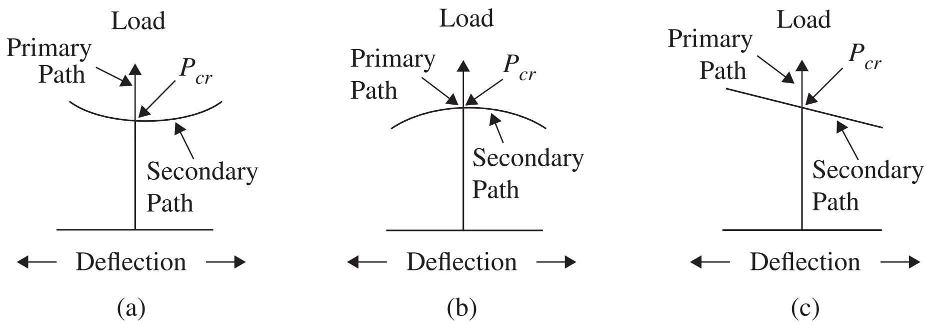

Figure 1.2 Bifurcation equilibrium paths: (a) Symmetric stable bifurcation; (b) Symmetric unstable bifurcation; (c) Asymmetric bifurcation.

the secondary or post-buckling path [3]. The point of intersection between the primary and secondary paths is called the point of bifurcation, and the load corresponding to this point is called the critical load. In Figures 1.2a and 1.2b, the secondary paths are symmetrical. In the symmetric bifurcation the post-buckling load deflection behavior remains the same irrespective of the direction in which the structure bends. It is a stable bifurcation in Figure 1.2a because the load increases with deflection after buckling, axially loaded columns and thin plates subjected to in-plane forces exhibit this behavior. The load decreases below the critical as the deflection increases in the post-buckling stage in Figure 1.2b, and the structure has an unstable bifurcation at the critical load. Guyed towers exhibit this behavior because some of the cables come under compression and are unable to sustain the external forces. If the post-buckling load deflection diagram is affected by the direction of buckling, then the bifurcation is asymmetric as shown in Figure 1.2c. Some framed structures show this kind of behavior.

1.2.2 Limit Load Instability

This type of instability is also called snap-through buckling. In this type of buckling, the primary path is nonlinear and once the load reaches a maximum, the point P in Figure 1.3a jumps to Q on another branch of the curve. The load at point P is the critical load in this type of instability. The structure snaps through to a nonadjacent equilibrium position represented by point Q. Spherical caps and shallow arches exhibit this behavior.

1.2.3 Finite Disturbance Instability

This type of instability occurs in cylindrical shells under the action of axial forces shown in Figure 1.3b. The load capacity of the structure drops suddenly at the critical load in Figure 1.3c. The structure takes a non-cylindrical shape after the critical load. The structure continues to take more axial compression in Figure 1.3c after taking another equilibrium configuration. In this type of instability, a finite disturbance of the cylinder or imperfection in the cylinder will lower the critical load considerably and the structure will change equilibrium configuration upon reaching the ideal critical load.

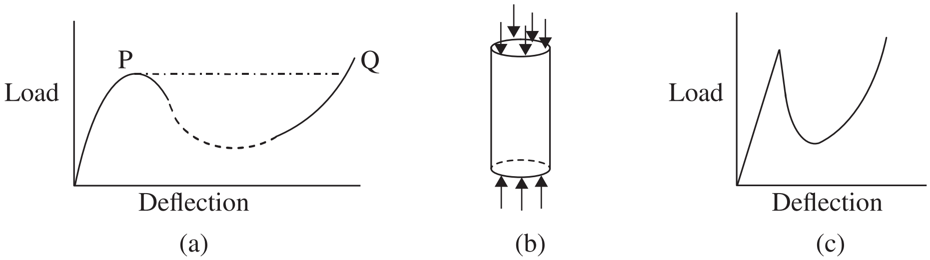

Figure 1.3 Post-buckling equilibrium paths: (a) Limit load instability; (b) Cylindrical shell under axial compression; (c) Finite difference instability.

1.3 Rigid Bar Columns

Columns consisting of rigid bars supported by springs and acted on by axial compression are studied by the displaced shape equilibrium, or by energy methods. At first, the small deflection analysis is considered. The study of rigid bar columns provides a good background on the nature of stability problems and the different methods used to solve them because these systems have limited degrees of freedom.

1.3.1 Rigid Bar Supported by a Translational Spring

1.3.1.1 The Displaced Shape Equilibrium Method

Consider a perfect rigid vertical column supported by a hinge at the bottom and a linear spring of stiffness "k" at the top. The bar is acted on by an axial load shown in Figure 1.4. If there is an accidental lateral disturbance, the spring force, $kL \sin\theta$, will bring it back to the vertical position for small axial loads. In this case the restoring moment due to spring force is larger than the overturning moment due to the force P as shown in Eq. (1.2a):

$$k L^2 \sin\theta \cos\theta > PL \sin\theta \tag{1.2a}$$

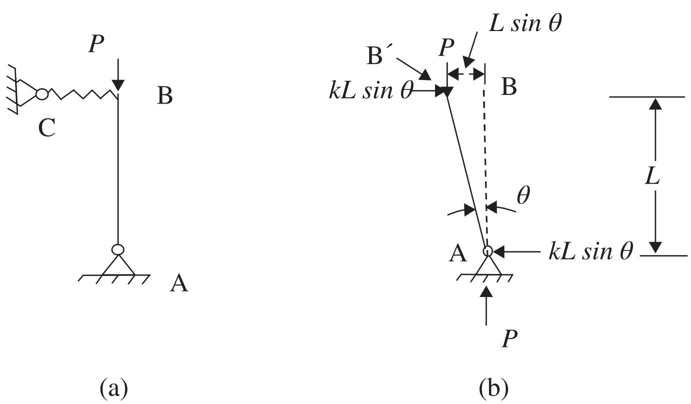

Figure 1.4 Rigid bar under axial force: (a) Rigid bar with axial load; (b) Free-body diagram of displaced shape.

and the vertical position of the bar is stable. The spring force will not be able to bring back the rigid bar to its vertical position for large axial force, because the overturning moment will be larger than the restoring moment shown below

$$k\,L^2 \sin\theta\ \cos\ \theta < PL\ \sin\theta \tag{1.2b}$$

and the vertical position of the bar is unstable. The minimum axial force at which the bar becomes unstable is called the critical load. It is the force at which the equilibrium changes from stable to unstable, and

$$k\,L^2 \sin\theta\ \cos\ \theta = PL\ \sin\theta$$
$$or\ \mathrm{P} = kL\ \cos\theta \tag{1.2c}$$

The critical load, $\mathrm{P_{cr}}$, can be found by considering the equilibrium of the slightly displaced position of the bar by taking moments of all forces about A in Figure 1.4b as follows:

$$\sum M_A = 0$$
$$PL\ \sin\theta - kL^2 \sin\theta\ \cos\theta = 0$$
$$\text{or } P = kL\ \cos\theta \tag{1.2d}$$

The same result is obtained from Eqs. (1.2c and 1.2d), hence the critical load can be found by considering the equilibrium of the slightly displaced shape. For small deflections, $\cos\theta \approx 1$, therefore,

$$P_{cr} = kL \tag{1.2e}$$

1.3.1.2 The Energy Method

The first law of thermodynamics can be used to derive equations used in the energy method. This law, which is a statement of the law of conservation of energy, can be stated as "The work that is performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase of kinetic energy plus the increase of internal energy."

$$W_e + Q = \Delta T + \Delta U \tag{1.3a}$$

Here, W_e, is the work performed on the system by the external forces, Q is the heat that flows into the system, ΔT is the increase of kinetic energy, and ΔU is the increase of internal energy [4]. For an adiabatic change, $Q = 0$, and for a body in equilibrium, $\Delta T = 0$. This reduces Eq. (1.3a) to

$$W_e = \Delta U \tag{1.3b}$$

The change in internal energy of an elastic body is determined by the strains, and is called the strain energy. If the system is subjected to conservative forces, W_e is independent of the path the system takes from the configuration X_0 to another configuration X. In this case, the W_e depends only on the two terminal configurations, and is denoted by $-V(X_0, X)$. The function $V(X_0, X)$

is called the potential energy of the external forces, and it is always measured as the change in the potential energy, ΔV, from one configuration to another configuration of the system.

