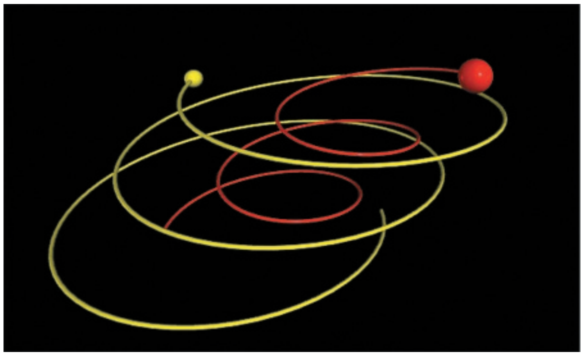
Interactions and Motion



OBJECTIVES

After studying this chapter, you should be able to:

- deduce from observations of an object's motion whether or not it has interacted with its surroundings.
- mathematically describe position and motion in three dimensions.
- mathematically describe momentum and change of momentum in three dimensions.
- read and modify a simple computational model of motion at constant velocity.

This textbook deals with the nature of matter and its interactions. The main goal of this textbook is for you to engage in a process central to science: constructing and applying physical models based on a small set of powerful fundamental physical principles and the atomic structure of matter. The variety of phenomena that you will be able to model, explain, and predict is very wide, including the orbit of stars around a black hole, nuclear fusion, the formation of sparks in air, and the speed of sound in a solid. This first chapter deals with the physical idea of interactions.

1.1 Kinds of Matter

OBJECTIVES

After studying this section, you should be able to:

- recall the approximate (order of magnitude) radius of an atom and the radius of an atomic nucleus.
- state the assumptions made in treating a system as a point particle.

In this course we will deal with material objects of many sizes, from subatomic particles to galaxies. All of these objects have certain things in common.

1.1.1 Atoms and Nuclei

Ordinary matter is made up of tiny atoms. An atom isn't the smallest type of matter, because it is composed of even smaller objects (electrons, protons, and neutrons). However, many of the ordinary everyday properties of ordinary matter can be understood in terms of atomic properties and interactions. As you probably know from studying chemistry, atoms have a very small, very dense core, called the nucleus, around which is found a cloud of electrons. The nucleus contains protons and neutrons, collectively called nucleons. Electrons are kept close to the nucleus by electric attraction to the protons (the neutrons hardly interact with the electrons).

QUESTION Recall your previous studies of chemistry. How many protons and electrons are there in a hydrogen atom? In a helium or carbon atom?

When you encounter a question in the text, you should think for a moment before reading on. Active reading contributes to significantly greater understanding.

Hydrogen is the simplest atom, with just one proton and one electron. A helium atom has two protons and two electrons. A carbon atom has six protons and six electrons. Near the other end of the chemical periodic table, a uranium atom has 92 protons and 92 electrons. **Figure 1.1.A** shows the relative sizes of the electron clouds in atoms of several elements but cannot show the nucleus to the same scale; the tiny dot marking the nucleus in the figure is much larger than the actual nucleus would be.

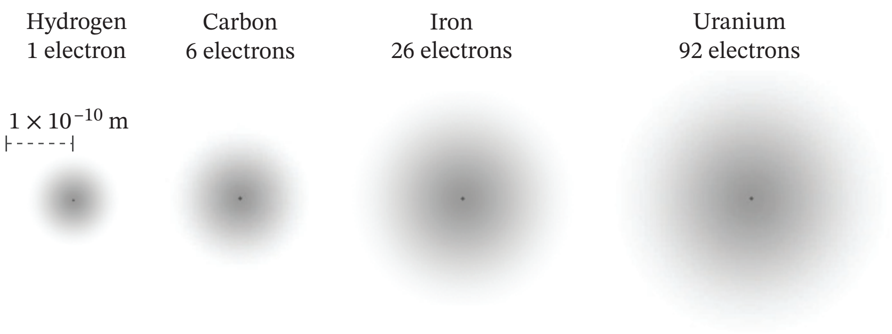


FIGURE 1.1.A Atoms of hydrogen, carbon, iron, and uranium. The gray blur represents the electron cloud surrounding the nucleus. The black dot shows the location of the nucleus. On this scale, however, the nucleus would be much too small to see.

The radius of the electron cloud for a typical atom is about 1×10^{-10} meter. The reason for this size can be understood using the principles of quantum mechanics, a major development in physics in the early 20th century. The radius of a proton is about 1×10^{-15} meter, very much smaller than the radius of the electron cloud.

Nuclei contain neutrons as well as protons (**Figure 1.1.B**). The most common form (isotope) of hydrogen has no neutrons in the nucleus. However, there exist isotopes of hydrogen with one or two neutrons in the nucleus (in addition to the proton). Hydrogen atoms containing one or two neutrons are called deuterium or tritium. The most common isotope of helium has two neutrons (and two protons) in its nucleus, but a rare isotope has only one neutron; this is called helium-3.

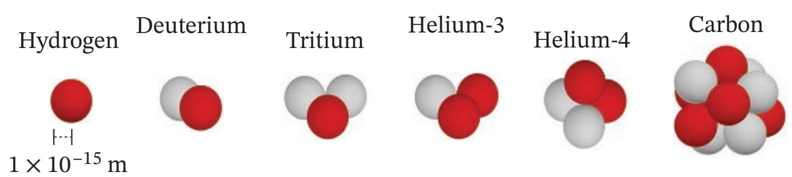


FIGURE 1.1.B Nuclei of hydrogen isotopes, helium isotopes, and carbon. Red spheres represent protons and gray spheres represent neutrons. Note the very much smaller scale than in **Figure 1.1.A**!

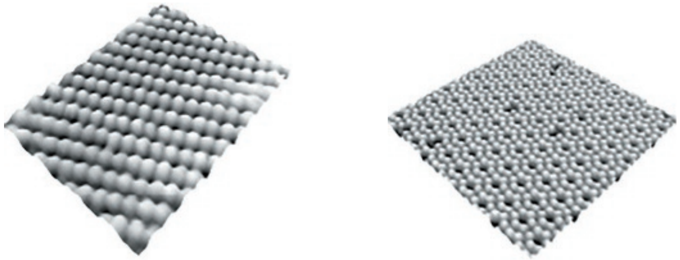
The most common isotope of carbon has six neutrons together with the six protons in the nucleus (carbon-12), whereas carbon-14 with eight neutrons is an isotope that plays an important role in dating archaeological objects.

Near the other end of the periodic table, uranium-235, which can undergo a fission chain reaction, has 92 protons and 143 neutrons, whereas uranium-238, which does not undergo a fission chain reaction, has 92 protons and 146 neutrons.

1.1.2 Molecules and Solids

When atoms come in contact with each other, they may stick (bond) to each other. Several atoms bonded together can form a molecule—a substance whose physical and chemical properties differ from those of the constituent atoms. For example, water molecules (H_2O) have properties quite different from the properties of hydrogen atoms or oxygen atoms.

An ordinary-sized rigid object made of bound-together atoms and big enough to see and handle is called a solid, such as a bar of aluminum. An instrument called a scanning tunneling microscope (STM) is able to map the locations of atoms on the surface of a solid, which has provided new techniques for investigating matter at the atomic level. Two such images appear in **Figure 1.1.C**. You can see that atoms in a crystalline solid are arranged in a regular three-dimensional array. The arrangement of atoms on the surface depends on the direction along which the crystal is cut. The irregularities in the right image reflect defects, such as missing atoms, in the crystal structure.



Courtesy of Randall Feenstra, IBM Corp.

FIGURE 1.1.C Two different surfaces of a crystal of pure silicon. The images were made with a scanning tunneling microscope.

1.1.3 Liquids and Gases

When a solid is heated to a higher temperature, the atoms in the solid vibrate more vigorously about their normal positions. If the temperature is raised high enough, this thermal agitation may destroy the rigid structure of the solid. The atoms may become able to slide over each other, in which case the substance is a liquid.

At even higher temperatures, the thermal motion of the atoms or molecules may be so large as to break the interatomic or intermolecular bonds completely, and the liquid turns into a gas. In a gas, the atoms or molecules are quite free to move around, only occasionally colliding with each other or the walls of their container.

We will learn how to analyze many aspects of the behavior of solids and gases. We won't have much to say about liquids because their properties are much harder to analyze. Solids are simpler to analyze than liquids because the atoms stay in one place (though with thermal vibration about their usual positions). Gases are simpler to analyze than liquids because between collisions the gas molecules are approximately unaffected by the other molecules. Liquids are the awkward intermediate state, where the atoms move around rather freely but are always in contact with other atoms. This makes the analysis of liquids very complex.

1.1.4 Planets, Stars, Solar Systems, and Galaxies

In our brief survey of the kinds of matter that we will study, we make a giant leap in scale from atoms all the way up to planets and stars, such as our Earth and Sun. We will see that many of the same principles that apply to atoms apply to planets and stars. By making this leap, we bypass an important physical science, geology, whose domain of interest includes the formation of mountains and continents. We will study objects that are much bigger than mountains, and we will study objects that are much smaller than mountains, but we don't have time to apply the principles of physics to every important kind of matter.

Our Sun and its accompanying planets constitute our Solar System. It is located in the Milky Way Galaxy, a giant rotating disk-shaped system of stars. On a clear dark night you can see a band of light (the Milky Way) coming from the huge number of stars lying in this disk, which you are looking at from a position in the disk, about two-thirds of the way out from the center of the disk. Our galaxy is a member of a cluster of galaxies that move around each other much as the planets of our Solar System move around the Sun (**Figure 1.1.D**). The Universe contains many such clusters of galaxies.

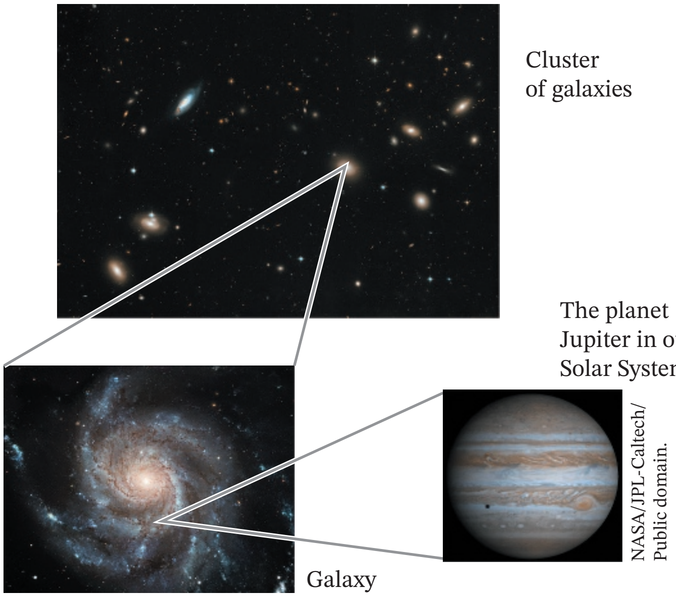


FIGURE 1.1.D Our Solar System exists inside a galaxy, which itself is a member of a cluster of galaxies.

1.1.5 Point Particles

It is common in physics to talk about the motion of a *point particle*. What we mean by a particle is an object whose size, shape, and internal structure are not important to us in the current context, and which we can consider to be located at a single point in space. In modeling the motion of a real object (whether it is a galaxy or a proton), we often choose to make the simplifying assumption that it is a point particle, as if Superman or a giant space alien had come along and squeezed the object until it was compressed into a very tiny, structureless microscopic speck with the full mass of the original object.

Of course, there are many situations in which it would be absurd to use this approximation. The Earth, for example, is a large, complex object, with a core of turbulent molten rock, huge moving continents, and massive sloshing oceans. Radioactivity keeps its core hot; electromagnetic radiation from the Sun warms its surface; and thermal energy is also radiated away into space. If we are interested in energy flows or continental motion or earthquakes we need to consider the detailed structure and composition of the Earth. However, if what we want to do is model the motion of the Earth as it interacts with other objects in our Solar System, it works quite well to ignore this complexity, and to treat the Earth, the Sun, the Moon, and the other planets as if they were point particles.

Even most very tiny objects, such as atoms, protons, and neutrons, are not truly point particles—they do have finite size, and they have internal structure, which can influence their interactions with other objects. By contrast, electrons may really be point particles. They appear to have no internal structure, and attempts to measure the radius of an electron have not produced a definite number (recent experiments indicate only that the radius of an electron is less than 10^{-18} m, much smaller than a proton).

As we consider various aspects of matter and its interactions, it will be important for us to state explicitly whether or not we are modeling material objects as point particles or as extended and perhaps deformable macroscopic chunks of matter. In Chapters 1–3, we will emphasize systems that can usefully be modeled as particles. In Chapter 4, we will begin to consider the detailed internal structure of material objects.

Checkpoint 1.1-C-01

a. Which of the following is the approximate radius of an atom?

1. 10^{-8} m
2. 10^{-10} m
3. 10^{-12} m
4. 10^{-15} m

b. Which of the following is the approximate radius of a nucleus?

1. 10^{-8} m
2. 10^{-10} m
3. 10^{-12} m
4. 10^{-15} m

c. In which of the following situations is it appropriate to treat the whole Earth as though it were a point particle?

1. Estimating the temperature at the center of the Earth.
2. Calculating the orbit of the Earth around the Sun.
3. Calculating the probability of an earthquake in Italy.
4. Predicting tides in the Pacific Ocean.

1.2 Detecting Interactions

OBJECTIVE

After studying this section, you should be able to explain how the motion diagram for a point particle, a series of positions marked at equal time intervals, indicates the presence or absence of interaction with another object.

Objects made of different kinds of matter interact with each other in various ways: gravitationally, electrically, magnetically, and through nuclear interactions. How can we detect that an interaction has occurred? In this section, we consider various kinds of observations that indicate the presence of interactions.

QUESTION Before you read further, take a moment to think about your own ideas of interactions. How can you tell that two objects are interacting with each other?

1.2.1 Change of Direction of Motion

Suppose that you observe a proton moving through a region of outer space, far from almost all other objects. The proton moves along a path like the one shown in **Figure 1.2.A**.

FIGURE 1.2.A A proton moves through space, far from almost all other objects. The initial direction of the proton's motion is indicated by the green arrow. The ×s represent the position of the proton at equal time intervals.

QUESTION Do you see evidence in **Figure 1.2.A** that the proton is interacting with another object?

Evidently a change in direction is a vivid indicator of interactions. If you observe a change in direction of the motion of a proton, you will find another object somewhere that has interacted with this proton. As we'll see in Chapter 3, protons are electrically charged, so perhaps the interaction may have been an electric interaction.

1.2.2 Change of Speed

Suppose that you observe an electron traveling in a straight line through outer space far from almost all other objects (**Figure 1.2.B**). The path of the electron is shown as though a camera had taken multiple exposures at equal time intervals.