$$W_e = -\Delta V \tag{1.3c}$$

Equations (1.3b and 1.3c) can be combined to write

$$\Delta V + \Delta U = 0 \tag{1.3d}$$

or

$$\Delta(V + U) = 0 \tag{1.3e}$$

V is the potential energy due to external forces, and U is considered the potential energy of the internal forces. Total potential energy of the system is

$$\Pi = V + U \tag{1.3f}$$

The total potential energy of a system is a minimum in the position of stable equilibrium, whereas it is a maximum for unstable equilibrium. The critical load can be obtained by equating the first derivative of the total potential energy equal to zero. In Figure 1.4

$$U = \tfrac{1}{2}k(L\ \sin\theta)^2 \tag{1.3g}$$
$$V = -P\,L(1 - \cos\theta) \tag{1.3h}$$
$$\Pi = -P\,L(1 - \cos\theta) + \tfrac{1}{2}k(L\ \sin\theta)^2 \tag{1.3i}$$
$$\frac{d\Pi}{d\theta} = -PL\ \sin\theta + k\,L^2\sin\theta\ \cos\theta \tag{1.3j}$$

Substituting $\frac{d\Pi}{d\theta} = 0$, we get

$$P = kL\cos\theta \tag{1.3k}$$

$$\cos\theta \approx 1 \text{ for small values of } \theta$$

or

$$P_{cr} = kL \tag{1.3l}$$

giving the same critical load as in Eq. (1.2e). From Eq. (1.3j)

$$\frac{d^2\Pi}{d\theta^2} = -PL\cos\theta + kL^2\cos 2\theta \tag{1.3m}$$

For the initial position,

$$\theta = 0,\ \frac{d^2\Pi}{d\theta^2} = -PL + kL^2 \tag{1.3n}$$

For $P < P_{cr}$, $\frac{d^2\Pi}{d\theta^2} > 0$, and for $P > P_{cr}$, $\frac{d^2\Pi}{d\theta^2} < 0$ in Eq. (1.3n). So the system is in stable equilibrium if $P < P_{cr}$, and is in unstable equilibrium for $P > P_{cr}$, in the initial position.

1.3.2 Two Rigid Bars Connected by Rotational Springs

1.3.2.1 The Displaced Shape Equilibrium Method

Consider two rigid bars as shown in Figure 1.5. The lower bar is connected to a pin support and a linear rotational spring of stiffness c_1 at the bottom. At the top the lower bar is connected to another bar by a linear rotational spring of stiffness c_2. The upper bar is free at the top, and the bars are subjected to an axial force of P.

Taking the equilibrium of the lower bar in Figure 1.5c, the sum of the moments of all forces about A is equal to zero,

$$c_1\theta_1 - c_2\,(\theta_2 - \theta_1) - PL_1 \sin\theta_1 = 0 \tag{1.4a}$$

From Figure 1.5d, sum the moments of all the forces about B and equate it to zero,

$$c_2(\theta_2 - \theta_1) - PL_2 \sin\theta_2 = 0 \tag{1.4b}$$

$\sin\theta \approx \theta$ in radians for small values of θ, and Eqs. (1.4a and 1.4b) can be written in the matrix form as

$$\begin{bmatrix} c_1 + c_2 - PL & -c_2 \\ -c_2 & c_2 - PL_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.4c}$$

$$\text{or} \left\{ \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} - P \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \right\} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.4d}$$

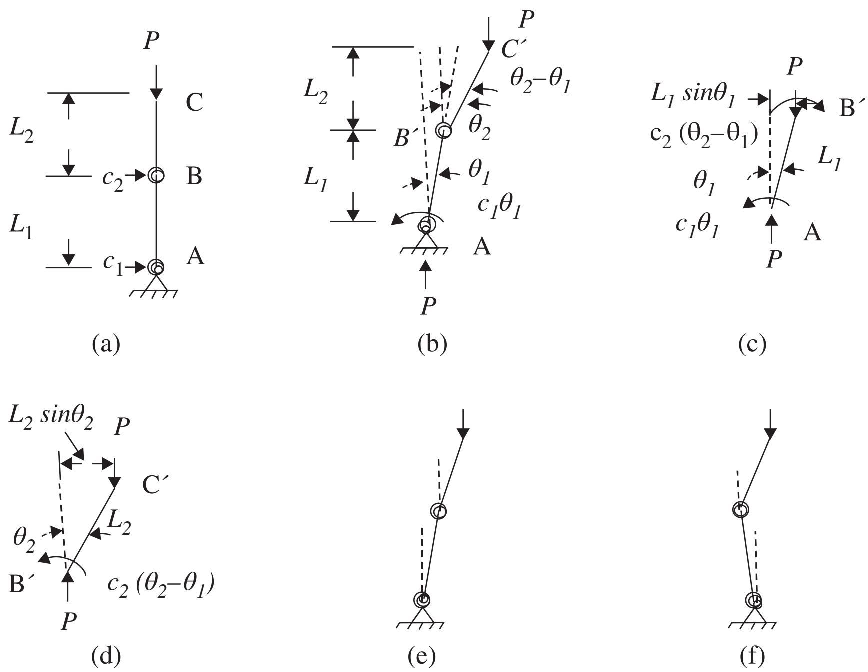

Figure 1.5 Two rigid bars under axial load: (a) Two rigid bars with axial force; (b) Displaced shape; (c) Free body diagram of lower bar; (d) Free body diagram of upper bar; (e) First buckling mode; (f) Second buckling mode.

Equation (1.4d) is an eigenvalue problem. The critical loads P are the eigenvalues, and the angular displacements, θ_1 and θ_2 are given as eigenvectors. For a nontrivial solution the determinant of the coefficient matrix is zero [5],

$$\begin{vmatrix} c_1 + c_2 - PL_1 & -c_2 \\ c_2 & c_2 - PL_2 \end{vmatrix} = 0 \tag{1.4e}$$

or

$$(c_1 + c_2 - PL_1)(c_2 - PL_2) - (c_2)^2 = 0$$

or

$$P^2 - P\left(\frac{c_1}{L_1} + \frac{c_2}{L_2} + \frac{c_2}{L_1}\right) + \frac{c_1 c_2}{L_1 L_2} = 0 \tag{1.4f}$$

The solution of the quadratic Eq. (1.4f) is given by

$$P = \frac{\frac{c_1}{L_1} + \frac{c_2}{L_2} + \frac{c_2}{L_1} \pm \sqrt{\left(\frac{c_1}{L_1} + \frac{c_2}{L_2} + \frac{c_2}{L_1}\right)^2 - 4\left(\frac{c_1 c_2}{L_1 L_2}\right)}}{2} \tag{1.4g}$$

If $c_1 = c_2 = c$, and $L_1 = L_2 = L$

$$P = \frac{\frac{3c}{L} \pm \sqrt{5}\frac{c}{L}}{2}$$

$$P = 0.382\frac{c}{L}, \quad \text{or } 2.618\frac{c}{L} \tag{1.4h}$$

The corresponding eigenvectors are:

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix} \quad \text{and}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix} \tag{1.4i}$$

1.3.2.2 The Energy Method

The critical load for the two rigid bars shown in Figure 1.5a subjected to an axial force P can be found by using the principle of stationery potential energy. The strain energy of the system in the displaced shape is given by

$$\mathrm{U} = {}^1\!/\!_2 c_1(\theta_1)^2 + {}^1\!/\!_2 c_2(\theta_2 - \theta_1)^2 \tag{1.5a}$$

The potential energy of the external force P is

$$V = -P\left[\mathrm{L}_1(1 - \cos\theta_1) + L_2(1 - \cos\theta_2)\right] \tag{1.5b}$$

Total potential energy of the system is

$$\Pi = {}^1\!/\!_2 c_1(\theta_1)^2 + {}^1\!/\!_2 c_2(\theta_2 - \theta_1)^2 - P\left[L_1(1 - \cos\theta_1) + L_2(1 - \cos\theta_2)\right] \tag{1.5c}$$

The potential energy of the system must be stationary for equilibrium. The first derivatives of the potential energy function, Π, with respect to θ_1 and θ_2 are:

$$\frac{\partial\Pi}{\partial\theta_1} = c_1\theta_1 + c_2(\theta_2 - \theta_1)(-1) - PL_1\sin\theta_1 = 0 \tag{1.5d}$$

$$\frac{\partial \Pi}{\partial \theta_2} = c_2(\theta_2 - \theta_1) - PL_2 \sin \theta 2 = 0 \tag{1.5e}$$

For small values of θ, $\sin\theta \approx \theta$ in radians, and Eqs. (1.5d and 1.5e) can be written in matrix form as:

$$\begin{bmatrix} c_1 + c_2 - PL_1 & -c_2 \\ -c_2 & c_2 - PL_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.5f}$$

Equations (1.4c and 1.5f) are the same, giving the same solution for the critical load, P_{cr}, by the energy method as given before by the displaced shape equilibrium method. For $c_1 = c_2 = c$, and $L_1 = L_2 = L$, from Eqs. (1.5d and 1.5e)

$$\frac{\partial^2 \prod}{\partial {\theta_1}^2} = c + c + -PL \cos \theta_1 \tag{1.5g}$$

$$\frac{\partial^2 \prod}{\partial {\theta_2}^2} = c - PL \cos \theta_2 \tag{1.5h}$$

$$\frac{\partial^2 \prod}{\partial \theta_1 \partial \theta_2} = -c \tag{1.5i}$$

For the two degrees of freedom systems from Eq. (1.1g),

$$2\delta^2\Pi = K_{11}\delta {q_1}^2 + 2K_{12}\delta q_1 \delta q_2 + K_{22}\delta {q_2}^2 \tag{1.5j}$$

For the initial position, $\theta_1 = \theta_2 = 0$, from Eqs. (1.1f, 1.5g, 1.5h, and 1.5i),

$$K_{11} = \frac{\partial^2 \prod}{\partial {\theta_1}^2} = 2c - PL, K_{22} = \frac{\partial^2 \prod}{\partial {\theta_2}^2} = c - PL, K_{12} = K_{21} = \frac{\partial^2 \prod}{\partial \theta_1 \partial \theta_2} = -c \tag{1.5k}$$

The two degrees of freedom system in Figure 1.5 is in stable equilibrium if $\delta^2\Pi > 0$, or from Eq.(1.1i) we have

$$D_1 = K_{11} > 0, \text{or } 2c–PL > 0, \quad \text{or } P < \frac{2c}{L} \tag{1.5l}$$

$$\text{and } |D_2| = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = \begin{vmatrix} 2c - PL & -c \\ -c & c - PL \end{vmatrix} > 0$$

$$\text{or } P^2 - \frac{3Pc}{L} + \frac{c^2}{L^2} > 0$$

$$\text{and } \left(P - \frac{0.382c}{L}\right)\left(\mathrm{P} - \frac{2.618c}{L}\right) > 0 \tag{1.5m}$$

Therefore, the two degrees of freedom system is in stable equilibrium if $P < P_{cr} = \frac{0.382c}{L}$, because the inequalities (1.5l and 1.5m) are satisfied in the initial position. It is unstable if

$$P > P_{cr} = \frac{0.382c}{L}.$$

1.3.3 Three-Member Truss

1.3.3.1 The Energy Method

Consider a three-member truss where the bars AB and AC are rigid. These bars are pin-connected at A, and the truss is simply supported at B and C as shown in Figure 1.6a. Points B and C are connected by a linear spring of stiffness k. The bars make an initial angle of θ_0 with the horizontal initially. When a vertical force of P is applied at A, the truss deforms, and the bars make an angle of θ with the horizontal in Figure 1.6b.

The strain energy of the system after deformation is given by

$$U = \tfrac{1}{2}k[\,2L(\cos\theta - \cos\theta_0)]^2 = 2kL^2(\cos\theta - \cos\theta_0)^2 \tag{1.6a}$$

Potential energy of the external force P is

$$V = -PL(\sin\theta_0 - \sin\theta) \tag{1.6b}$$

Total potential energy of the system is

$$\Pi = -PL(\sin\theta - \sin\theta_0) + 2kL^2(\cos\theta - \cos\theta_0)^2$$

$$\frac{d\Pi}{d\theta} = PL\cos\theta - 4kL^2(\cos\theta - \cos\theta_0)\,\sin\theta \tag{1.6c}$$

By making $d\Pi/d\theta = 0$, the equilibrium equation is

$$\frac{P}{4kL} = \sin\theta - \cos\theta_0\,\tan\theta \tag{1.6d}$$

$$\frac{d^2\Pi}{d\theta^2} = -PL\,\sin\theta - 4kL^2[\cos 2\theta - \cos\theta_0\cos\theta] \tag{1.6e}$$

Substitute (1.6d) into (1.6e)

$$\frac{d^2\Pi}{d\theta^2} = -4kL[\sin\theta - \cos\theta_0\tan\theta]L\,\sin\theta - 4kL^2[\,\cos 2\theta - \cos\theta_0\cos\theta]$$

Simplifying the above expression gives

$$\frac{d^2\Pi}{d\theta^2} = \frac{4kL^2}{\cos\theta}(\cos\theta_0 - \cos^3\theta) \tag{1.6f}$$

For stable equilibrium, $\dfrac{d^2\Pi}{d\theta^2} > 0$, or $\cos\theta_0 > \cos^3\theta$ is the desired condition.

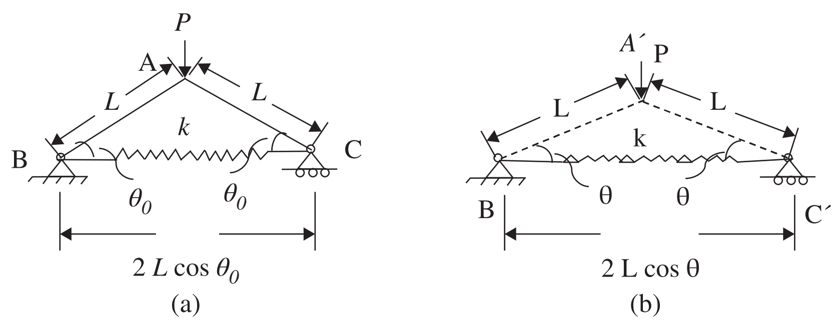

Figure 1.6 Three-member truss with rigid bars: (a) Three member truss; (b) Displaced shape.

Therefore, at $\theta_c < \theta < -\theta_c$, the truss is in stable equilibrium. Where $\cos\theta_0 = \cos^3\theta_c$ at the critical equilibrium, and θ_c is the angle the rigid bars make at the critical equilibrium with the horizontal.

For unstable equilibrium, $\frac{d^2\Pi}{d\theta^2} < 0$, or $\cos\theta_0 < \cos^3\theta$ is the desired condition. Therefore, when $-\theta_c < \theta < \theta_c$, the truss is in unstable equilibrium. We can also look at the stability of the truss with respect to the load P. At the critical equilibrium, $\theta = \theta_c$, now substituting ṅ in Eq. (1.6d) gives

$$\frac{P}{4kL} = \sin\theta_c - \cos^3\theta_c \tan\theta_c = \sin^3\theta_c$$

Therefore, at $\frac{P}{kL} < 4\sin^3\theta_c$, the equilibrium is stable and at $\frac{P}{kL} > 4\sin^3\theta_c$, the equilibrium is unstable. From Eq. (1.6d)

$$\frac{P}{4kL} = \sin\theta - \cos\theta_0 \tan\theta$$

$$\text{or} \quad \frac{P}{kL} = 4\sin\theta\left(1 - \frac{\cos\theta_0}{\cos\theta}\right)$$

$$\text{or} \quad \frac{P}{kL} = 0 \text{ for } \theta = 0, \text{ and } \theta = \pm\theta_0. \tag{1.6g}$$

Assume the initial inclination of the truss members is $\theta_c = 20^0$, then for critical equilibrium

$$\cos 20^0 = \cos^3\theta_c, \quad \text{or } \theta_c = 11.62^0.$$

For stable equilibrium,

$$\cos\theta_0 > \cos^3\theta, \quad \text{or} \quad \cos\theta < (\cos\theta_0)^{\frac{1}{3}}$$

or

$$\theta > \left\{\cos^{-1}\left[\cos(\theta_0)^{\frac{1}{3}}\right]\right\}$$

hence,

$$\theta > \theta_c = 11.62^0 \quad \text{and} \quad \theta < \theta_c = -11.62^0$$

For unstable equilibrium, $\cos\theta_0 < \cos^3\theta$, or $\cos\theta > (\cos\theta_0)^{\frac{1}{3}}$
or

$$\theta < \left\{\cos^{-1}\left[\cos(\theta_0)^{\frac{1}{3}}\right]\right\}$$

hence,

$$\theta < \theta_c = 11.62^0 \quad \text{and} \quad \theta > \theta_c = -11.62^0$$

A plot of Eq. (1.6g) is shown for $\theta_0 = 20^0$ in Figure 1.7.

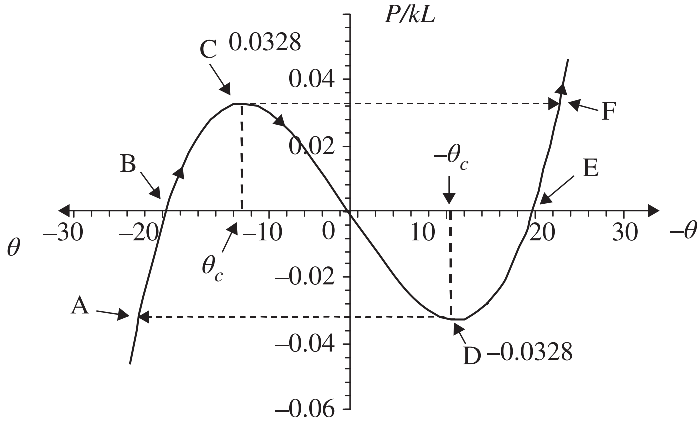

Figure 1.7 Displacement path of three-member truss.