FIGURE 1.2.B An electron moves through space, far from almost all other objects. The initial direction of the electron's motion is to the right, as indicated by the arrow. The ×s represent the position of the electron at equal time intervals.

QUESTION Where is the electron's speed largest? Where is the electron's speed smallest?

The speed is largest at the right, where the ×s are farther apart, which means that the electron has moved farthest during the time interval between exposures. The speed is smallest at the left, where the ×s are closer together, which means that the electron has moved the least distance during the time interval between exposures.

QUESTION As we'll see in Chapter 3, electrons, which have negative electric charges, repel each other. Suppose that the only other object nearby was another electron. What was the approximate initial location of this other electron?

The other electron must have been located to the left of the starting location. Evidently a change in speed is an indicator of interactions. If you observe a change in speed of an electron, you will find another object somewhere that has interacted with the electron.

1.2.3 Velocity Includes Both Speed and Direction

In physics, the word *velocity* has a special technical meaning that is different from its meaning in everyday speech. In physics, *velocity* denotes a combination of speed and direction. Even if the speed or direction of motion is changing, the velocity has a precise value (speed and direction) at any instant. In contrast, in everyday speech, "speed" and "velocity" are often used as synonyms. In physics and other sciences, however, words have rather precise meanings and there are few synonyms.

For example, consider an airplane that at a particular moment is flying with a speed of 1000 kilometers per hour in a direction that is due east. We say the velocity is 1000 km/h, east, where we specify both speed and direction. An airplane flying west with a speed of 1000 km/h would have the same speed but a different velocity. In Section 1.6, we will define velocity in a mathematical formalism that includes both speed and direction.

We have seen that a change in an object's speed, or a change in the direction of its motion, indicates that the object has interacted with at least one other object. These two indicators of interaction, change of speed and change of direction, can be combined into one compact statement:

A change of velocity (speed or direction or both) indicates the existence of an interaction.

In physics diagrams, the velocity of an object is represented by an arrow: a line with an arrowhead. The tail of the arrow is placed at the location of the object at a particular instant, and the arrow points in the direction of the motion of the object at that instant. The length of the arrow is proportional to the speed of the object. **Figure 1.2.C** shows two successive positions of a particle at two different times, with velocity arrows indicating a change in speed of the particle (it's slowing down).

FIGURE 1.2.C Two successive positions of a particle (indicated by a dot), with arrows indicating the velocity of the particle at each location. The shorter arrow indicates that the speed of the particle at location 2 is less than its speed at location 1.

Figure 1.2.D shows three successive positions of a different particle at three different times, with velocity arrows indicating a change in direction but no change in speed. Note that the arrows themselves are straight; even if the path of the particle curves over time, at any instant the particle may be considered to be traveling in a specific direction.

The velocity of an object is tangent to its path at every point in its travel (**Figure 1.2.E**).

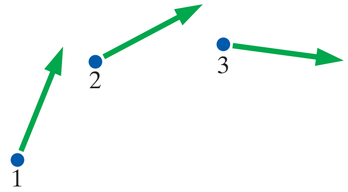


FIGURE 1.2.D Three successive positions of a particle (indicated by a dot), with arrows indicating the velocity of the particle at each location. The arrows are the same length, indicating the same speed, but they point in different directions, indicating a change in direction and therefore a change in velocity.

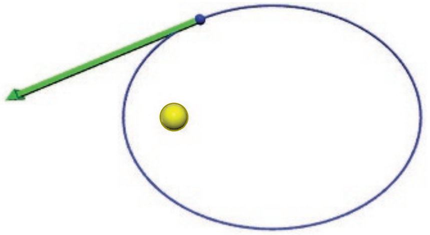


FIGURE 1.2.E An elliptical path followed by a planet orbiting a star. The planet's velocity is a vector (green arrow) whose direction is always tangent to the path.

Interactive

We will see a little later that velocity is only one example of a physical quantity that has a magnitude (an amount or a size) and a direction. Other examples of such quantities are position relative to an origin in 3D space, changes in position or velocity, and force. In Section 1.4, we will see how to represent such quantities as vectors: single mathematical entities that combine information about magnitude and direction.

1.2.4 Constant Velocity

Suppose that you observe a rock moving along in outer space far from all other objects. We don't know what made it start moving in the first place; presumably a long time ago an interaction gave it some velocity and it has been coasting through the vacuum of space ever since.

It is an observational fact that such an isolated object moves at constant, unchanging speed, in a straight line. Its velocity does not change (neither its direction nor its speed changes). Sometimes we call motion with unchanging velocity *uniform motion* (**Figure 1.2.F**). A simpler term is *constant velocity*, because velocity refers to both speed and direction.

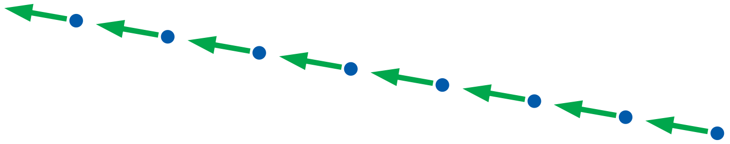


FIGURE 1.2.F Constant velocity (uniform motion)—no change in speed or direction. Green arrows represent velocity. Positions are recorded at equal time intervals.

QUESTION Does an object at rest have a constant velocity?

If an object remains at rest, then neither the speed nor direction of the object's velocity changes. This is a special case of constant velocity: the object's speed is constant (zero is a valid value of speed) and the direction of motion, while undefined, is not changing.

QUESTION If we observe an object with constant velocity, can we conclude that it has no interactions with its surroundings?

When we observe an object with constant velocity, one possibility is that it has no interactions at all with its surroundings. However, there is another possibility: the object may be experiencing multiple interactions that cancel each other out. In either case, we can correctly deduce that the net (total) interaction of the object with its surroundings is zero.

1.2.5 Other Indicators of Interaction

A change in velocity is not the only indication that an object has interacted with its surroundings, but it is the only change possible for a single object that is modeled as a point particle, which has neither shape nor internal structure. In later chapters, we will examine other kinds of changes, such as change of temperature, change of shape or configuration, and change of identity (for example, in nuclear reactions). In Chapters 1–3, however, we will concentrate on how interactions change motion.

Checkpoint 1.2-C-01

a. Which of the following is/are moving with constant velocity?

1. A ship sailing northeast at a speed of 5 meters per second.
2. The Moon orbiting the Earth.
3. A tennis ball traveling across the court after having been hit by a tennis racket.
4. A can of soda sitting on a table.
5. A person riding on a Ferris wheel that is turning at a constant rate.

b. In which of the following situations is there observational evidence for significant interaction between two objects? For each situation, explain how you can you tell.

1. A ball bounces off a wall with no change in speed.
2. A baseball that was hit by a batter flies toward the outfield.
3. A communications satellite orbits the Earth, traveling in a circle at constant speed.
4. A space probe travels in a straight line at constant speed toward a distant star.
5. A charged particle detected in a particle detector moves at constant speed, leaving a curving track behind it.

1.3 Newton's First Law of Motion

OBJECTIVES

After studying this section, you should be able to apply Newton's first law to:

- identify situations in which the net (total) interaction of an object with its surroundings is zero.
- identify situations in which the net (total) interaction of an object with its surroundings is nonzero.

The basic relationship between change of velocity and interaction is summarized qualitatively by what is known as Newton's first law of motion, though it was originally discovered by Galileo. In his original Latin, Newton said, "Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare." A literal translation is "Every body persists in its state of resting or of moving uniformly in a direction, except to the extent that it is compelled to change that state by forces pressed upon it." Expressing this in more modern language, we have this:

NEWTON'S FIRST LAW OF MOTION

Every body persists in its state of rest or of moving with constant speed in a constant direction, except to the extent that it is compelled to change that state by forces acting on it.

Force is the way in which the amount of interaction is quantified, and we'll discuss force in detail in Chapter 2. The words "except to the extent" imply that the stronger the interaction, the more change there will be in direction and/or speed. The weaker the interaction, the less change.

It is the *net*, or total, interaction with the surroundings that determines whether an object's motion will be uniform (constant speed and direction). If there is no net (total) interaction at all, the object's velocity will be constant (constant speed and direction). There are two ways this could happen:

- There may be no interactions at all with the surroundings.
- There may be several interactions that cancel each other.

Newton's first law of motion is only qualitative because it doesn't give us a way to calculate quantitatively how much change in speed or direction will be produced by a certain amount of interaction, a subject we will take up in the next chapter. Nevertheless, Newton's first law of motion is important in providing a conceptual framework for thinking about the relationship between interaction and motion.

This law represented a major break with ancient tradition, which assumed that constant pushing was required to keep something moving. This law says something radically different: no interactions at all are needed to keep something moving!

QUESTION To move a box across a table at constant speed in a straight line, you must keep pushing it. Does this contradict Newton's first law?

Because a constant interaction is required to keep the box moving, we might be tempted to conclude that Newton's first law of motion does not apply in many everyday situations. However, what matters is the net interaction of the box with its surroundings, which could be zero if there are multiple interactions that cancel each other out.

QUESTION In addition to your hand, what other objects in the surroundings interact with the box?

The table also interacts with the box, in a way that we call friction. If you push just hard enough to compensate exactly for the table friction, the sum of all the interactions is zero, and the box moves at constant speed as predicted by Newton's first law (**Figure 1.3.A**). (If you push harder than the table does, the box's speed steadily increases.)

It is difficult to observe motion without friction in everyday life because objects almost always interact with many other objects, including air, flat surfaces, and so on. You may be able to think of situations in which you have seen an object keep moving at constant (or nearly constant) velocity, without being pushed or pulled. One example of a nearly friction-free situation is a hockey puck sliding on ice. The puck slides a long way at nearly constant speed in a straight line (constant velocity) because there is little friction with the ice. An even

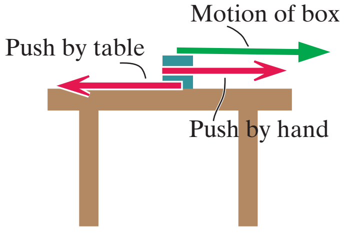


FIGURE 1.3.A The red arrows represent the magnitude and direction of the pushes the box gets from your hand and from the friction with the table. If these pushes add up to zero, the box moves with constant speed in a straight line, indicated by the green arrow.

better example is the constant velocity motion of an object in outer space, far from all other objects.

QUESTION Is a change of position an indicator of an interaction?

Not necessarily. If the change of position occurs simply because a particle is moving at constant speed and direction, then a mere change of position is not an indicator of an interaction because constant velocity is an indicator of zero net interaction. We need to know the object's velocity at each observation to be able to make further deductions.

QUESTION If you observe an object at rest in one location, and later you observe it again at rest but in a different location, can you conclude that an interaction took place?

Yes. You can infer that there must have been an interaction to give the object a nonzero velocity to move the object toward the new position and another interaction to slow the object to a stop in its new position.

QUESTION Is it possible to deduce the existence of an interaction even though you do not observe a change?

As we saw when we considered pushing a box across a table at constant speed, sometimes we may find indirect evidence for an additional interaction. When something doesn't change, although we would normally expect a change due to a known interaction, we can logically deduce that an additional interaction must be occurring. For example, consider a helium filled balloon that hovers motionless in the air despite the downward gravitational pull of the Earth. Evidently, there is some additional kind of interaction that opposes the gravitational interaction. In this case, interactions with air molecules have the net effect of pushing up on the balloon (buoyancy). The lack of change implies that the effect of the air molecules exactly compensates for the gravitational interaction with the Earth.

The stability of the nucleus of an atom is another example of indirect evidence for an additional interaction. The nucleus contains positively charged protons that repel each other electrically, yet the nucleus remains intact. We conclude that there must be some other kind of interaction present, a nonelectric attractive interaction that overcomes the electric repulsion. This is evidence for a nonelectric interaction called the *strong interaction*, which, as we will see, acts among protons and neutrons to hold the nucleus together. We will discuss the strong interaction in Chapter 3.

Checkpoint 1.3-C-01

a. Apply Newton's first law to each of the following situations. In which situations can you conclude that the object is undergoing a net interaction with one or more other objects?

1. A book slides across the table and comes to a stop.
2. A proton in a particle accelerator moves faster and faster.
3. A car travels at constant speed around a circular race track.
4. A spacecraft travels at a constant speed in a straight line toward a distant star.
5. A hydrogen atom remains at rest in outer space.

b. A spaceship far from all other objects uses its rockets to attain a speed of 1×10^4 m/s. The crew then shuts off the power. According to Newton's first law, which of the following statements about the motion of the spaceship after the power is shut off are correct? (Choose all statements that are correct.)

1. The spaceship will move in a straight line.
2. The spaceship will travel on a curving path.
3. The spaceship will enter a circular orbit.
4. The speed of the spaceship will not change.
5. The spaceship will gradually slow down.
6. The spaceship will stop suddenly.

1.4 Describing the 3D World: Vectors

OBJECTIVES

After studying this section, you should be able to use 3D vectors to describe and analyze physical systems. You will:

- carry out vector addition, subtraction, and multiplication by a scalar both numerically and graphically.
- calculate the magnitude of a vector.
- find the unit vector in the direction of a given vector.
- choose the appropriate vector operation to perform in order to determine position, displacement, and relative position.
- determine the angles between a vector and the coordinate axes, and calculate the unit vector from these angles.

1.4.1 Vectors in 3D

Physical phenomena take place in the 3D world around us. To make quantitative predictions and give detailed, quantitative explanations, we need tools for describing precisely the positions and velocities of objects in 3D and the changes in position and velocity due to interactions. These tools are mathematical entities called 3D *vectors*. A symbol denoting a vector is written with an arrow over it:

$$\vec{r} \text{ is a vector}$$

In three dimensions, a vector is a triple of numbers $\langle x, y, z\rangle$. Quantities such as the position or velocity of an object can be represented as vectors:

$$\vec{r}_1 = \langle 3.2, -9.2, 66.3\rangle \text{ m (a position vector)}$$
$$\vec{v}_1 = \langle -22.3, 0.4, -19.5\rangle \text{ m/s (a velocity vector)}$$

Many vectors have units associated with them, such as meters or meters per second. In this course, we will work with important physical quantities that are vectors, including position, velocity, rate of change of velocity (acceleration), momentum, rate of change of momentum, force, angular momentum, torque, electric field, magnetic field, and momentum flow. All of these vectors have associated physical units.

We use the notation $\langle x, y, z\rangle$ for vectors because it emphasizes the fact that a vector is a single entity and because it is easy to work with. This notation appears in many calculus textbooks; you will probably encounter other ways of expressing vectors mathematically as well.