In Figure 1.7 the stable equilibrium paths lie on the lines ABC and DEF, while the unstable equilibrium path lie on the segment CD. As the load P increases from zero, first the stable equilibrium path BC is followed until the critical load at point C is reached. At point C the structure snaps through from point C to F as shown by the dashed line in Figure 1.7. This occurs because as the load is increased infinitesimally from the peak point C, the stable equilibrium available at that load is corresponding to point F. Therefore, there is a large deformation for a small change in the load until the state corresponding to point F is reached. The structure is in stable equilibrium beyond F in new configuration. The change of state from point C to F does not occur through equilibrium paths but occurs dynamically and the structure is unstable during this change. This type of instability is called snap through or limit point instability. If the load P is decreased, the structure follows the path FED, and at point D snaps through to point A. The load deflection curve in Figure 1.7 also shows that this problem is nonlinear even at small deformations. We cannot obtain meaningful results if linearization simplification is used for angles θ and θ_0, even if these angles are small.

1.3.4 Three Rigid Bars with Two Linear Springs

1.3.4.1 The Displaced Shape Equilibrium Method

Three rigid bars are shown in Figure 1.8a. the system is supported by a hinge at A and a roller support at B. The bars are joined by pins C and D, the supports at C and D consist of two linear springs each of stiffness k. The system is subjected to an axial force P as shown. As the force increases, the system deflects as shown in Figure 1.8b, the vertical deflections at C and D are δ_1 and δ_2 respectively. It is a two degrees of freedom system because these two deflections are needed to define the displaced shape. The deflections are assumed to be small.

In Figure 1.8b,

$$\Sigma M_B = 0$$

$$V_A(3L) – k\,\delta_1(2L) – k\,\delta_2(L) = 0$$

$$V_A = \frac{2}{3}k\delta_1 + \frac{1}{3}\,k\delta_2, \text{and}$$

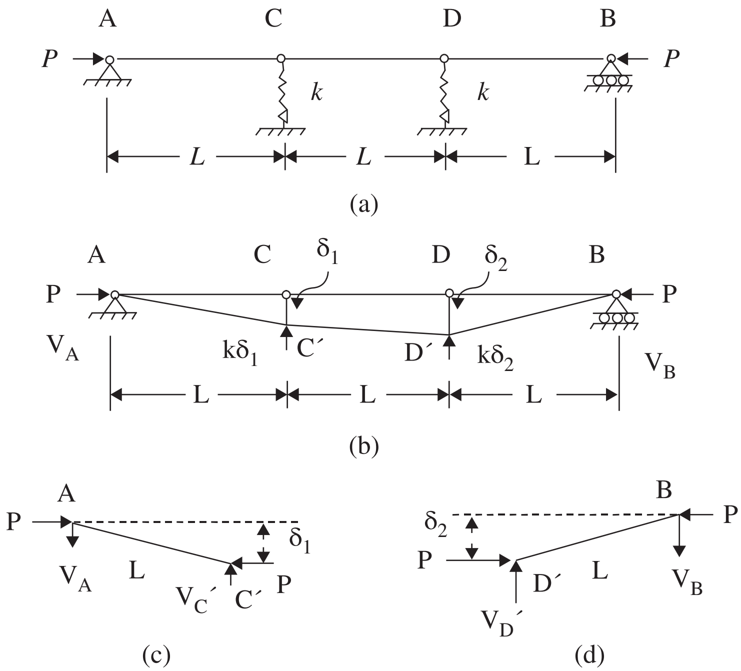

Figure 1.8 Three rigid bars with two linear springs: (a) Three rigid bars; (b) Displaced shape; (c) Free body diagram of AC'; (d) Free body diagram of BD'.

$$\Sigma F_{vertical} = 0$$

$$\frac{2}{3}k\,\delta_1 + \frac{1}{3}k\,\delta_2 - k\delta_1 - k\,\delta_2 + V_B = 0$$

$$V_B = \frac{1}{3}k\delta_1 + \frac{2}{3}k\,\delta_2$$

From Figures 1.8c and 1.8d, and small deformations

$$\Sigma M_{C\prime} = 0$$

$$\left(\frac{2}{3}k\delta_1 + \frac{1}{3}k\delta_2\right)L - P\delta_1 = 0 \tag{1.7a}$$

$$\Sigma M_{D\prime} = 0$$

$$\left(\frac{1}{3}k\delta_1 + \frac{2}{3}k\delta_2\right)L - P\delta_2 = 0 \tag{1.7b}$$

Equations (1.7a and 1.7b) can be written in the matrix form as

$$\begin{bmatrix} \frac{2}{3}kL - P & \frac{1}{3}kL \\ \frac{1}{3}kL & \frac{2}{3}kL - P \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.7c}$$

For a nontrivial solution the determinant of the coefficient matrix is zero,

$$\begin{vmatrix} \frac{2}{3}kL-P & \frac{1}{3}kL \\ \frac{1}{3}kL & \frac{2}{3}kL-P \end{vmatrix} = 0 \tag{1.7d}$$

The characteristic equation is

$$P^2 - \frac{4}{3}k\,L\,P + \frac{1}{3}k^2\,L^2 = 0 \tag{1.7e}$$

The two roots of Eq. (1.7e) are $P_1 = \frac{1}{3}kL$, and $P_2 = kL$.

The first eigenvector is

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

for $P_1 = P_{cr} = \frac{1}{3}kL$, and the deflected shape is the buckling mode as given in Figure 1.9a.

The second eigenvector is

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

for $P_2 = kL$, and the deflected shape is symmetric as shown in Fig. 1.9b.

1.3.4.2 The Energy Method

The strain energy of the system in Figure 1.8b is given by

$$U = \frac{1}{2}k{\delta_1}^2 + \frac{1}{2}k\delta_2^2 \tag{1.8a}$$

The potential energy of the external force is

$$V = -P\left[\left(L - \left(1 - \cos\frac{\delta_1}{L}\right) + L\left(1 - \cos\frac{\delta 2}{L}\right) + L\left[1 - \cos\frac{\delta_2 - \delta_1}{L}\right]\right.\right. \tag{1.8b}$$

Total potential energy of the system is

$$\Pi = U + V \tag{1.8c}$$

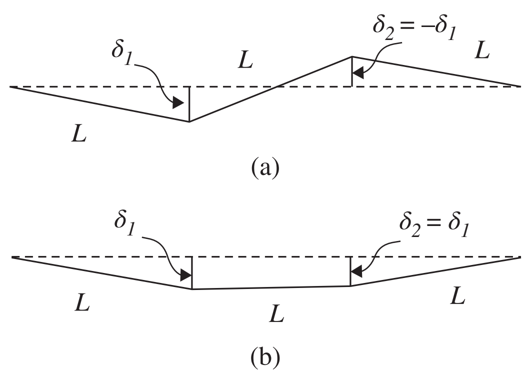

Figure 1.9 Mode shapes of the three rigid bars with linear springs: (a) Asymmetrical deflected shape; (b) Symmetrical deflected shape.

or

$$\Pi = \frac{1}{2}k\delta_1{}^2 + \frac{1}{2}k\delta_2^2 - PL\left[3 - \cos\frac{\delta_1}{L} - \cos\frac{\delta 2}{L} - \cos\frac{\delta_2 - \delta_1}{L}\right] \tag{1.8d}$$

The first derivatives of the potential energy function, Π, with respect to δ_1 and δ_2 must be zero for potential energy to be stationary. Therefore,

$$\frac{\partial \Pi}{\partial \delta_1} = k\delta_1 - PL\left[\frac{1}{L}\sin\frac{\delta_1}{L} - \frac{1}{L}\sin\frac{\delta_2 - \delta_1}{L}\right] \tag{1.8e}$$

$$\frac{\partial \Pi}{\partial \delta_1} = k\delta_2 - PL\left[\frac{1}{L}\sin\frac{\delta_2}{L} + \frac{1}{L}\sin\frac{\delta_2 - \delta_1}{L}\right] \tag{1.8f}$$

For small angle approximation, $\sin\frac{\delta_1}{L} \approx \frac{\delta_1}{L}$, $\sin\frac{\delta_2}{L} \approx \frac{\delta_2}{L}$, and $\sin\frac{\delta_2 - \delta_1}{L} \approx \frac{\delta_2 - \delta_1}{L}$, therefore,

$$\frac{\partial \Pi}{\partial \delta_1} = k\delta_1 - P\left(\frac{\delta_1}{L} - \frac{\delta_2 - \delta_1}{L}\right) = 0 \tag{1.8g}$$

$$\frac{\partial \Pi}{\partial \delta_2} = k\delta_2 - P\left(\frac{\delta_2}{L} + \frac{\delta_2 - \delta_1}{L}\right) = 0 \tag{1.8h}$$

Equations (1.8g and 1.8h) can be written in the matrix form as:

$$\begin{bmatrix} k - P\frac{2}{L} & \frac{P}{L} \\ \frac{P}{L} & k - P\frac{2}{L} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.8i}$$

For a nontrivial solution the determinant of the coefficient matrix is zero,

$$\begin{bmatrix} k - P\frac{2}{L} & \frac{P}{L} \\ \frac{P}{L} & k - P\frac{2}{L} \end{bmatrix} = 0 \tag{1.8j}$$

or

$$P^2 - \frac{4}{3}PLk + \frac{L^2}{3}k^2 = 0 \tag{1.8k}$$

Equation (1.8k) is the same characteristic equation as Eq. (1.7e), giving the same two roots of $P_1 = P_{cr} = \frac{kL}{3}$, and $P_2 = kL$ as before by the displaced shape equilibrium method.

$$\frac{\partial^2 \prod}{\partial \delta_1^2} = K_{11} = k - 2\frac{P}{L} \tag{1.8l}$$

$$\frac{\partial^2 \prod}{\partial \delta_2^2} = K_{22} = k - 2\frac{P}{L} \tag{1.8m}$$

$$\frac{\partial^2 \prod}{\partial \delta_1 \partial \delta_2} = K_{12} = K_{21} = \frac{P}{L} \tag{1.8n}$$

The three rigid bar system in Figure 1.8 is in stable equilibrium if $\delta^2\Pi > 0$, therefore from Eq. (1.1i)

$$D_1 = K_{11} = k - 2\frac{P}{L} > 0, \text{or } P < \frac{kL}{2} \tag{1.8o}$$

$$\text{and} \quad |D_2| = \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = \begin{vmatrix} k - \frac{2P}{L} & \frac{P}{L} \\ \frac{P}{L} & k - \frac{2P}{L} \end{vmatrix} > 0$$

$$\text{or} \quad P^2 - \frac{4}{3}PkL + \frac{k^2L^2}{3} > 0$$

$$\text{and} \quad \left(P - \frac{kL}{3}\right)(P - kL) > 0 \tag{1.8p}$$

Therefore, the three bar rigid system is in stable equilibrium if $P < P_{cr} = \frac{kL}{3}$, because the inequalities in Eqs. (1.8o and 1.8p) are satisfied. It is unstable if $P > P_{cr} = \frac{kL}{3}$.

1.4 Large Displacement Analysis

So far, the analysis has been limited to the linear, small deflection theory that applies to infinitely small deformations from the initial stressed state of the structure. The small deflection theory gives information about the critical load and it is also possible to determine the state of equilibrium in the initial position by studying the second derivatives of the total potential energy by this theory. This is sufficient for most structural engineering problems. However, nonlinear finite displacement theory is needed to gain a full understanding of the post-buckling behavior of a system. We can plot the post buckling equilibrium path using this large displacement theory. It also gives us an indication of the stability of bifurcation.

1.4.1 Rigid Bar Supported by a Translational Spring

1.4.1.1 The Displaced Shape Equilibrium Method

The rigid bar given in Figure 1.4 will be considered here without making the assumption of small deformation. The equilibrium equation is

$$P = kL\,\cos\theta \tag{1.2d}$$

and the critical load is given by $P_{cr} = k\,L$. The equilibrium diagram, $\frac{P}{P_{cr}}$ versus θ, giving the post-buckling path is plotted in Figure 1.10 using Eqs. (1.2d and 1.2e). The initial inclination of the column to the right or left causes a decrease in the load capacity of the column and values of $\frac{P}{P_{cr}}$ continually decrease with increasing θ. The post-buckling displacement path is also symmetric about the initial position of the column, therefore, the bifurcation is called symmetric unstable bifurcation.

1.4.1.2 The Energy Method

This method can also be used to find the critical load and the load deflection graph as shown in Figure 1.10. In addition, it can give the nature of equilibrium in the system initially when the applied load reaches the critical load value as well as during post-buckling. Equations (1.3i–1.3m) can be rewritten from Figure 1.4 as

$$\Pi = -P\,L(1 - \cos\theta) + {}^1\!/\!{}_2 k(L\,\sin\theta)^2 \tag{1.3i}$$

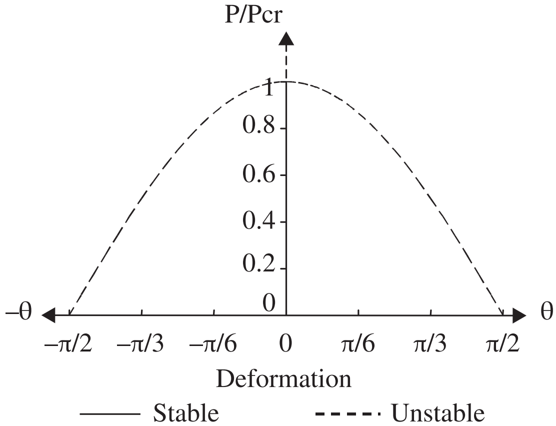

Figure 1.10 Equilibrium path of rigid bar in Figure 1.4.

$$\frac{d\Pi}{d\theta} = -PL\ \sin\theta + k\,L^2 \sin\theta\ \cos\theta \tag{1.3j}$$

$\frac{d\Pi}{d\theta} = 0$, and we get the equilibrium equation as

$$P = kL\ \cos\theta \tag{1.3k}$$

$\cos\theta \approx 1$ for small values of θ
or

$$P_{cr} = kL \tag{1.3l}$$

$$\frac{d^2\Pi}{d\theta^2} = -PL\cos\theta + kL^2\cos 2\theta \tag{1.3m}$$

From Eq. (1.3k) and Eq. (1.3m)

$$\frac{d^2\Pi}{d\theta^2} = -kL^2\cos\theta + kL^2\cos 2\theta \tag{1.9a}$$

Substituting $P = P_{cr} = kL,\ and\ \theta = 0$ in Eq. (1.9a), $\frac{d^2\Pi}{d\theta^2} = 0$. This does not give us an idea of the nature of equilibrium at the bifurcation. Therefore, to determine the initial post-critical behavior near bifurcation we may write total potential energy, Π, as a Taylor series as follows:

$$\Pi = \Pi\Big|_{\theta=0} + \frac{d\Pi}{d\theta}\Big|_{\theta=0}\theta + \frac{1}{2!}\frac{d^2\Pi}{d\theta^2}\Big|_{\theta=0}\theta^2 + \frac{1}{3!}\frac{d^3\Pi}{d\theta^3}\Big|_{\theta=0}\theta^3 + \frac{1}{4!}\frac{d^4\Pi}{d\theta^4}\Big|_{\theta=0}\theta^4 + --- \tag{1.9b}$$

$$\frac{d^3\Pi}{d\theta^3} = PL\sin\theta - 2kL^2\sin 2\theta \tag{1.9c}$$

At

$$P = P_{cr} = kL, \text{and}\ \theta = 0, \frac{d^3\Pi}{d\theta^3} = 0$$

$$\frac{d^4\Pi}{d\theta^4} = PL\cos\theta - 4kL^2\cos 2\theta \tag{1.9d}$$

$$\text{At } P = P_{cr} = kL \text{ and } \theta = 0, \frac{d^4 \prod}{d\theta^4} = -3kL^2 \tag{1.9e}$$

Therefore,

$$\Pi = \frac{-3kL^2}{24}\theta^4 = -\frac{1}{8}kL^2\theta^4 \tag{1.9f}$$

This indicates that the total potential energy, Π, is negative or it decreases with increasing θ, at the initial position, $\theta = 0$ and $P = P_{cr}$. The bifurcation is symmetric and unstable from Eqs. (1.3k and 1.9f), as shown in Figure 1.10.

During the post-buckling path when $\theta \neq 0$, Eqs. (1.3k and 1.3m) give

$$\frac{d^2 \prod}{d\theta^2} = -kL^2\cos^2\theta + kL^2 \cos 2\theta$$

$$\text{or} \quad \frac{d^2 \prod}{d\theta^2} = -kL^2\sin^2\theta \tag{1.9g}$$

Therefore, $\frac{d^2 \prod}{d\theta^2} < 0$ for different values of θ, and the post-buckling path is unstable.