1.4.2 Position Vectors

A position vector is a simple example of a physical vector quantity. We will use a 3D Cartesian coordinate system to specify positions in space and other vector quantities. Usually, we will orient the axes of the coordinate system as shown in **Figure 1.4.A**: $+x$ axis to the right, $+y$ axis upward, and $+z$ axis coming out of the page toward you. This is a *right-handed* coordinate system: if you hold the thumb, first, and second fingers of your right hand perpendicular to each other and align your thumb with the x axis and your first finger with the y axis, your second finger points along the z axis. In some textbook discussions of 3D coordinate systems, the x axis points out, the y axis points to the right, and the z axis points up. This is the same right-handed coordinate system viewed from a different perspective. Because we will sometimes consider motion in a single plane, it makes sense to orient the xy plane in the plane of a vertical page or computer display, so we will use the viewpoint in which the y axis points up.

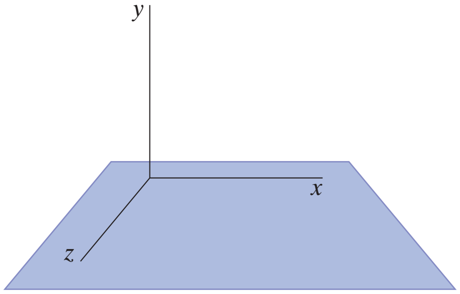


FIGURE 1.4.A Right-handed 3D coordinate system. The xy plane is in the plane of the page, and the z axis projects out of the page toward you.

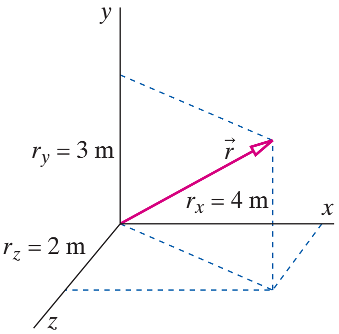


FIGURE 1.4.B A position vector $\vec{r} = \langle 4, 3, 2 \rangle$ m and its x, y, and z components.

A position in 3D space can be considered to be a vector, called a position vector, pointing from an origin to that location. **Figure 1.4.B** shows a position vector, represented by an arrow with its tail at the origin, that might represent your final position if you started at the origin and walked 4 meters along the x axis, then 2 meters parallel to the z axis, then climbed a ladder so you were 3 meters above the ground. Your new position relative to the origin is a vector that can be written like this:

$$\vec{r} = \langle 4, 3, 2 \rangle \text{ m}$$

Each of the numbers in the triple is called a **component** of the vector and is associated with a particular axis. Usually the components of a vector are denoted symbolically by the subscripts x, y, and z:

$$\vec{v} = \langle v_x, v_y, v_z \rangle \quad \text{(a velocity vector)}$$

$$\vec{r} = \langle r_x, r_y, r_z \rangle \quad \text{(a position vector)}$$

$$\vec{r} = \langle x, y, z \rangle \quad \text{(alternative notation for a position vector)}$$

Interactive

The components of the position vector $\vec{r} = \langle 4, 3, 2 \rangle$ m are:

$$r_x = 4 \text{ m} \quad \text{(the } x \text{ component)}$$

$$r_y = 3 \text{ m} \quad \text{(the } y \text{ component)}$$

$$r_z = 2 \text{ m} \quad \text{(the } z \text{ component)}$$

The x component of the vector $\vec{v}$ is the number v_x. The z component of the vector $\vec{v}_1 = \langle -22.3, 0.4, -19.5 \rangle$ m/s is -19.5 m/s. A component such as v_x is not a vector because it is only one number.

QUESTION Can a vector be zero?

The zero vector $\langle 0, 0, 0 \rangle$ is a legal vector, which we will sometimes write as $\vec{0}$. A zero position vector describes the position of an object located at the origin. A zero velocity vector describes the velocity of an object that is at rest at a particular instant.

1.4.3 Drawing Vectors

A position vector is special in that its tail is always at the origin of a coordinate system, but this is not the case for other vectors. It is important to note that the x component of a vector specifies the difference between the x coordinate of the tail of the vector and the x coordinate of the tip of the vector. It does not give any information about the location of the tail of the vector (compare **Figures 1.4.B** and **1.4.C**). By convention, arrows representing vector quantities such as velocity are usually drawn with the tail of the arrow at the location of the object.

In **Figure 1.4.B**, we represented your position vector relative to the origin graphically by an arrow whose tail is at the origin and whose arrowhead is at your position. The length of the arrow represents the distance from the origin, and the direction of the arrow represents the direction of the vector, which is the direction of a direct path from the initial position to the final position (the *displacement*; by walking and climbing, you displaced yourself from the origin to your final position).

Because it is difficult to draw a 3D diagram on paper, you will usually be asked to draw vectors that all lie in a single plane. **Figure 1.4.D** shows an arrow in the xy plane representing the vector $\langle -3, -1, 0 \rangle$.

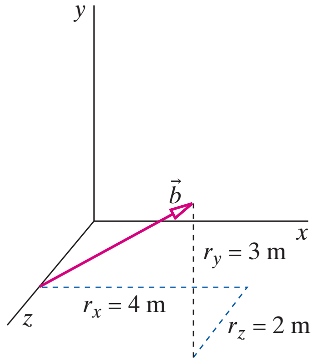


FIGURE 1.4.C The arrow represents the vector $\vec{b} = \langle 4, 3, 2 \rangle$ m drawn with its tail at location $\langle 0, 0, 2 \rangle$ m.

1.4.4 Scalars

A quantity that is represented by a single number is called a scalar. A scalar quantity does not have a direction. Examples include the mass of an object, such as 50 kg, or temperature, such as −20° C. Vectors and scalars are very different entities: a vector can never be equal to a scalar, and a scalar cannot be added to a vector. Scalars can be positive, negative, or zero:

$$m = 50 \text{ kg}$$
$$T = -20^\circ \text{ C}$$

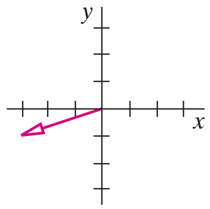


FIGURE 1.4.D The position vector $\langle -3, -1, 0 \rangle$ drawn at the origin in the xy plane. The components of the vector specify the displacement from the tail to the tip. The z axis, which is not shown, comes out of the page toward you.

1.4.5 Vector Operations

Vectors are mathematical entities and have their own mathematical operations. Some of these operations are the same as those you already know for scalars. Others, such as multiplication, are quite different, and division by a vector is not legal. Here are the vector operations that we will discuss and use in this textbook:

VECTOR OPERATIONS

Mathematical operations that are defined for vectors:

- Multiply or divide a vector by a scalar: $2\vec{a}, \vec{v}/5$
- Find the magnitude of a vector: $|\vec{a}|$
- Find a unit vector giving direction: $\hat{a}$
- Add one vector to another: $\vec{a} + \vec{b}$
- Subtract one vector from another: $\vec{a} - \vec{b}$
- Differentiate a vector: $d\vec{r}/dt$
- Dot product of two vectors (result is a scalar): $\vec{a} \cdot \vec{b}$
- Cross product of two vectors (result is a vector): $\vec{a} \times \vec{b}$

The vector dot product will be introduced in Chapter 5 and the vector cross product in Chapter 10.

There are certain operations that are neither legal nor meaningful for vectors:

- A vector cannot be set equal to a scalar.
- A vector cannot be added to or subtracted from a scalar.
- A vector cannot occur in the denominator of an expression. (Although you can't divide by a vector, note that you can legally divide by the magnitude of a vector, which is a scalar.)
- As with scalars, you can't add or subtract vectors that have different units.

1.4.6 Multiplying a Vector by a Scalar

A vector can be multiplied (or divided) by a scalar. If a vector is multiplied by a scalar, each of the components of the vector is multiplied by the scalar:

$$\text{If } \vec{r} = \langle x, y, z \rangle, \text{ then } a\vec{r} = \langle ax, ay, az \rangle$$

$$\text{If } \vec{v} = \langle v_x, v_y, v_z \rangle, \text{ then } \frac{\vec{v}}{b} = \left\langle \frac{v_x}{b}, \frac{v_y}{b}, \frac{v_z}{b} \right\rangle$$

$$\frac{1}{2}\langle 6, -20, 9 \rangle = \langle 3, -10, 4.5 \rangle$$

Multiplication by a scalar "scales" a vector, keeping its direction the same but making its magnitude larger or smaller (**Figure 1.4.E**). Multiplying by a negative scalar reverses the direction of a vector.

$$(-1)\langle 0, 0, 4 \rangle = \langle 0, 0, -4 \rangle$$

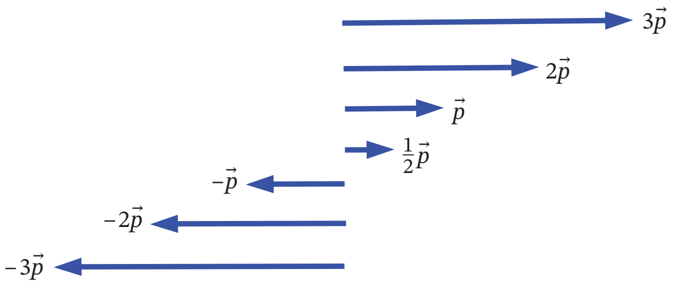


FIGURE 1.4.E Multiplying a vector by a scalar changes the magnitude of the vector. Multiplying a vector by a negative scalar reverses the direction of the vector.

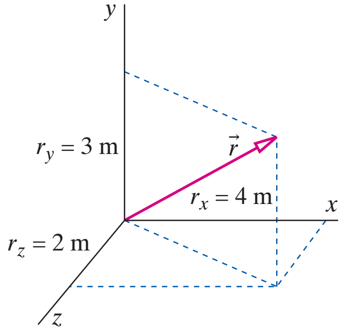


FIGURE 1.4.F A vector representing a displacement from the origin.

Checkpoint 1.4-C-01

You stand at location $\vec{r} = \langle 2, -3, 5 \rangle$ m. Your friend stands at location $\vec{r}/2$. What is your friend's position vector?

1. $\langle 4, -6, 10 \rangle$ m
2. $\langle 1, -3, 5 \rangle$ m
3. $\langle 0.5, -0.5, 0.5 \rangle$ m
4. $\langle 1, -1.5, 2.5 \rangle$ m

1.4.7 Magnitude

Figure 1.4.F shows a vector representing a displacement of $\langle 4, 3, 2 \rangle$ m from the origin. What is the distance from the tip of this vector to the origin?

Using a 3D extension of the Pythagorean theorem for right triangles (**Figure 1.4.G**), we find that

$$\sqrt{(4\text{ m})^2 + (3\text{ m})^2 + (2\text{ m})^2} = \sqrt{29}\text{ m} = 5.39\text{ m}$$

We say that the magnitude $|\vec{r}|$ of the position vector $\vec{r}$ is

$$|\vec{r}| = 5.39\text{ m}$$

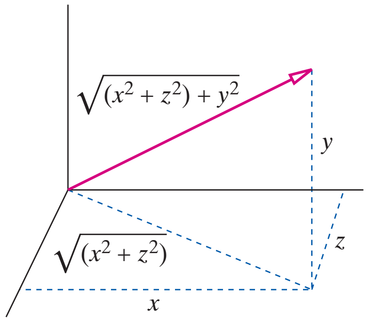


FIGURE 1.4.G The magnitude of a vector is the square root of the sum of the squares of its components (3D version of the Pythagorean theorem).

The magnitude of a vector is written either with absolute-value bars around the vector as $|\vec{r}|$ or simply by writing the symbol for the vector without the little arrow above it, r.

MAGNITUDE OF A VECTOR

$$\text{If the vector } \vec{r} = \langle r_x, r_y, r_z \rangle \text{ then } |\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} \text{ (a scalar).}$$

The magnitude of a vector is always a positive number. The magnitude of a vector is a single number, not a triple of numbers, and it is a scalar, not a vector.

You may wonder about a quantity like $-3\vec{r}$, which involves the product of a scalar and a vector. This expression can be factored:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}|$$

The magnitude of a scalar is its absolute value, so:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}| = 3\sqrt{r_x^2 + r_y^2 + r_z^2}$$

Checkpoint 1.4-C-02

If $\vec{v} = \langle 2, -3, 5 \rangle$ m/s, what is $\left|-\frac{1}{2}\vec{v}\right|$?

1. 12.3 m/s
2. −12.3 m/s
3. 6.16 m/s
4. −3.08 m/s
5. 3.08 m/s

1.4.8 Unit Vectors

One way to describe the direction of a vector is by specifying a unit vector. A unit vector is a vector of magnitude 1, pointing in some direction. A unit vector is written with a "hat" (caret) over it instead of an arrow. The unit vector $\hat{a}$ is called "a-hat."

QUESTION Is the vector $\langle 1, 1, 1 \rangle$ a unit vector?

The magnitude of $\langle 1, 1, 1 \rangle$ is $\sqrt{1^2 + 1^2 + 1^2} = 1.73$, so this is not a unit vector.

The vector $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$ is a unit vector because its magnitude is 1:

$$\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = 1$$

Note that every component of a unit vector must be less than or equal to 1.

In our 3D Cartesian coordinate system, there are three special unit vectors, oriented along the three axes. They are called i-hat, j-hat, and k-hat, and they point along the x, y, and z axes, respectively (**Figure 1.4.H**):

$$\hat{\imath} = \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

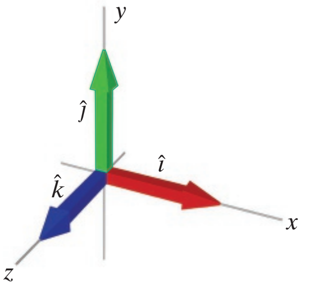


FIGURE 1.4.H The unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$

One way to express a vector is in terms of these special unit vectors:

$$\langle 0.02, -1.7, 30.0\rangle = 0.02\hat{\imath} + (-1.7)\hat{\jmath} + 30.0\hat{k}$$

Not all unit vectors point along an axis, as shown in **Figure 1.4.I**. For example, the vectors

$$\hat{g} = \langle 0.5774, 0.5774, 0.5774\rangle \text{ and } \hat{r} = \langle 0.424, 0.566, 0.707\rangle$$

are both approximately unit vectors because the magnitude of each is approximately equal to 1. Again, note that every compon*ent of a unit vector is less than or equal to 1.*

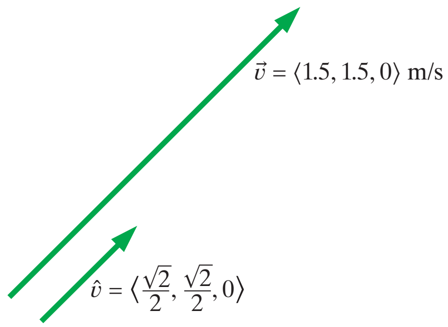


FIGURE 1.4.I The unit vector $\hat{v}$ has the same direction as the vector $\vec{v}$, but its magnitude is 1, and it has no physical units.

Any vector may be factored into the product of a unit vector in the direction of the vector, multiplied by a scalar equal to the magnitude of the vector.

$$\vec{w} = |\vec{w}| \cdot \hat{w}$$

For example, a vector of magnitude 5, aligned with the y axis, could be written as:

$$\langle 0, 5, 0\rangle = 5\langle 0, 1, 0\rangle$$

Therefore, to find a unit vector in the direction of a particular vector, we just divide the vector by its magnitude:

FINDING A UNIT VECTOR

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z\rangle}{\sqrt{(x^2 + y^2 + z^2)}}$$

$$\hat{r} = \left\langle \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{y}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \right\rangle$$

EXAMPLE 1.4.A | Magnitude and Direction

Factor the vector $\vec{v} = \langle -22.3, 0.4, -19.5\rangle$ m/s into a magnitude times a unit vector.

Solution

$$|\vec{v}| = \sqrt{(-22.3)^2 + (0.4)^2 + (-19.5)^2} \text{ m/s} = 29.6 \text{ m/s}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -22.3, 0.4, -19.5\rangle \text{ m/s}}{29.6 \text{ m/s}} = \langle -0.753, 0.0135, -0.658\rangle$$

$$\vec{v} = (29.6 \text{ m/s})\langle -0.753, 0.0135, -0.658\rangle$$

We can now explain algebraically why multiplying a vector by a scalar changes the magnitude but not the direction of a vector. If we write the original vector as the product of a magnitude and a unit vector, after multiplying by a scalar the unit vector is unchanged, but the magnitude is increased or decreased:

$$\vec{a} = \langle 3, -2, 4 \rangle = (5.385)\langle 0.577, -0.371, 0.743 \rangle$$
$$2\vec{a} = (2)(5.385)\langle 0.577, -0.371, 0.743 \rangle$$
$$= 10.770\langle 0.577, -0.371, 0.743 \rangle$$
$$= \langle 6, -4, 8 \rangle$$

1.4.9 Equality of Vectors

In order for two vectors to be equal, they must satisfy these conditions:

Equality of Vectors

A vector is equal to another vector if and only if all components of the vectors are equal.

$$\vec{w} = \vec{r} \text{ means that } w_x = r_x \text{ and } w_y = r_y \text{ and } w_z = r_z$$

The magnitudes and directions of two equal vectors are the same:

$$|\vec{w}| = |\vec{r}| \quad \text{and} \quad \hat{w} = \hat{r}$$

Checkpoint 1.4-C-03

Consider the vectors $\vec{r}_1$ and $\vec{r}_2$ represented by arrows in **Figure 1.4.J**.

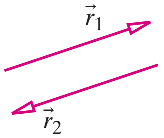


FIGURE 1.4.J Are these two vectors equal?

a. Is $\vec{r}_1 = \vec{r}_2$?

1. Yes
2. No

b. If $\vec{a} = \langle 400, 200, -100 \rangle$ m/s^2 and $\vec{c} = \vec{a}$, what is the unit vector $\hat{c}$ in the direction of $\vec{c}$?

1. $\langle 0.4, 0.2, -0.1 \rangle$
2. $\langle 0.25, 0.5, 1 \rangle$
3. $\langle -0.873, -0.436, 0.218 \rangle$
4. $\langle 0.873, 0.436, -0.218 \rangle$

1.4.10 Vector Addition and Subtraction

Vectors may be added, and one vector may be subtracted from another vector. However, a scalar cannot be added to or subtracted from a vector.

ADDING AND SUBTRACTING VECTORS

The sum or difference of two vectors is another vector, obtained by adding or subtracting the components of the vectors. Given two vectors:

$$\vec{A} = \langle A_x, A_y, A_z \rangle$$
$$\vec{B} = \langle B_x, B_y, B_z \rangle$$

then

$$\vec{A} + \vec{B} = \langle (A_x + B_x), (A_y + B_y), (A_z + B_z) \rangle$$
$$\vec{A} - \vec{B} = \langle (A_x - B_x), (A_y - B_y), (A_z - B_z) \rangle$$

EXAMPLE 1.4.B | Vector Addition and Subtraction

If $\vec{A} = \langle 1, 2, 3 \rangle$ and $\vec{B} = \langle -4, 5, 6 \rangle$,

a. find $\vec{A} + \vec{B}$.

b. find $\vec{A} - \vec{B}$.

Solution

a. $\vec{A} + \vec{B} = \langle (1 + (-4)), (2 + 5), (3 + 6) \rangle = \langle -3, 7, 9 \rangle$

b. $\vec{A} - \vec{B} = \langle (1 - (-4)), (2 - 5), (3 - 6) \rangle = \langle 5, -3, -3 \rangle$

If $\vec{C} = \vec{A} + \vec{B}$, then $\vec{C} - \vec{B} = \vec{A}$ and so on, just as in scalar addition and subtraction. Note also that $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$, which is sometimes useful in the context of graphical subtraction (see below).

QUESTION Is adding the magnitudes of two vectors equivalent to adding two vectors and then taking the magnitude?

No. The magnitude of a vector is not in general equal to the sum of the magnitudes of the two original vectors! For example, the magnitude of the vector $\langle 3, 0, 0 \rangle$ is 3, and the magnitude of the vector $\langle -2, 0, 0 \rangle$ is 2, but the magnitude of the vector $(\langle 3, 0, 0 \rangle + \langle -2, 0, 0 \rangle)$ is 1, not 5!

Checkpoint 1.4-C-04

If $\vec{F}_1 = \langle 300, 0, -200 \rangle$ and $\vec{F}_2 = \langle 150, -300, 0 \rangle$, calculate the following quantities and make the requested comparisons:

a. $\vec{F}_1 + \vec{F}_2$

1. $\langle 1.27, -0.894, -0.555 \rangle$
2. $\langle 0.385, 0.894, -0.555 \rangle$
3. $\langle 450, -300, -200 \rangle$
4. $\langle 150, 300, -200 \rangle$

b. $|\vec{F}_1 + \vec{F}_2|$

1. 577
2. 25.1
3. 696
4. 391

c. $|\vec{F}_1| + |\vec{F}_2|$

1. 577
2. 25.1
3. 696
4. 391

d. $\vec{F}_1 - \vec{F}_2$

1. $\langle 1.27, -0.894, -0.555 \rangle$
2. $\langle 0.385, 0.894, -0.555 \rangle$
3. $\langle 450, -300, -200 \rangle$
4. $\langle 150, 300, -200 \rangle$

e. $|\vec{F}_1 - \vec{F}_2|$

1. 577
2. 25.1
3. 696
4. 391

f. $|\vec{F}_1| - |\vec{F}_2|$

1. 577
2. 25.1
3. 696
4. 391

1.4.11 Graphical Addition and Subtraction

The sum of two vectors has a geometric interpretation. Imagine that you are walking in a straight line in hilly terrain. In **Figure 1.4.K**, frame (a), you first walk along displacement vector $\vec{A}$, and then walk along displacement vector $\vec{B}$. What is your net displacement vector $\vec{C} = \vec{A} + \vec{B}$? The x component C_x of your net displacement is the sum of A_x and B_x. Similarly, $C_y = A_y + B_y$, etc.

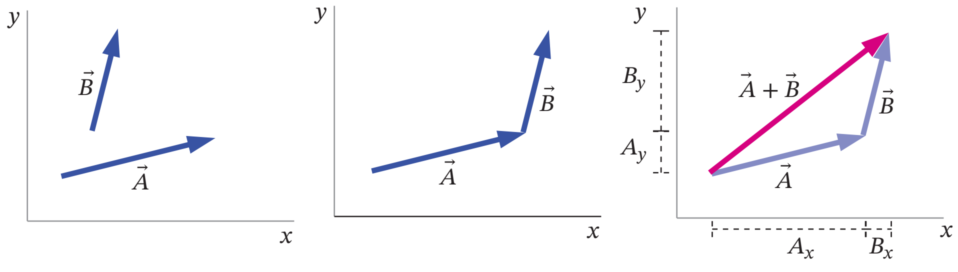


FIGURE 1.4.K The procedure for adding $\vec{A} + \vec{B}$ graphically: (a) Original vectors. (b) Move $\vec{B}$ so the tail of $\vec{B}$ is at the tip of $\vec{A}$. (c) Draw a new arrow starting at the tail of $\vec{A}$ and ending at the tip of $\vec{B}$.

GRAPHICAL ADDITION OF VECTORS

To add two vectors $\vec{A}$ and $\vec{B}$ graphically (**Figure 1.4.K**):

- Draw the first vector $\vec{A}$.
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the tip of the first vector.
- Draw a new vector from the *tail* of vector $\vec{A}$ to the *tip* of vector $\vec{B}$.

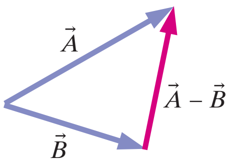


FIGURE 1.4.L The procedure for subtracting vectors graphically: Draw vectors tail to tail; draw a new vector from the tip of the second vector to the tip of the first vector.

GRAPHICAL SUBTRACTION OF VECTORS

To subtract one vector $\vec{B}$ from another vector $\vec{A}$ graphically (**Figure 1.4.L**):

- Draw the first vector $\vec{A}$.
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the tail of the first vector.
- Draw a new vector from the *tip* of vector $\vec{B}$ to the *tip* of vector $\vec{A}$.

Note that you can check this algebraically and graphically. As shown in **Figure 1.4.L**, because the tail of $\vec{A} - \vec{B}$ is located at the tip of $\vec{B}$, then the vector $\vec{A}$ should be the sum of $\vec{B}$ and $\vec{A} - \vec{B}$, as indeed it is:

$$\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$$

Collinear Vectors Graphical addition and subtraction of collinear vectors would be messy and difficult to interpret if we actually drew the arrows on top of each other. To make diagrams easier to interpret, we typically offset arrows slightly so we can see the results (**Figure 1.4.M**).

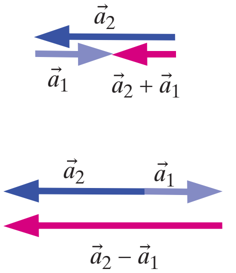


FIGURE 1.4.M To add (top diagram) and subtract (bottom diagram) collinear vectors graphically, we offset the arrows slightly for clarity.

Checkpoint 1.4-C-05

Which of the following statements about the three vectors in **Figure 1.4.N** are correct?

1. $\vec{s} = \vec{t} - \vec{r}$
2. $\vec{r} = \vec{t} - \vec{s}$
3. $\vec{r} + \vec{t} = \vec{s}$
4. $\vec{s} + \vec{t} = \vec{r}$
5. $\vec{r} + \vec{s} = \vec{t}$

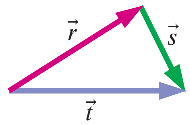


FIGURE 1.4.N Three vectors

1.4.12 Commutativity and Associativity

Vector addition is commutative:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Vector subtraction is not commutative:

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

The associative property holds for vector addition and subtraction:

$$(\vec{A} + \vec{B}) - \vec{C} = \vec{A} + (\vec{B} - \vec{C})$$

1.4.13 Applications of Vector Subtraction

Because we are interested in changes caused by interactions, we will frequently need to calculate the change in a vector quantity. For example, we may want to know the change in a moving object's position or the change in its velocity during some time interval. Finding such changes requires vector subtraction.

The Greek letter Δ (capital delta, signifying *D* for *Difference*) is traditionally used to denote the change in a quantity (either a scalar or a vector). We use the subscript i to denote an initial value of a quantity and the subscript f to denote the final value of a quantity.

Δ (DELTA) IS THE SYMBOL FOR A CHANGE

The symbol Δ (Greek uppercase delta) means *final minus initial*. If a vector $\vec{r}_i$ denotes the initial position of an object relative to the origin (its position at the beginning of a time interval), and $\vec{r}_f$ denotes the final position of the object, then

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$\Delta\vec{r}$ means change of $\vec{r}$ or $\vec{r}_f - \vec{r}_i$ (displacement).

Δt means change of t or $t_f - t_i$ (time interval).

Because subtraction is not commutative, the order of the quantities matters: the symbol Δ (delta) always means final minus initial, *not* initial minus final. For example, when a child's height changes from 1.1 m to 1.2 m, the change is $\Delta y = +0.1$ m, a positive number. If your bank account dropped from \$150 to \$130, what was the change in your balance? Δ (bank account) = −20 dollars.

Another important application of vector subtraction is the calculation of relative position vectors, vectors that represent the position of one object relative to another object.

RELATIVE POSITION VECTOR

If object 1 is at location $\vec{r}_1$ and object 2 is at location $\vec{r}_2$ (**Figure 1.4.O**), the position of 2 relative to 1 is:

$$\vec{r}_{2\text{ relative to }1} = \vec{r}_2 - \vec{r}_1$$

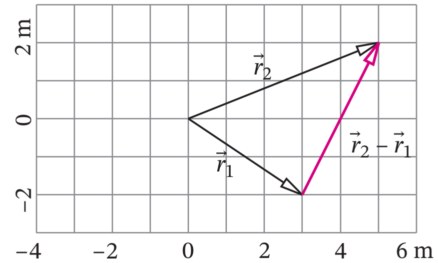


FIGURE 1.4.O Relative position vector.

Checkpoint 1.4-C-06

At 10:00 AM you are at location ⟨−3, 2, 5⟩ m. By 10:02 AM, you have walked to location ⟨6, 4, 25⟩ m.

a. What is $\Delta\vec{r}$, the change in your position?

1. ⟨6, 4, 25⟩ m
2. ⟨9, 2, 20⟩ m
3. ⟨3, 6, 30⟩ m
4. 22.0 m
5. 26.0 m

b. What is Δt, the time interval during which your position changed?

1. 120 s
2. 2.0 s
3. 10.2 s
4. 60 s

1.4.14 Unit Vectors and Angles

Suppose that a taut string is at an angle θ_x to the $+x$ axis and we need a unit vector in the direction of the string. **Figure 1.4.P** shows a unit vector $\hat{A}$ pointing along the string. What is the x component of this unit vector? Consider the triangle whose base is A_x and whose hypotenuse is $|\hat{A}| = 1$. From the definition of the cosine of an angle we have this:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A_x}{1}, \text{ so } A_x = \cos\theta_x$$

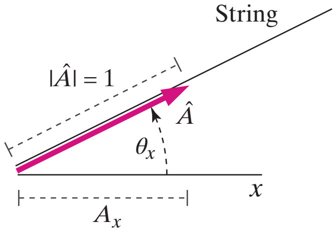


FIGURE 1.4.P A unit vector whose direction is at a known angle from the $+x$ axis.

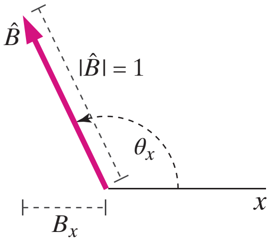


FIGURE 1.4.Q A unit vector in the second quadrant from the $+x$ axis.