1.4.2 Rigid Bar Supported by Translational and Rotational Springs

1.4.2.1 The Displaced Shape Equilibrium Method

A rigid bar connected to a translational spring at the top and a rotational spring at the bottom is acted on by an axial force P as shown in Figure 1.11a. The free body diagram of the deflected system is shown in Figure 1.11b. Taking the moment of all the forces acting on the system in Figure 1.11b about A and equating to zero, we have

$$PL \sin\theta - kL \sin\theta\,(L\cos\theta) - c\theta = 0 \tag{1.10a}$$

or

$$P = kL \cos\theta + \frac{c}{L}\frac{\theta}{\sin\theta} \tag{1.10b}$$

For small values of θ, $\cos\theta \approx 1$, $\sin\theta \approx \theta$, hence, the critical load is

$$P_{cr} = kL + \frac{c}{L}. \tag{1.10c}$$

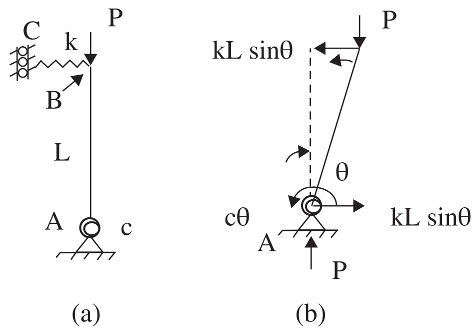

Figure 1.11 Rigid bar connected to translational and rotational springs: (a) Rigid bar with two springs; (b) Free body diagram of displaced shape.

1.4.2.2 The Energy Method

The strain energy of the system in Figure 1.11 is

$$U = \frac{1}{2}k(L\ \sin\theta)^2 + \frac{1}{2}c\,\theta^2 \tag{1.11a}$$

and the potential energy of the external forces is

$$V = -P(L - L\ \cos\theta) \tag{1.11b}$$

Total potential energy is given by

$$\Pi = \frac{1}{2}k(L\ \sin\theta)^2 + \frac{1}{2}c\,\theta^2 - P\,(L - L\ \cos\theta) \tag{1.11c}$$

Taking the first derivative of the total potential energy with respect to θ and equating it to zero gives the equilibrium equation

$$\frac{d\prod}{d\theta} = kL^2 \sin\theta\ \cos\theta + c\theta - PL\ \sin\theta = 0 \tag{1.11d}$$

giving the same relation between the force P and θ as in Eq. (1.10b) and the same critical load P_{cr} as before.

$$\text{or}\frac{d\prod}{d\theta} = \frac{kL^2}{2}\sin 2\theta + c\theta - PL\sin\theta$$

$$\frac{d^2\prod}{d\theta^2} = kL^2\cos 2\theta + c - PL\cos\theta \tag{1.11e}$$

Substituting $P = P_{cr} = kL + \frac{c}{L}$, and $\theta = 0$ in Eq. (1.11e) gives

$$\frac{d^2\prod}{d\theta^2} = kL^2\cos(\theta) + c - \left(kL + \frac{c}{L}\right)L\cos(\theta) = 0$$

Therefore, use higher terms in the Taylor series in Eq. (1.9b) to know whether the total potential energy is relative maximum or minimum at the bifurcation.

$$\frac{d^3\prod}{d\theta^3} = -2kL^2\sin 2\theta + PL\sin\theta \tag{1.11f}$$

At the bifurcation,

$$P = P_{cr} = kL + \frac{c}{L}, \theta = 0,$$

$$\frac{d^3\prod}{d\theta^3} = 0$$

and

$$\frac{d^4\prod}{d\theta^4} = -4kL^2\cos 2\theta + PL\cos\theta$$

$$= -4\,k\,L^2 + \left(k\,L + \frac{c}{L}\right)L = -3k\,L^2 + c \tag{1.11g}$$

Therefore, from the Taylor series of Eq. (1.9b) we have

$$\Pi = \frac{1}{24}(-3kL^2 + c)\theta^4 = \left(-\frac{kL^2}{8} + \frac{c}{24}\right)\theta^4 \tag{1.11h}$$

Π is positive, if $\frac{c}{24} > \frac{kL^2}{8}$, or $\frac{kL^2}{c} < \frac{1}{3}$ for stable equilibrium at the bifurcation. On the other hand, Π is negative if $\frac{kL^2}{c} > \frac{1}{3}$, and the equilibrium at the bifurcation is unstable.

During the post-buckling path when $\theta \neq 0$, from Eqs. (1.10a and 1.11e)

$$\frac{d^2 \prod}{d\theta^2} = kL^2\cos^2\theta - kL^2\sin^2\theta + c - \left(kL\cos\theta + \frac{c\theta}{L\sin\theta}\right) L\cos\theta$$

or $\frac{d^2 \prod}{d\theta^2} = -kL^2 \sin^2\theta + c - c\theta\cot\theta$

For $\frac{d^2 \prod}{d\theta^2} > 0$, the post-buckling path is stable, and at $\frac{d^2 \prod}{d\theta^2} < 0$, it is unstable. Therefore,

$$\text{if } \frac{kL^2}{c} < \frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} \tag{1.11i}$$

the post-buckling path is stable, and for

$$\frac{kL^2}{c} > \frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} \tag{1.11j}$$

the post-buckling path is unstable.

At $\theta = 0$, $\frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} = \frac{0}{0}$. Therefore, differentiate numerator and denominator with respect to θ, and applying Le Hospital's rule, we get $\frac{\sin\theta - \theta\cos\theta}{\sin^3\theta} = \frac{1}{3}$ for $\theta = 0$. Hence, if $\frac{kL^2}{c} < \frac{1}{3}$, it is stable bifurcation, and for $\frac{kL^2}{c} > \frac{1}{3}$, it is unstable at $\theta = 0$ as shown before.

Let $\frac{kL^2}{c} = 0.35$, or $k\,L = 0.35\frac{c}{L}$, and from Eq. (1.10c)

$$P_{cr} = 0.35\frac{c}{L} + \frac{c}{L} = \frac{1.35c}{L}$$

From Eq. (1.10b)

$\frac{PL}{c} = \frac{kL^2}{c}\cos\theta + \frac{\theta}{\sin\theta}$, dividing both sides of the equation by 1.35

$$\frac{PL}{1.35c} = \frac{P}{P_{cr}} = \frac{0.35\cos\theta + \frac{\theta}{\sin\theta}}{1.35} \tag{1.11k}$$

$\frac{P}{P_{cr}}$ versus θ graph is plotted in Figure 1.12, and it shows that post-buckling path is unstable at the bifurcation because $\frac{kL^2}{c} = 0.35 > \frac{1}{3}$ and it continues to be unstable until $\frac{kL^2}{c} = 0.35 < \frac{\sin\theta - \theta\cos\theta}{\sin^3\theta}$, when it becomes stable.

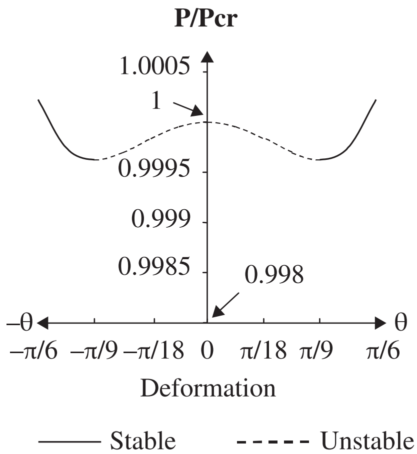

Figure 1.12 Displacement path of rigid bar supported by translational and rotational springs.