In **Figure 1.4.P**, the angle θ_x is shown in the first quadrant (θ_x less than 90°), but this works for larger angles as well. For example, in **Figure 1.4.Q** the angle from the $+x$ axis to a unit vector $\hat{B}$ is in the second quadrant (θ_x greater than 90°); here $\cos\theta_x$ is negative, which corresponds to a negative value of B_x.

What is true for x is also true for y and z. **Figure 1.4.R** shows a 3D unit vector $\hat{r}$ and indicates the angles between the unit vector and the x, y, and z axes. Evidently, we can write

$$\hat{r} = \langle \cos\theta_x, \cos\theta_y, \cos\theta_z \rangle$$

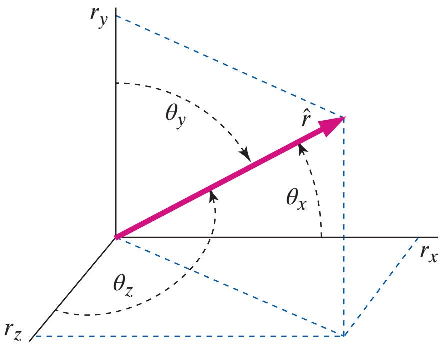


FIGURE 1.4.R A 3D unit vector and its angles to the x, y, and z axes.

The three cosines of the angles between a vector (or unit vector) and the coordinate axes shown in **Figure 1.4.R** are called the *direction cosines* of the vector. The cosine function is never greater than 1, just as no component of a unit vector can be greater than 1.

A common special case is that of a unit vector lying in the xy plane, with zero z component (**Figure 1.4.S**). In this case $\theta_x + \theta_y = 90°$, so that $\cos\theta_y = \cos(90° - \theta_x) = \sin\theta_x$, therefore you can express the cosine of θ_y as the sine of θ_x, which is often convenient. However, in the general 3D case shown in **Figure 1.4.R**, there is no such simple relationship among the direction angles or among their cosines.

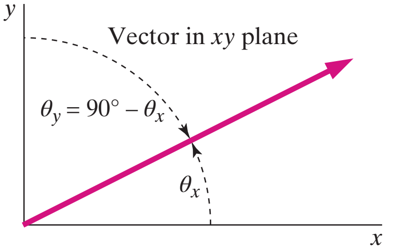


FIGURE 1.4.S If a vector lies in the xy plane, $\cos\theta_y = \sin\theta_x$.

FINDING A UNIT VECTOR FROM ANGLES

To find a unit vector if angles are given:

- Redraw the vector of interest with its tail at the origin and determine the angles between this vector and the axes.
- Imagine the vector $\langle 1, 0, 0\rangle$, which lies on the $+x$ axis. θ_x is the angle through which you would rotate the vector $\langle 1, 0, 0\rangle$ until its direction matched that of your vector. θ_x is positive, and $\theta_x \leq 180°$.

- θ_y is the angle through which you would rotate the vector $\langle 0, 1, 0 \rangle$ until its direction matched that of your vector. θ_y is positive, and $\theta_y \le 180°$.
- θ_z is the angle through which you would rotate the vector $\langle 0, 0, 1 \rangle$ until its direction matched that of your vector. θ_z is positive, and $\theta_z \le 180°$.

EXAMPLE 1.4.C | From Unit Vector to Angles Video

A vector $\vec{r}$ points from the origin to the location $\langle -600, 0, 300 \rangle$ m. What is the angle that this vector makes to the x axis? To the y axis? To the z axis?

Solution

$$\hat{r} = \frac{\langle -600, 0, 300 \rangle \text{ m}}{\sqrt{(-600)^2 + (0)^2 + (300)^2} \text{ m}} = \langle -0.894, 0, 0.447 \rangle$$

But we also know that $\hat{r} = \langle \cos\theta_x, \cos\theta_y, \cos\theta_z \rangle$, so $\cos\theta_x = -0.894$, and the arccosine gives $\theta_x = 153.4°$. Similarly,

$$\cos\theta_y = 0, \text{ so } \theta_y = 90° \quad \text{(which checks; no } y \text{ component)}$$
$$\cos\theta_z = 0.447, \text{ so } \theta_z = 63.4°$$

Looking down on the xz plane in **Figure 1.4.T**, you can see that the difference between $\theta_x = 153.4°$ and $\theta_z = 63.4°$ is 90°, as it should be.

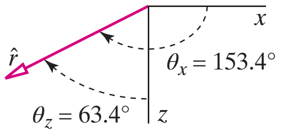


FIGURE 1.4.T Look down on the xz plane. The difference in the two angles is 90°, as it should be.

EXAMPLE 1.4.D | From Angle to Unit Vector Video

A rope lying in the xy plane, pointing up and to the right, supports a climber at an angle of 20° to the vertical (**Figure 1.4.U**). What is the unit vector pointing up along the rope?

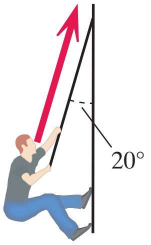


FIGURE 1.4.U A climber supported by a rope.

Solution Follow the procedure given above for finding a unit vector from angles. In **Figure 1.4.V**, we redraw the vector with its tail at the origin determine the angles between the vector and the axes.

If we rotate the unit vector $\langle 1, 0, 0 \rangle$ from along the $+x$ axis to the vector of interest, we see that we have to rotate through an angle $\theta_x = 70°$. To rotate the unit vector $\langle 0, 1, 0 \rangle$ from along the $+y$ axis to the vector of interest, we have to rotate through an angle of $\theta_y = 20°$. The angle from the $+z$ axis to our vector is $\theta_z = 90°$. Therefore, the unit vector that points along the rope is:

$$\langle \cos 70°, \cos 20°, \cos 90° \rangle = \langle 0.342, 0.940, 0 \rangle$$

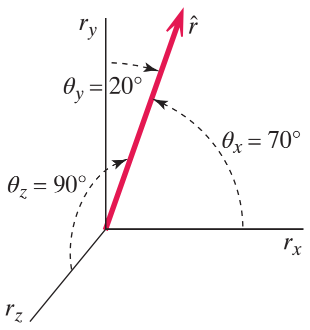


FIGURE 1.4.V Redraw the vector with its tail at the origin. Identify the angles between the positive axes and the vector. In this example, the vector lies in the xy plane.

You may have noticed that the y component of the unit vector can also be calculated as $\sin 70° = 0.940$, and it can be useful to recognize that a vector component can be obtained using sine instead of cosine. There is, however, an advantage in consistently calculating in terms of direction cosines. This is a method that always works, especially in 3D, and that helps avoid errors due to choosing the wrong trig function.

Checkpoint 1.4-C-07

a. A unit vector lies in the xy plane, at an angle of 160° from the $+x$ axis, with a positive y component. What is the unit vector? (It helps to draw a diagram.)

1. $\langle -0.940, -0.342, 0\rangle$
2. $\langle -0.940, 0.940, 0\rangle$
3. $\langle -0.342, 0.940, 0\rangle$
4. $\langle -0.940, 0.342, 0\rangle$

b. A string runs up and to the left in the xy plane, making an angle of 40° to the vertical. Determine the unit vector that points along the string.

1. $\langle 0.643, 0.766, 0\rangle$
2. $\langle -0.766, 0.643, 0\rangle$
3. $\langle -0.643, 0.766, 0\rangle$
4. $\langle 0.766, 0.643, 0\rangle$

1.4.15 Reorienting the Coordinate Axes

In order to describe position and displacement, we had to choose an origin and a set of axes. What if we had made different choices? Certain quantities related to vectors change when a different orientation is chosen for the coordinate axes, but others remain the same. Scalar quantities such as mass and temperature do not change. The magnitude of a vector remains the same when axes are oriented differently, even though the components of the vector do change—the x component of velocity will have a different value if the x axis is chosen to have a different orientation. (Because of this, a vector component is not considered mathematically to be a true scalar, even though it is a single number. However, this distinction is not going to be important in the context of the physics we will study in this course.)

1.5 SI Units

OBJECTIVES

After studying this section, you should be able to use SI units. To do this, you will:

- interpret metric prefixes such as nano- and kilo-.
- convert units to SI units, given appropriate conversion factors.

In this book, we use the SI (Système Internationale) unit system, as is customary in technical work. The SI unit of mass is the kilogram (kg), the unit of distance is the meter (m), and the unit of time is the second (s) (**Table 1.1**). In later chapters, we will encounter other SI units, such as the newton (N), which is a unit of force. Many quantities combine SI units (for example, velocity, which has units of m/s).

It is essential to use SI units in physics equations; this may require that you convert from some other unit system to SI units. Common metric prefixes are shown in **Table 1.2**. If mass is known in grams, you need to divide by 1000 and use the mass in kilograms. If a distance is given in centimeters, you need to divide by 100 to convert the distance to meters. If the time is measured in minutes, you need to multiply by 60 to use a time in seconds. A convenient way to do such conversions is to multiply by factors that are equal to 1, such as (1 min)/(60 s) or

(100 cm)/(1 m). As an example, consider converting 60 miles per hour to SI units, meters per second. Start with 60 mi/h and multiply by factors of 1:

$$\left(60\,\frac{\cancel{\text{mi}}}{\cancel{\text{h}}}\right)\left(\frac{1\,\cancel{\text{h}}}{60\,\cancel{\text{min}}}\right)\left(\frac{1\,\cancel{\text{min}}}{60\,\text{s}}\right)\left(\frac{5280\,\cancel{\text{ft}}}{1\,\cancel{\text{mi}}}\right)\left(\frac{12\,\cancel{\text{in}}}{1\,\cancel{\text{ft}}}\right)\left(\frac{2.54\,\cancel{\text{cm}}}{1\,\cancel{\text{in}}}\right)\left(\frac{1\,\text{m}}{100\,\cancel{\text{cm}}}\right) = 26.8\,\frac{\text{m}}{\text{s}}$$

Observe how most of the units cancel, leaving final units of m/s.

TABLE 1.1 Basic SI units

Quantity	Unit	Symbol
mass	kilogram	kg
distance	meter	m
time	second	s
charge	coulomb	C

TABLE 1.2 Common metric prefixes

1×10^{3}	kilo	1×10^{-3}	milli
1×10^{6}	mega	1×10^{-6}	micro
1×10^{9}	giga	1×10^{-9}	nano
1×10^{12}	tera	1×10^{-12}	pico

Checkpoint 1.5-C-01

A snail moves at a rate of 16 cm per minute. What is its speed in SI units? Write out the factors as was done above.

1. 2.67×10^{-3} m/s
2. 0.267 m/s
3. 0.16 m/s
4. 16 m/s

1.6 Speed and Velocity

OBJECTIVES

After studying this section, you should be able to:

- distinguish between speed and velocity.
- apply the definition of average velocity.

There are several different quantities that may be used to express how fast an object is moving. In everyday conversation these are often used interchangeably, but in physics they are distinct concepts and must not be treated as synonyms.

1.6.1 Average Speed

Speed is a scalar quantity. It may be somewhat surprising to reflect on the fact that speed cannot be measured directly in a single measurement. Even the radar guns commonly used to measure the speed of cars or baseballs do not make instantaneous measurements. The wavelength of radar waves reflected from a moving object is different from that of the original waves; the original and reflected waves must be compared over a period of time to determine the speed of the moving object.

Because the measurement interval is finite, it is of course possible that the moving object is speeding up or slowing down during that interval. So what speed do we actually measure, if we are not measuring the exact speed at one instant? We refer to the results of our measurements as average speed.

To determine the average speed of a moving object, it is necessary to measure two times and two positions and to calculate the speed by dividing the distance traveled by the time elapsed. The *average speed* of an object can be found using the familiar expression

$$s_{avg} = d/\Delta t$$

where d is the total distance traveled and Δt is the elapsed time.

EXAMPLE 1.6.A | A Sprinter's Average Speed

At the 2008 Summer Olympics in Beijing, Jamaican sprinter Usain Bolt won the gold medal in the 100 m race, finishing in a time of 9.69 s and setting a new world record for the event. What was his average speed? From videos of the event one can determine that he reached the 60 meter mark at a time 5.73 s after the start of the race. Was his speed constant?

Solution Bolt's average speed over the entire 100 m was

$$s_{\text{avg}} = \frac{100\text{ m}}{9.69\text{ s}} = 10.32\text{ m/s}$$

However, his average speed over the first 60 m was

$$s_{\text{avg}} = \frac{60\text{ m}}{5.73\text{ s}} = 10.47\text{ m/s}$$

Bolt clearly slowed down at the end of the race, beginning to celebrate his win even before reaching the finish line.

If we wanted to know Bolt's speed as accurately as possible at the instant he reached the 60 m mark, we would need to know his position at two times: just before he reached the mark and just afterward. The smaller the time interval, the closer the average speed we calculate will be to the actual instantaneous speed.

1.6.2 Average Velocity

Because it has a magnitude and a direction, velocity is a vector quantity. As with speed, to determine velocity we need two measurements separated in time. With this data, we can find the displacement, or change of position $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$ of an object during a time interval, where $\vec{r}_i$ is the initial 3D position and $\vec{r}_f$ is the final 3D position. Displacement can be thought of as the position of an object relative to its initial position. For example, **Figure 1.6.A** shows the displacement of a comet over a 5 year period.

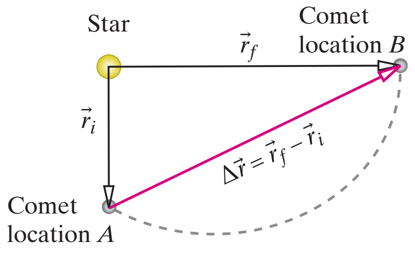


FIGURE 1.6.A A comet orbiting a star moves from location A to location B in 5 years. The vector $\Delta\vec{r}$ is the displacement of the comet during this time period. The dotted line indicates the actual path taken by the comet.

Dividing the (vector) displacement by the (scalar) time interval (final time minus initial time) gives the average (vector) velocity of the object:

DEFINITION: AVERAGE VELOCITY

$$\vec{v}_{\text{avg}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

A more compact way of writing this expression, using the symbol Δ (capital Greek delta, defined in Section 1.4.13), is

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t}$$

Note that we are using vector subtraction here to find the displacement of a single object from one time to another, while earlier we used vector subtraction to find the relative position of one object with respect to a second object at a single time. The mathematical operation is the same, but the physical meaning is different.

EXAMPLE 1.6.B | Average Velocity of a Bee

Consider a bee in flight. At time $t_i = 15.0$ s after 9:00 AM, the bee's position vector was $\vec{r}_i = \langle 2, 4, 0\rangle$ m. At time $t_f = 15.1$ s after 9:00 AM, the bee's position vector was $\vec{r}_f = \langle 3, 3.5, 0\rangle$ m. What was the average velocity of the bee during this interval? Express this vector as the product of the magnitude of the velocity (speed) and a unit vector in the direction of the velocity.