1.4.3 Two Rigid Bars Connected by Rotational Springs

1.4.3.1 The Energy Method

The two rigid bars of equal length L and connected by rotational springs of equal spring stiffness c shown in Figure 1.5 are analyzed here by large displacement analysis. The first and second derivatives of the total potential energy function, Π, from Eqs. (1.5d, 1.5e, 1.5g, 1.5h, and 1.5i) are as follows:

$$\frac{\partial \Pi}{\partial \theta_1} = 2c\,\theta_1 - c\,\theta_2 - PL\,\sin\theta_1 = 0 \tag{1.12a}$$

$$\frac{\partial \Pi}{\partial \theta_1} = -c\theta_1 + c\theta_2 - PL\,\sin\theta_2 = 0 \tag{1.12b}$$

$$\frac{\partial^2 \prod}{\partial {\theta_1}^2} = 2c - PL\cos\theta_1 \tag{1.12c}$$

$$\frac{\partial^2 \prod}{\partial {\theta_2}^2} = c - PL\cos\theta_2 \tag{1.12d}$$

$$\frac{\partial^2 \prod}{\partial \theta_1 \partial \theta_2} = -c \tag{1.12e}$$

Equations (1.12a and 1.12b) are equilibrium equations of the system. These are solved by eliminating θ_2. From Eq. (1.12a)

$$\theta_2 = \frac{2c\theta_1 - PL\sin\theta_1}{c} \tag{1.12f}$$

Substituting Eq. (1.12f) in Eq. (1.12b), we get

$$-c\theta_1 + 2c\theta_1 - PL\sin\theta_1 - PL\sin\frac{2c\theta_1 - PL\sin\theta_1}{c} = 0$$

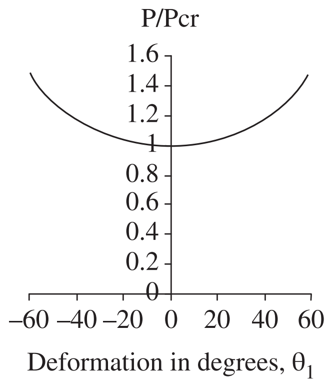

Figure 1.13 Displacement path of two rigid bars connected by rotational springs.

or

$$\frac{P}{\frac{c}{L}} = \frac{\theta_1}{\sin\theta_1 + \sin\left(2\theta_1 - \frac{PL}{c}\sin\theta_1\right)} \tag{1.12g}$$

$P_{cr} = 0.382\frac{c}{L}$, from Eq. (1.5i).
Therefore,

$$\frac{P}{P_{cr}} = \frac{1}{0.382}\left[\frac{\theta_1}{\sin\theta_1 + \sin\left(2\theta_1 - \frac{PL}{c}\sin\theta_1\right)}\right] \tag{1.12h}$$

$\frac{P}{P_{cr}}$ vs. θ_1 graph is plotted in Figure 1.13, and it shows that the post-buckling path is stable.

1.5 Imperfections

So far, it has been assumed that the rigid bars considered were geometrically perfect. In general, the columns may be imperfect, having a certain amount of deformation present in the initial state when the springs are unrestrained at the load P = 0.

1.5.1 Rigid Bar Supported by a Rotational Spring at the Base

1.5.1.1 The Displaced Shape Equilibrium Method

Consider a rigid bar of length L supported by a rotational spring of stiffness c. The column is initially imperfect and inclined by an angle α as shown in Figure 1.14. From the equilibrium of the column in the displaced position making an angle of θ with the vertical, we have

$$PL\ \sin\ \theta - c\,(\theta - \alpha) = 0 \tag{1.13a}$$

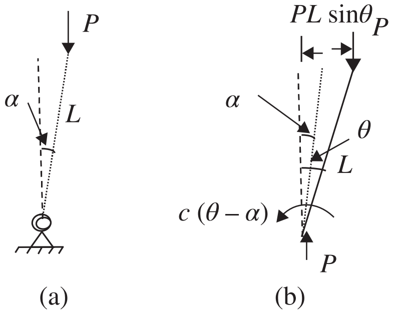

Figure 1.14 Imperfect rigid-bar column with rotational spring at the base: (a) Imperfect rigid bar; (b) Free body diagram of displaced shape.

or

$$P = \frac{c(\theta - \alpha)}{L \sin\theta} \tag{1.13b}$$

1.5.1.2 The Energy Method

Strain energy of the imperfect column in Figure 1.14 is

$$U = \frac{1}{2}c(\theta - \alpha)^2 \tag{1.14a}$$

and the potential energy of the external forces is

$$V = -P\,L\,(\cos\alpha - \cos\theta) \tag{1.14b}$$

The total potential energy is given by

$$\Pi = \frac{1}{2}c(\theta - \alpha)^2 - P\,L(\cos\alpha - \cos\theta) \tag{1.14c}$$

$$\frac{d\Pi}{d\theta} = c(\theta - \alpha) - P\,L\sin\theta \tag{1.14d}$$

Setting $\frac{d\Pi}{d\theta} = 0$, we have the equilibrium condition, and $P = \frac{c(\theta - \alpha)}{L\sin\theta}$ as before. For a perfect column and small θ, $P_{cr} = \frac{c}{L}$

The equilibrium diagrams, $\frac{P}{P_{cr}}$ versus θ, are plotted in Figure 1.15 for initial imperfections of $\alpha = -10, -5, 5$, and $10°$ of inclination with the vertical. The points where columns change from stable to unstable state lie on the critical curve defined by $\frac{d^2\Pi}{d^2\theta} = 0$.

$$\frac{d^2\Pi}{d\theta^2} = c - PL\cos\theta \tag{1.14e}$$

If $\frac{d^2\Pi}{d^2\theta} = 0, P = \frac{c}{L\cos\theta}$, or $\frac{P}{P_{cr}} = \frac{1}{\cos\theta}$, and $\frac{P}{P_{cr}}$ versus θ critical curve is plotted in Figure 1.15.

The column is stable if $\frac{d^2\Pi}{d^2\theta} > 0$, and it is unstable if $\frac{d^2\Pi}{d^2\theta} < 0$. Substituting the value of $P = P_{cr}$ from Eq. (1.13b) into Eq. (1.14e), we have

$$\frac{d^2\Pi}{d\theta^2} = c\left[1 - \frac{(\theta - \alpha)}{\tan\theta}\right] \tag{1.14f}$$

So the equilibrium path is stable if $\tan\theta > \theta - \alpha$ if $\alpha < \theta < \pi/2$, and $\tan\theta < \theta - \alpha$ if $-\pi/2 < \theta < 0$. This can also be seen from the slopes of the equilibrium curves in Figure 1.15.

$$\begin{aligned} P &= \frac{c}{L}\left(\frac{\theta - \alpha}{\sin\theta}\right) \\ \frac{dP}{d\theta} &= \frac{c}{L\sin\theta}\left[1 - \frac{\theta - \alpha}{\tan\theta}\right] \end{aligned} \tag{1.14g}$$

For $\alpha > 0$ and $\theta > \alpha$, $\tan\theta > \theta - \alpha$; and from Eq. (1.14g), $\frac{dP}{d\theta} > 0$. Similarly, for $\alpha > 0$ and $\theta < 0$, $\tan\theta < \theta - \alpha$; and from Eq. (1.14g), $\frac{dP}{d\theta} < 0$. Therefore, equilibrium curves are stable when their slope is positive in the bottom right and negative in the top left in Figure 1.15. The same way it can be proved that for $\alpha < 0$, the equilibrium curves are stable when their slope is negative for $\theta < 0$ in the bottom left; and the slope is positive for $\theta > 0$ in the top right in Figure 1.15. The results are symmetrical. For the critical state, $\frac{d^2\Pi}{d^2\theta} = 0$, therefore, from Eqs. (1.14f and 1.14g), $\frac{dP}{d\theta} = 0$ and

$$\tan\theta = \theta - \alpha \tag{1.14h}$$

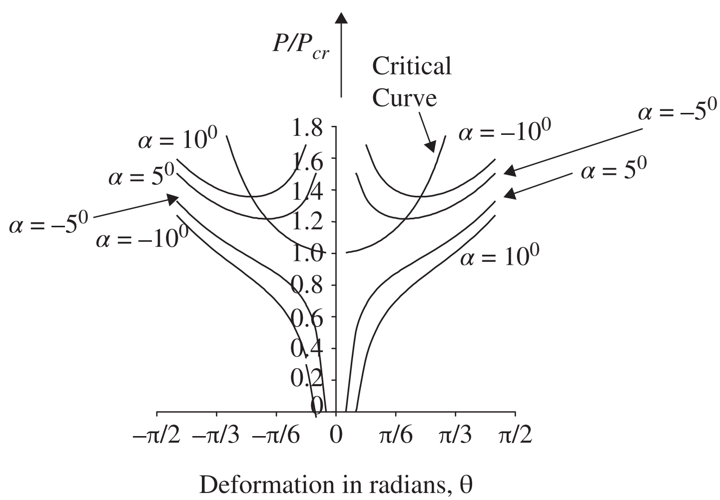

Figure 1.15 Equilibrium path of the rigid bar imperfect column with rotational spring at the base.

The zero slope point on each equilibrium curve gives the critical state. Figure 1.15 also shows that the imperfect column can be stable at loads higher than that of the perfect column.

1.5.2 Two Rigid Bars Connected by Rotational Springs

1.5.2.1 The Displaced Shape Equilibrium Method

Consider the column shown in Figure 1.16a that has two degrees of freedom. The deflected shape of the column is defined by the angles θ_1 and θ_2. Initially the column is imperfect shown by the angles of inclination α_1 and α_2 of the two bars with the vertical.