Solution On the diagram shown in **Figure 1.6.B**, we draw and label three arrows representing the vectors $\vec{r}_i$, $\vec{r}_f$, and $\vec{r}_f - \vec{r}_i$. The tail of the latter arrow is placed at the bee's initial position. The vector $\vec{r}_f - \vec{r}_i$, which points in the direction of the bee's motion, is the displacement of the bee during this time interval.

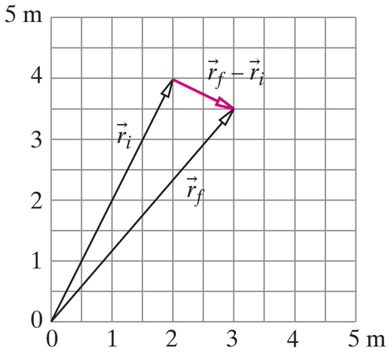


FIGURE 1.6.B The displacement vector points from initial position to final position.

We calculate the bee's displacement vector numerically by taking the difference of the two vectors, final minus initial:

$$\vec{r}_f - \vec{r}_i = \langle 3, 3.5, 0\rangle \text{ m} - \langle 2, 4, 0\rangle \text{ m} = \langle 1, -0.5, 0\rangle \text{ m}$$

This numerical result should be consistent with our graphical construction. Look at the components of $\vec{r}_f - \vec{r}_i$ in **Figure 1.6.B**. Do you see that this vector has an x component of $+1$ m and a y component of -0.5 m?

The average velocity of the bee, a vector quantity, is the (vector) displacement $\vec{r}_f - \vec{r}_i$ divided by the (scalar) time interval $t_f - t_i$.

$$\vec{v}_{\text{avg}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} = \frac{\langle 1, -0.5, 0\rangle \text{ m}}{(15.1 - 15.0) \text{ s}} = \frac{\langle 1, -0.5, 0\rangle \text{ m}}{0.1 \text{ s}} = \langle 10, -5, 0\rangle \text{ m/s}$$

Because we divided $\vec{r}_f - \vec{r}_i$ by a scalar $(t_f - t_i)$, the average velocity $\vec{v}_{\text{avg}}$ points in the direction of the bee's motion, assuming that the bee flew in a straight line.

The average speed of the bee is the magnitude of its velocity:

$$|\vec{v}_{\text{avg}}| = \sqrt{10^2 + (-5)^2 + 0^2} \text{ m/s} = 11.18 \text{ m/s}$$

The direction of the bee's motion, expressed as a unit vector, is:

$$\hat{v}_{\text{avg}} = \frac{\vec{v}_{\text{avg}}}{|\vec{v}_{\text{avg}}|} = \frac{\langle 10, -5, 0\rangle \text{ m/s}}{11.18 \text{ m/s}} = \langle 0.894, -0.447, 0\rangle$$

Note that the "m/s" units cancel; the result is dimensionless. We can check that this really is a unit vector:

$$\sqrt{0.894^2 + (-0.447)^2 + 0^2} = 0.9995$$

This is not quite 1.0 due to rounding the velocity coordinates and speed to three significant figures. To check, we can put the pieces back together and see what we get. The original vector factors into the product of the magnitude times the unit vector:

$$|\vec{v}|\hat{v} = (11.18 \text{ m/s})\langle 0.894, -0.447, 0\rangle = \langle 10, -5, 0\rangle \text{ m/s}$$

This is the same as the original vector $\vec{v}$.

Average Speed vs. Magnitude of Average Velocity Note that if an object changes direction, its average speed may not be equal to the magnitude of its average velocity over a given interval.

For example, suppose that a runner makes one lap around a 400 m track in 50 s. Her average speed is

$$s_{avg} = \frac{400\text{ m}}{50\text{ s}} = 8\text{ m/s}$$

However, her displacement over this interval is actually zero because she starts and ends at the same location. Therefore, her average velocity was $\vec{0}$ and the magnitude of her average velocity was 0.

1.6.3 Scaling a Vector to Fit on a Graph

We can plot the average velocity vector on the same graph that we use for showing the vector positions of the bee (**Figure 1.6.C**). However, note that velocity has units of meters per second whereas positions have units of meters, so we are mixing quantities on this diagram.

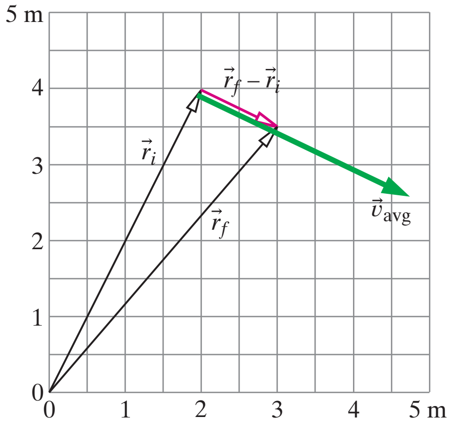


FIGURE 1.6.C Average velocity vector: displacement divided by time interval.

Moreover, the magnitude of the vector, 11.18 m/s, doesn't fit on a graph that is only 5 units wide (in meters). It is standard practice in such situations to scale down the arrow representing the vector to fit on the graph, preserving the correct direction. In **Figure 1.6.C**, we've scaled down the arrow representing the velocity vector by about a factor of 3 to make the arrow fit on the graph. Of course, if there is more than one velocity vector, we use the same scale factor for all the velocity vectors. The same kind of scaling is used with other physical quantities that are vectors, such as force and momentum, which we will encounter later.

Checkpoint 1.6-C-01

At a time 0.2 s after it has been hit by a tennis racket, a tennis ball is located at $\langle 5, 7, 2 \rangle$ m, relative to an origin in one corner of a tennis court. At a time 0.7 s after being hit, the ball is located at $\langle 9, 2, 8 \rangle$ m.

a. What is the average velocity of the tennis ball?

1. $\langle 18, 4, 16 \rangle$ m/s
2. $\langle 28, 18, 20 \rangle$ m/s
3. $\langle 8, -10, 12 \rangle$ m/s
4. $\langle 4, -5, 6 \rangle$ m/s

b. What is the magnitude of the average velocity of the tennis ball?

1. 10 m/s
2. 17.5 m/s
3. 24.4 m/s
4. 8.77 m/s

c. What is the unit vector in the direction of the ball's average velocity?

1. $\langle 1, -1, 1\rangle$
2. $\langle 0.721, -0.463, 0.515\rangle$
3. $\langle 0.737, 0.164, 0.655\rangle$
4. $\langle 0.456, -0.570, 0.684\rangle$

1.7 Predicting a New Position

OBJECTIVES

After studying this section, you should be able to solve problems involving velocity. You should be able to:

- use average velocity to update the position of an object.
- explain the difference between average and instantaneous velocity.
- given an object's path, draw an arrow representing its instantaneous velocity at a particular location.

1.7.1 The Position Update Equation

We can rewrite the velocity relationship in the form

$$\left(\vec{r}_f - \vec{r}_i\right) = \vec{v}_{\text{avg}}\left(t_f - t_i\right)$$

That is, the (vector) displacement of an object is its average (vector) velocity times the time interval. This is just the vector version of the simple notion that if you run at a speed of 7 m/s for 5 s, you move a distance of (7 m/s)(5 s) = 35 m, or that a car going 50 mi/h for 2 h goes (50 mi/h)(2 h) = 100 mi.

Is $\left(\vec{r}_f - \vec{r}_i\right) = \vec{v}_{\text{avg}}\left(t_f - t_i\right)$ a valid vector relation? Yes, multiplying a vector $\vec{v}_{\text{avg}}$ times a scalar $\left(t_f - t_i\right)$ yields a vector. We make a further rearrangement to obtain a relation for updating the position when we know the velocity:

THE POSITION UPDATE EQUATION

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\left(t_f - t_i\right)$$

or

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\,\Delta t$$

This equation says that if we know the starting position, the average velocity, and the time interval, we can predict the final position. This equation will be important throughout our work.

The position update equation $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\,\Delta t$ is a vector equation, so we can write out its full component form:

$$\langle x_f, y_f, z_f\rangle = \langle x_i, y_i, z_i\rangle + \left\langle v_{avg,x}, v_{avg,y}, v_{avg,z}\right\rangle \Delta t$$

Because the x component on the left of the equation must equal the x component on the right (and similarly for the y and z components), this compact vector equation represents three separate component equations:

$$x_f = x_i + v_{avg,x}\Delta t$$
$$y_f = y_i + v_{avg,y}\Delta t$$
$$z_f = z_i + v_{avg,z}\Delta t$$

EXAMPLE 1.7.A | Predicting the Position of a Ball

At time $t_i = 12.18$ s after 1:30 PM, a ball's position vector is $\vec{r}_i = \langle 20, 8, -12\rangle$ m. The ball's velocity at that moment is $\vec{v} = \langle 9, -4, 6\rangle$ m/s. At time $t_f = 12.21$ s after 1:30 PM, where will the ball be, assuming that its velocity hardly changes during this short time interval?

Solution

$$\vec{r}_f = \langle 20, 8, -12\rangle \text{ m} + (\langle 9, -4, 6\rangle \text{ m/s})(12.21 - 12.18) \text{ s}$$
$$\vec{r}_f = \langle 20, 8, -12\rangle \text{ m} + \langle 0.27, -0.12, 0.18\rangle \text{ m}$$
$$\vec{r}_f = \langle 20.27, 7.88, -11.82\rangle \text{ m}$$

Checkpoint 1.7-C-01

A proton traveling with a velocity of $\left\langle 3\times 10^5, 2\times 10^5, -4\times 10^5\right\rangle$ m/s passes the origin at a time 9.0 s after a proton detector is turned on. Assuming that the velocity of the proton does not change, what will be its position at time 9.7 s?

1. $\left\langle 2.91\times 10^6, 1.94\times 10^6, -3.88\times 10^6\right\rangle$ m
2. $\left\langle 2.1\times 10^5, 1.4\times 10^5, -2.8\times 10^5\right\rangle$ m
3. $\left\langle 2.7\times 10^6, 1.8\times 10^6, -3.6\times 10^6\right\rangle$ m
4. $\left\langle 3.33\times 10^4, 2.22\times 10^4, -4.44\times 10^4\right\rangle$ m

1.7.2 Instantaneous Velocity

Figure 1.7.A shows the path of a ball, with positions marked at 1-s intervals, and **Table 1.3** lists the position information. While the ball is in the air, its velocity is constantly changing due to interactions with the Earth (gravity) and with the air (air resistance).

TABLE 1.3 Elapsed time and position of the ball at each location shown in Figure 1.7.A.

Loc.	t (s)	Position (m)
A	0.0	$\langle 0, 0, 0\rangle$
B	1.0	$\langle 22.3, 26.1, 0\rangle$
C	2.0	$\langle 40.1, 38.1, 0\rangle$
D	3.0	$\langle 55.5, 39.2, 0\rangle$
E	4.0	$\langle 69.1, 31.0, 0\rangle$
F	5.0	$\langle 80.8, 14.8, 0\rangle$

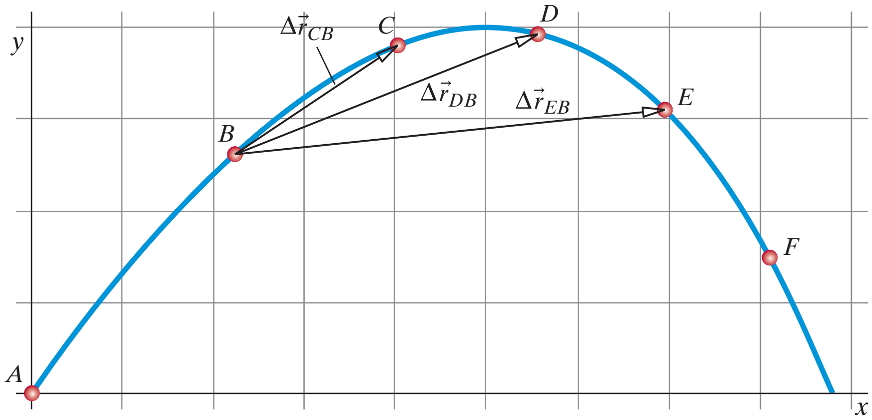


FIGURE 1.7.A The trajectory of a ball through air. The axes represent the x and y distances from the ball's initial location; each square on the grid corresponds to 10 meters. The position of the ball at intervals of 1 s is represented by the colored dots. Three different displacements, corresponding to three different time intervals, are indicated by arrows on the diagram.

Suppose we ask: What is the velocity of the ball at the precise instant that it reaches location B? This quantity would be called the *instantaneous velocity* of the ball. We can start by approximating the instantaneous velocity of the ball by finding its average velocity over some larger time interval.

We can use the position and time data in **Table 1.3** to calculate the average velocity of the ball over three different intervals, by finding the ball's displacement during each interval, and dividing by the appropriate Δt for that interval:

$$\vec{v}_{EB} = \frac{\Delta\vec{r}_{EB}}{\Delta t} = \frac{\vec{r}_E - \vec{r}_B}{t_E - t_B} = \frac{(\langle 69.1, 31.0, 0\rangle - \langle 22.3, 26.1, 0\rangle)\text{ m}}{(4.0 - 1.0)\text{s}}$$
$$= \langle 15.6, 1.6, 0\rangle \frac{\text{m}}{\text{s}}$$

$$\vec{v}_{DB} = \frac{\Delta\vec{r}_{DB}}{\Delta t} = \frac{\vec{r}_D - \vec{r}_B}{t_D - t_B} = \frac{(\langle 55.5, 39.2, 0\rangle - \langle 22.3, 26.1, 0\rangle)\text{ m}}{(3.0 - 1.0)\text{s}}$$
$$= \langle 16.6, 6.55, 0\rangle \frac{\text{m}}{\text{s}}$$

$$\vec{v}_{CB} = \frac{\Delta\vec{r}_{CB}}{\Delta t} = \frac{\vec{r}_C - \vec{r}_B}{t_C - t_B} = \frac{(\langle 40.1, 38.1, 0\rangle - \langle 22.3, 26.1, 0\rangle)\text{ m}}{(2.0 - 1.0)\text{s}}$$
$$= \langle 17.8, 12.0, 0\rangle \frac{\text{m}}{\text{s}}$$

Not surprisingly, the average velocities over these different time intervals are not the same, because both the direction of the ball's motion and the speed of the ball were changing continuously during its flight. The three average velocity vectors that we calculated are shown in **Figure 1.7.B**.

QUESTION Which of the three average velocity vectors depicted in **Figure 1.7.B** best approximates the instantaneous velocity of the ball at location B?

Simply by looking at the diagram, we can tell that $\vec{v}_{CB}$ is closest to the actual instantaneous velocity of the ball at location B, because its direction is closest to the direction in which the ball is actually traveling. Because the direction of the instantaneous velocity is the direction in which the ball is moving at a particular instant, the instantaneous velocity is tangent to the ball's path. Of the three average velocity vectors we calculated, $\vec{v}_{CB}$ best approximates a tangent to the path of the ball. Evidently $\vec{v}_{CB}$, the velocity calculated with the shortest time interval, $t_C - t_B$, is the best approximation to the instantaneous velocity at location B. If we used even smaller values of Δt in our calculation of average velocity, such as 0.1 s, or 0.01 s, or 0.001 s, we would presumably have better and better estimates of the actual instantaneous velocity of the object at the instant when it passes location B.

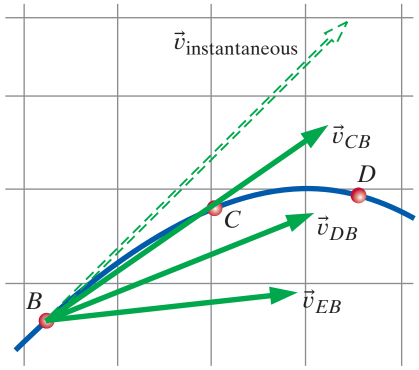


FIGURE 1.7.B A segment of the trajectory shown in **Figure 1.7.A**. The three different average velocity vectors calculated above are shown by three arrows, each with its tail at location *B*. The three arrows representing average velocities are drawn with their tails at the location of interest. The dashed arrow represents the actual instantaneous velocity of the ball at location *B*. Note that because the units of velocity are m/s, these arrows use a different scale from the distance scale used for the path of the ball.

We can draw two important conclusions about instantaneous velocity:

- The direction of the instantaneous velocity of an object is tangent to the path of the object's motion.
- Smaller time intervals yield more accurate estimates of instantaneous velocity.

Checkpoint 1.7-C-02

A comet travels in an elliptical path around a star, in the direction shown in **Figure 1.7.C**. Which arrow best indicates the direction of the comet's instantaneous velocity vector at each of the numbered locations in the orbit?

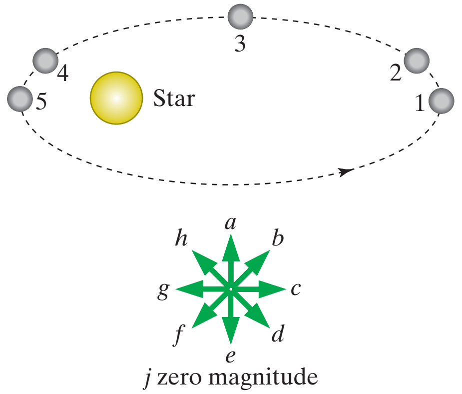


FIGURE 1.7.C A comet goes around a star.

1.7.3 Velocity Is a Rate of Change

You may already have learned about derivatives in calculus. Because instantaneous velocity is the rate of change of position, it is a derivative, the limit of $\Delta\vec{r}/\Delta t$ as the time interval Δt used in the calculation gets closer and closer to zero:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}, \text{which is written as } \vec{v} = \frac{d\vec{r}}{dt}$$

In **Figure 1.7.B**, the process of taking the limit is illustrated graphically. As smaller values of Δt are used in the calculation, the average velocity vectors approach the limiting value: the actual instantaneous velocity.

The rate of change (derivative) of a vector is also a vector. Differentiating a vector simply requires differentiating each component:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\langle x, y, z\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle v_x, v_y, v_z\rangle$$

The derivative of the position vector $\vec{r}$ gives components that are the components of the velocity, as we should expect.

Informally, you can think of $d\vec{r}$ as an infinitesimal displacement and dt as an infinitesimal time interval. It is as though we had continued the process illustrated in **Figure 1.7.B** to smaller and smaller time intervals, down to an extremely tiny time interval dt with a correspondingly tiny displacement $d\vec{r}$. The ratio of these tiny quantities is the instantaneous velocity.

The ratio of these two tiny quantities need not be small. For example, suppose that an object moves in the x direction a tiny distance of 1×10^{-15} m, the radius of a proton, in a very short time interval of 1×10^{-23} s :

$$\vec{v} = \frac{\langle 1 \times 10^{-15}, 0, 0\rangle \text{ m}}{1 \times 10^{-23} \text{ s}} = \langle 1 \times 10^{8}, 0, 0\rangle \text{ m/s}$$

which is one-third the speed of light $(3 \times 10^{8} \text{ m/s})$!

1.7.4 Acceleration

Velocity is the time rate of change of position: $\vec{v} = d\vec{r}/dt$. Similarly, we define *acceleration* as the time rate of change of velocity: $\vec{a} = d\vec{v}/dt$. Acceleration, which is itself a vector quantity, has units of meters per second per second, written as m/s/s or m/s^2.

DEFINITION: ACCELERATION

Instantaneous acceleration is the time rate of change of velocity:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Average acceleration can be calculated from a change in velocity:

$$\vec{a}_{\text{avg}} = \frac{\Delta\vec{v}}{\Delta t}$$

The units of acceleration are m/s^2.

EXAMPLE 1.7.B | Acceleration of a Car

A car traveling in the $+x$ direction speeds up from 20 m/s to 26 m/s in 3 s. What is its average acceleration?

Solution

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\langle 26, 0, 0\rangle \text{ m/s} - \langle 20, 0, 0\rangle \text{ m/s}}{3 \text{ s}} = \langle 2, 0, 0\rangle \text{ m/s/s}$$

For another example, if you drop a rock near the surface of the Earth, its speed increases 9.8 m/s every second, so the magnitude of its acceleration is 9.8 m/s/s, as long as air resistance is negligible.

1.8 Graphs of Motion

OBJECTIVES

After studying this section, you should be able to interpret and sketch graphs related to motion with constant velocity. To do this you will:

- describe the motion represented by a graph of position component (x, y, or z) versus time.
- describe the motion represented by a graph of a velocity component (v_x, v_y, or v_z) versus time.
- given a graph of a velocity component versus time, identify the corresponding position component versus time graph, and vice versa.

1.8.1 One Dimensional Motion

FIGURE 1.8.A An object moves to the right with constant velocity (uniform motion). Images of the object are shown at equal time intervals.

A graph is a compact way of representing motion that helps us infer an object's interactions with its surroundings and make predictions about its future motion. This section focuses on the reasoning involved in relating graphs of constant-velocity motion data to qualitative descriptions of the motion.

In Section 1.2, we found that if an object has no net interaction with its surroundings, its motion is uniform. We depicted its motion by showing the object at equal time intervals along its path and drawing an arrow for its velocity at each instant. **Figure 1.8.A** shows an object moving to the right with constant velocity (uniform motion).

Let's overlay a coordinate system onto the trajectory of the object, with the $+x$ axis in the direction of motion. **Figure 1.8.B** shows the object and the coordinate system, including the origin.

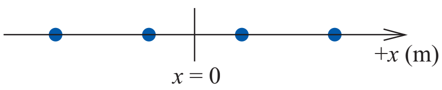


FIGURE 1.8.B Successive positions of an object traveling to the right with constant velocity.

To describe this in words, we might say "An object, initially located on the left side of the origin, travels to the right, in the $+x$ direction, with constant velocity. Its x-position increases at a uniform rate."

Another way to represent uniform motion is with graphs. Define $t = 0$ at the first image of the object; call this the initial x-position of the object. **Figure 1.8.C** shows a qualitative graph of x vs. t for the object.

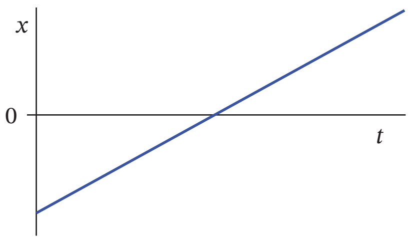


FIGURE 1.8.C A graph of x-position vs. time for an object initially left of the origin and traveling with a constant x-velocity to the right, in the $+x$ direction.

At $t = 0$, the object is at the left of the origin so its intial x-position is negative, as indicated by the vertical intercept on the graph. The slope of the graph is $\Delta x/\Delta t$, which is the x-component of the velocity of the object. In **Figure 1.8.C**, the slope of the line is positive and constant, showing v_x is positive and constant. We can also describe the object's motion by graphing its x velocity vs. time. **Figure 1.8.D** shows a qualitative graph of v_x vs. t for the object.

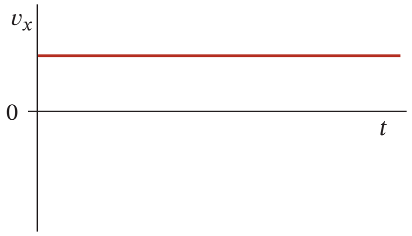


FIGURE 1.8.D A graph of x-velocity vs. time for an object traveling with a constant x-velocity in the $+x$ direction.

From the graph in **Figure 1.8.D**, we notice that the value of v_x is positive, so the object travels in the $+x$ direction. We also see that the value of v_x does not change (the graph is a flat line). The slope of the v_x vs. t graph is $\Delta v_x/\Delta t$, which is zero. Thus, $\Delta v_x = 0$, and we conclude that v_x is constant.

QUESTION Suppose an object has a positive initial x-position and travels to the left, in the $-x$ direction, with constant velocity. What would graphs of x vs. t and v_x vs. t look like?

A qualitative graph of x vs. t is shown in **Figure 1.8.E**. Because the object's initial x-position is positive, the value of x at $t = 0$ on the x vs. t graph is positive. Because the object has a constant velocity in the $-x$ direction, the slope of the x vs. t graph is negative.

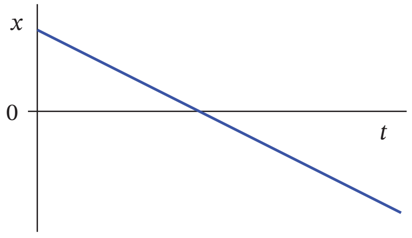


FIGURE 1.8.E A graph of x-position vs. time for an object initially to the right of the origin and traveling with a constant x-velocity to the left, in the $-x$ direction.

Because v_x is constant and negative, a graph of v_x vs. t is a flat line with a negative value of v_x, as shown in **Figure 1.8.F**.

FIGURE 1.8.F A graph of x-velocity vs. time for an object traveling with a constant velocity to the left in the $-x$ direction.

EXAMPLE 1.8.A | Interpreting a Graph

Consider the graph of x vs. t in **Figure 1.8.G**.

1. Verbally describe the motion of the object.
2. Sketch a qualitative v_x vs. t graph for this motion.

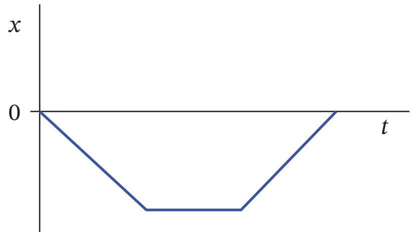


FIGURE 1.8.G A graph of x vs. t.

Solution (1) The graph in **Figure 1.8.G** does not depict uniform motion for the entire time interval in the graph. There are three distinct time intervals, or segments, during which the x-component of velocity is constant. The object is initially at the origin and moves in the $-x$ direction with a constant x-velocity. Then it stops and remains at rest for a certain time interval. Then it moves in the $+x$ direction with a constant x-velocity back to the origin. Interestingly, from the initial time to the final time given in the graph, the x-displacement of the object is zero.

(2) The verbal description of motion guides us in constructing a v_x vs. t graph for the object's motion (**Figure 1.8.H**).

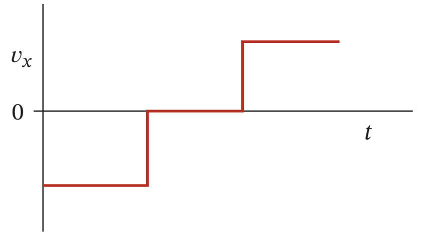


FIGURE 1.8.H The v_x vs. t graph that corresponds to the motion depicted in **Figure 1.8.G**.

Interactive

Note that the graph of v_x vs. t in **Figure 1.8.H** is useful for concluding that the object moves in the $-x$ direction with constant v_x, stops and remains at rest ($v_x = 0$), and then moves in the $+x$ direction with constant v_x. However, this graph alone would not tell us the object started at the origin ($x = 0$). It contains velocity information but no information about the initial x-position of the object. This is because velocity is independent of the choice of origin for the coordinate system.

Checkpoint 1.8-C-01

Figure 1.8.I presents a graph of v_x vs. t for a moving object.

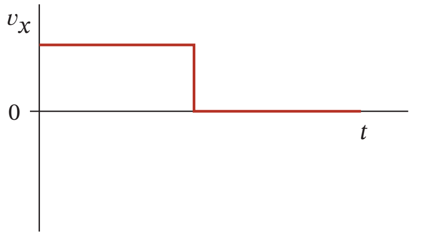


FIGURE 1.8.I A graph of v_x vs. t for an object.

a. Which of these statements correctly describes the motion of the object?

1. The object is at rest at a location on the $+x$ axis and then suddenly moves to the origin and remains at rest at the origin.
2. The object travels in the $+x$ direction and speeds up, then suddenly stops speeding up and travels with a constant speed in the $+x$ direction.
3. The object travels with a constant speed in the $+x$ direction, then suddenly stops and remains at rest.

b. Select the graphs of x vs. t in **Figure 1.8.J** that are consistent with the v_x vs. t graph in **Figure 1.8.I**. There may be more than one.

1. A
2. B
3. C
4. D

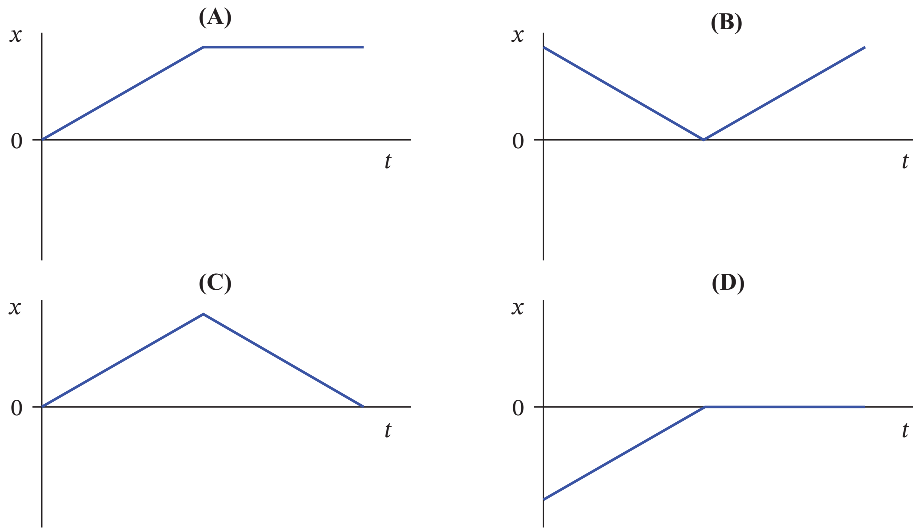


FIGURE 1.8.J Possible graphs of x vs. t.