Taking the sum of the moments about A in Figure 1.16b equal to zero,

$$PL\sin\theta_1 + c[(\theta_2 - \alpha_2) - (\theta_1 - \alpha_1)] - c(\theta_1 - \alpha_1) = 0$$

or

$$c\,\theta_2 = 2c\theta_1 - 2c\alpha_1 + c\alpha_2 - PL\ \sin\theta_1 \tag{1.15a}$$

Similarly taking the sum of the moments about B in Figure 1.16c equal to zero,

$$PL\ \sin\theta_2 - c[(\theta_2 - \alpha_2) - (\theta_1 - \alpha_1)] = 0 \tag{1.15b}$$

Eliminating θ_2 from Eqs. (1.15a and 1.15b), we have

$$\frac{P}{\frac{c}{L}} = \frac{\theta_1 - \alpha_1}{\sin\theta_1 + \sin\left(2\theta_1 - 2\alpha_1 + \alpha_2 - \frac{PL}{c}\sin\theta_1\right)} \tag{1.15c}$$

1.5.2.2 The Energy Method

The strain energy of the column in Figure 1.16 is given by

$$U = \frac{1}{2}c(\theta_1 - \alpha_1)^2 + \frac{1}{2}c[\theta_2 - \alpha_2 - (\theta_1 - \alpha_1)]^2 \tag{1.16a}$$

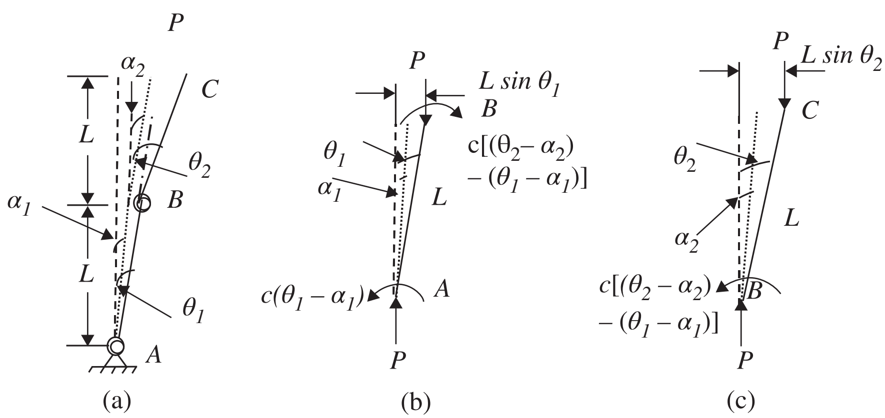

Figure 1.16 Imperfect column with two rigid bars and two rotational springs: (a) Displaced shape of the column; (b) Free body diagram of the lower bar; (c) Free body diagram of the upper bar.

$$V = -P[L\ \cos\alpha_2 - L\cos\theta_2 + L\cos\alpha_1 - L\cos\theta_1] \tag{1.16b}$$

The total potential energy is $\Pi = U + V$, or

$$\Pi = \frac{1}{2}c(\theta_1 - \alpha_1)^2 + \frac{1}{2}c[(\theta_2 - \alpha_2) - (\theta_1 - \alpha_1)]^2 - P[L\ \cos\alpha_2 - L\ \cos\theta_2 + L\ \cos\alpha_1 - L\ \cos\theta_1]$$

By differentiating, we obtain the equilibrium conditions:

$$\frac{\partial\Pi}{\partial\theta_1} = c(\theta_1 - \alpha_1) - c(\theta_2 - \alpha_2 - \theta_1 + \alpha_1) - PL\sin\theta_1 = 0 \tag{1.16c}$$

or $c\,\theta_2 = 2c\theta_1 - 2c\alpha_1 + c\alpha_2 - PL\sin\theta_1$, same as Eq. (1.15a).

$$\frac{\partial\Pi}{\partial\theta_2} = PL\ \sin\theta_2 - c[\theta_2 - \alpha_2 - (\theta_1 - \alpha_1)] = 0 \tag{1.16d}$$

Equation (1.16d) is the same as Eq. (1.15b), therefore, eliminating θ_2 from above equations will lead to the same P versus θ_1 relation as in Eq. (1.15c). For the column in Figure 1.16 if it is perfect, i.e. $\alpha_1 = \alpha_2 = 0$, and if the displacements, θ_1 and θ_2 are small, the critical load $P_{cr} = 0.382\frac{c}{L}$. From Eq. (1.15c) we get

$$\frac{P}{P_{cr}} = \frac{\theta_1 - \alpha_1}{0.382\left[\sin\theta_1 + \sin\left(2\theta_1 - 2\alpha_1 + \alpha_2 - \frac{PL}{c}\sin\theta_1\right)\right]} \tag{1.16e}$$

Assume

$$\alpha_1 = \alpha_2 = \alpha.$$

The equilibrium path given by Eq. (1.16e) is plotted in Figure 1.17.

Discrete systems with one or two degrees of freedom have been analyzed in this chapter. In the analysis for stability of discrete systems, algebraic equations were developed and solved. Differential equations are formed when the analysis of continuous systems such as beams and

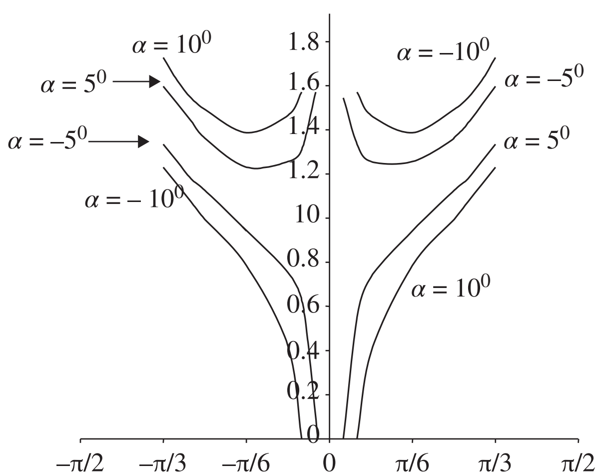

Figure 1.17 Displacement path of two imperfect rigid bars column connected by two rotational springs.

columns is performed. The solution of these differential equations is much more difficult than the algebraic equations. These differential equations can be converted to algebraic equations by discretizing a structure or assuming a Fourier series expansion for its displacements. So the analysis of discrete systems is also valuable to analyze the continuous systems. The methods for solving stability problems learned here will be useful in later chapters.

Problems

1.1 Find the critical load P_{cr} for the rigid bar column in Figure P1.1 by using the equilibrium method. The column is restricted by a rotational spring of stiffness c at the support

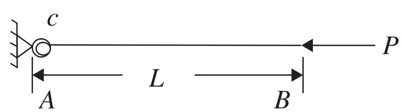

Figure P1.1

1.2 Solve Problem 1.1 by the energy method.

1.3 Determine the critical load P_{cr} for the rigid bar column in Figure P1.3 a, b by using the equilibrium method.

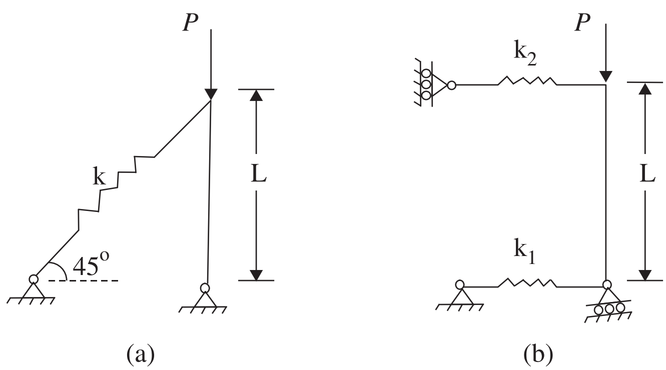

Figure P1.3

1.4 Solve Problem 1.3 by the energy method.

1.5 Analyze the stability behavior of the rigid bar system in Figure P1.5.

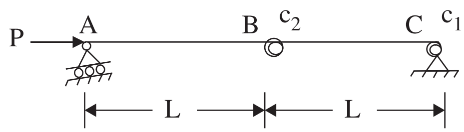

Figure P1.5

1.6 Analyze the stability behavior of the rigid bar and spring system in Figure P1.6. The column is initially imperfect and is inclined by an angle α.

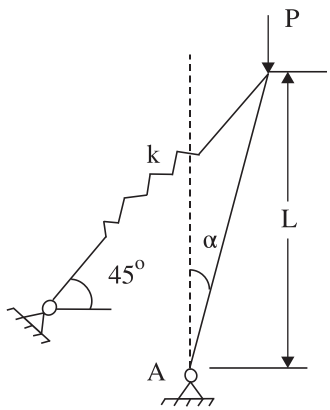

Figure P1.6

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