1.8.2 Three-Dimensional Motion

Up to now we used graphs of x-position vs. time and x-velocity vs. time to describe the motion of an object moving in one dimension along the x-axis. However, what if the object is moving in two or three dimensions and a graph of x-velocity vs. time shows that v_x is constant? Can we conclude that its y-velocity and z-velocity are also constant? Certainly not, because a graph of x-velocity vs. time contains no information about the y or z components of the velocity. To know whether an object's motion in truly uniform, we would also have to graph v_y and v_z vs. time. To be uniform motion, the velocity (a vector!) must be constant, meaning all three velocity components are constant.

It isn't possible to graph a vector. For general three-dimensional motion, to completely describe the motion of an object with graphs, you will need three graphs, one for each component.

1.9 Momentum

OBJECTIVES

After studying this section, you should be able to:

- calculate the momentum of an object.
- calculate the change of momentum of an object over a given interval, numerically and graphically.
- distinguish between momentum, change in momentum, and change in magnitude of momentum.
- given an object's path, draw an arrow representing its instantaneous momentum at a particular location.
- use momentum to predict the future position of an object.

1.9.1 Definition of Momentum (Approximate)

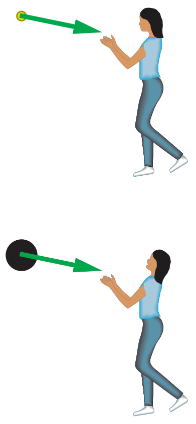


FIGURE 1.9.A Catching a bowling ball requires a larger interaction than catching a tennis ball with the same velocity.

In trying to model the real world, physicists look for powerful ideas that are very general—that is, that apply to a very large range of systems and phenomena. Some of the most powerful and general principles involve hidden quantities—things we do not perceive directly. Momentum is such a quantity.

We have discussed velocity, a vector quantity that describes motion and can be determined from measurements of position and time; position, time, speed, and now 3D velocity are all familiar quantities. However, velocity is not the whole story. Consider the following thought experiment.

Suppose you gently toss a tennis ball to a friend, in such a way that just before the ball reaches her hands, its velocity is $\langle 0.3, -0.2, 0 \rangle$ m/s. When your friend catches the ball, she must interact with the ball to stop its motion, changing its velocity from $\langle 0.3, -0.2, 0 \rangle$ m/s to $\vec{0}$ m/s.

Now suppose that you again toss a ball to your friend with the same velocity; this time, however, the ball is a bowling ball! When the ball reaches her hands with velocity $\langle 0.3, -0.2, 0 \rangle$ m/s, your friend must interact much more strongly with the ball to change its velocity to $\vec{0}$ m/s (**Figure 1.9.A**). Even though the change in the velocities of the two balls is identical, the amount of interaction needed to cause this change is very different. Evidently, the mass of the moving object must explicitly be taken into account.

The larger the mass of the object, the stronger the interaction required to change its motion. Because the same is true for the velocity of the object (your friend would have had to interact more strongly to stop a tennis ball with velocity $\langle 40, 0, 0 \rangle$ m/s), we will surmise that it is the product of mass and velocity that is important. This quantity is called *momentum;* Because it is the product of a scalar and a vector, momentum is a vector. For historical reasons, the symbol used to represent momentum is $\vec{p}$. Momentum is of fundamental importance not only in classical (prequantum) mechanics but also in relativity and quantum mechanics. In Chapter 3, we will discuss the fact that momentum is a *conserved* quantity; the total momentum of the universe is constant. If due to interactions the momentum of an object increases, the momentum of the rest of the universe must decrease.

APPROXIMATE DEFINITION OF MOMENTUM

$$\vec{p} \approx m\vec{v}$$

The units of momentum are kg · m/s. We will see in Section 1.11 that this expression is a good approximation for the momentum of objects traveling at speeds that are small compared to the speed of light.

The momentum is always in the same direction as the velocity.

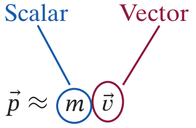


FIGURE 1.9.B The approximate expression for momentum is the product of a scalar times a vector. The scalar factor, mass, must be positive, so the direction of an object's momentum is the same as the direction of its velocity.

Although we can see and compare velocities, momentum is a quantity that we can't see directly. We will encounter other important quantities that aren't directly visible, such as energy, angular momentum, and electric and magnetic fields. Like velocity, momentum is a vector quantity, so it has a magnitude and a direction. Because the mass m must be a positive number, this scalar factor cannot change the direction of the vector (**Figure 1.9.B**). Therefore, the direction of an object's momentum is the same as the direction of its velocity.

1.9.2 Change of Momentum

Looking back at Newton's first law of motion, we can see that the idea that a body "persists in its state of rest or of moving with constant speed in a constant direction..." can be stated compactly as "the momentum of a body remains constant..." In Chapter 2, we will relate momentum change to interaction mathematically, using the concept of force to quantify interactions. This will allow us to predict quantitatively the motion of objects whose momentum is changed by interactions with their surroundings.

Change in momentum, therefore, is an important quantity. We have just noted that it was harder for the person to change the momentum of a bowling ball than to change the momentum of a tennis ball with the same velocity. Calculating a change of momentum requires vector subtraction.

EXAMPLE 1.9.A | Magnitude of Momentum Change

Consider the tennis ball and bowling ball discussed above. What is the magnitude of the change in the momentum of each ball when your friend catches it?

Solution The mass of a regulation tennis ball is about 58 g, or 0.058 kg in SI units. the velocity of the ball just before your friend catches it is $\langle 0.3, -0.2, 0\rangle$ m/s. The initial state is just before the ball touches her hands, and the final state is when the ball is no longer moving (**Figure 1.9.C**).

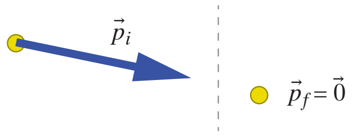


FIGURE 1.9.C The system is the ball. The initial state is just before touching your friend's hands, and the final state is just after the ball has come to a stop in her hands.

Tennis Ball
Assigning symbols to the given quantities, we have:

$$m_T = 0.058 \text{ kg}$$
$$\vec{v}_i = \langle 0.3, -0.2, 0\rangle \text{ m/s}$$
$$\vec{p}_i = m_T\vec{v}_i = \langle 0.0174, -0.0116, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\vec{p}_f = \langle 0, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \langle -0.0174, 0.0116, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$|\Delta\vec{p}| = \sqrt{(-0.0174)^2 + 0.0116^2} \text{ kg}\cdot\text{m/s} = 0.0209 \text{ kg}\cdot\text{m/s}$$

Bowling Ball
For a bowling ball of mass 5.8 kg (about 13 lb):

$$m_B = 5.8 \text{ kg}$$
$$\vec{v}_i = \langle 0.3, -0.2, 0\rangle \text{ m/s}$$
$$\vec{p}_i = m_T\vec{v}_i = \langle 1.74, -1.16, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\vec{p}_f = \langle 0, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \langle -1.74, 1.16, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$|\Delta\vec{p}| = 2.09 \text{ kg}\cdot\text{m/s}$$

Tennis Ball vs. Bowling Ball
The change in velocity of each ball was the same, but the magnitude of the change in momentum of the bowling ball was 100 times larger than the change of momentum of the tennis ball.

Momentum is a vector quantity and proportional to the velocity, so just as was the case with velocity, there are two aspects of momentum that can change: magnitude and direction. A mathematical description of change of momentum must include either a change in the magnitude of the momentum, or a change in the direction of the momentum, or both.

EXAMPLE 1.9.B | Change in Magnitude and Direction of Momentum

Figure 1.9.D shows a portion of the trajectory of a ball in air, subject to gravity and air resistance. At location B, the ball's momentum is $\vec{p}_B = \langle 3.03, 2.83, 0\rangle$ kg · m/s. At location C, the ball's momentum is $\vec{p}_C = \langle 2.55, 0.97, 0\rangle$ kg · m/s.

a. Find the change in the ball's momentum between these locations and show it on the diagram.

b. What changed: the direction of the ball's momentum, the magnitude of the ball's momentum, or both?

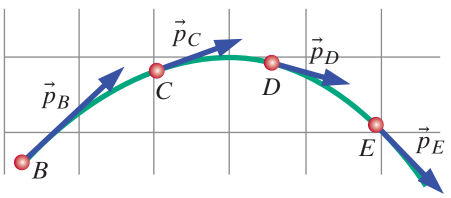


FIGURE 1.9.D A portion of the trajectory of a ball moving through air, subject to gravity and air resistance. The arrows represent the momentum of the ball at the locations indicated by letters. Notice again that the momentum (and velocity) is always tangent to the curving motion.

Solution

$$\Delta\vec{p} = \vec{p}_C - \vec{p}_B = \langle 2.55, 0.97, 0 \rangle \text{ kg}\cdot\text{m/s} - \langle 3.03, 2.83, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$= \langle -0.48, -1.86, 0\rangle \text{ kg}\cdot\text{m/s}$$

Both the x and y components of the ball's momentum decreased, so $\Delta\vec{p}$ has negative x and y components. This is consistent with the graphical subtraction shown in **Figure 1.9.E**.

It is clear from the diagram that both the magnitude and direction of the ball's momentum changed. The arrow representing $\vec{p}_B$ is longer than the arrow representing $\vec{p}_C$, and the directions of the arrows are different.

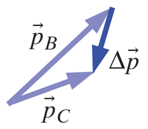


FIGURE 1.9.E Graphical calculation of $\Delta\vec{p}$.

EXAMPLE 1.9.C | A Ball Bounces Off a Wall

A tennis ball of mass 58 g travels with velocity $\langle 50, 0, 0\rangle$ m/s toward a wall. After bouncing off the wall, the tennis ball is observed to be moving at nearly the same speed in the opposite direction.

a. Draw a diagram showing the initial and final momentum of the tennis ball.

b. What is the change in the momentum of the tennis ball?

c. Compare the change in the magnitude of the tennis ball's momentum to the magnitude of the change of the ball's momentum.

Solution

a. The initial and final momenta of the ball are shown in **Figure 1.9.F**.

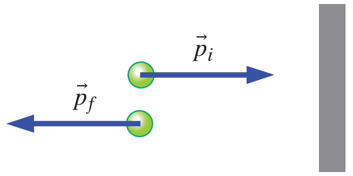


FIGURE 1.9.F The initial and final momentum of the tennis ball.

b. The change in the ball's momentum was:

$$\vec{p}_i = (0.058 \text{ kg})\langle 50, 0, 0\rangle \text{ m/s} = \langle 2.9, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\vec{p}_f = (0.058 \text{ kg})\langle -50, 0, 0\rangle \text{ m/s} = \langle -2.9, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\Delta\vec{p} = \langle -2.9, 0, 0\rangle \text{ kg}\cdot\text{m/s} - \langle 2.9, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$
$$\Delta\vec{p} = \langle -5.80, 0, 0\rangle \text{ kg}\cdot\text{m/s}$$

c. The change in the magnitude of the ball's momentum was:

$$\Delta|\vec{p}| = |\vec{p}_f| - |\vec{p}_i|$$
$$\Delta|\vec{p}| = \sqrt{-2.9^2 + 0^2 + 0^2} \text{ kg}\cdot\text{m/s} - \sqrt{2.9^2 + 0^2 + 0^2} \text{ kg}\cdot\text{m/s}$$
$$\Delta|\vec{p}| = 0 \text{ kg}\cdot\text{m/s}$$

The magnitude of the change in the ball's momentum was:

$$|\Delta\vec{p}| = \sqrt{-5.8^2 + 0^2 + 0^2} = 5.8 \text{ kg}\cdot\text{m/s}$$

How do we make sense of this difference? The interaction with the wall made a large change in the (vector) momentum of the ball; the magnitude of this change is twice as large as the magnitude of the ball's original momentum. However, because the change in the ball's speed was negligible, the change in the magnitude of its momentum was also negligible. We will see in Chapter 2 that this distinction is important because it is the change in the vector momentum that is proportional to the strength of an interaction with the surroundings. In discussing momentum change, we will almost always be interested in $\Delta\vec{p}$ and its magnitude ($|\Delta\vec{p}|$) rather than in the change in the magnitude ($\Delta|\vec{p}|$).

Checkpoint 1.9-C-01

The planet Mars has a mass of 6.4×10^{23} kg and travels in a nearly circular orbit around the Sun, as shown in **Figure 1.9.G**. When it is at location A, the velocity of Mars is $\langle 0, 0, -2.5 \times 10^4 \rangle$ m/s. When it reaches location B, the planet's velocity is $\langle -2.5 \times 10^4, 0, 0\rangle$ m/s. We're looking down on the orbit from above the north poles of the Sun and Mars, with $+x$ to the right and $+z$ down the page. Again, notice that the momentum is always tangent to the path.

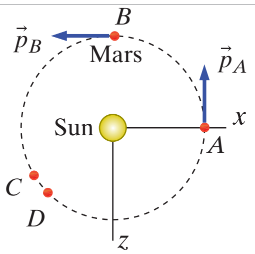


FIGURE 1.9.G The nearly circular orbit of Mars around the Sun. NOTE: This is viewed from above the orbital plane ($+x$ to the right, $+z$ down the page). This diagram is not to scale; the sizes of the Sun and Mars are exaggerated.

a. What is $\Delta\vec{p}$, the change in the momentum of Mars between locations A and B?

1. $\langle -1.6\times10^{28}, 0, 1.6\times10^{28}\rangle$ kg · m/s
2. $\langle -1.6\times10^{28}, 1.6\times10^{28}, 0\rangle$ kg · m/s
3. $\langle -1.6\times10^{28}, 0, 0\rangle$ kg · m/s
4. $\langle 0, 1.6\times10^{28}, 0\rangle$ kg · m/s

b. Which arrow in **Figure 1.9.H** points in the direction of the change in the momentum of Mars between locations C and D? (It will help to make a diagram showing initial and final momenta.)

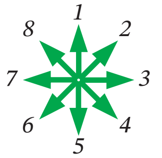


FIGURE 1.9.H Possible directions.

1.9.3 Using Momentum to Update Position

If you know the momentum of an object, you can calculate the change in position of the object over a given time interval. This is straightforward if the object is traveling at a speed low enough that the approximate expression for momentum can be used, because

$$\vec{v} \approx \vec{p}/m \quad \text{if} \quad v << c$$

EXAMPLE 1.9.D | Displacement of an Ice Skater

An ice skater whose mass is 50 kg glides across the ice with constant momentum $\langle 400, 0, 300\rangle$ kg · m/s. At a particular instant in her skating program, she passes location $\langle -24, 0, -15\rangle$ m. If she continues on with the same momentum, what will be her position in 2 s?

Solution

$$\vec{v} \approx \frac{\vec{p}}{m} = \frac{\langle 400, 0, 300\rangle \text{ kg}\cdot\text{m/s}}{50 \text{ kg}} = \langle 8, 0, 6\rangle \text{ m/s}$$

$$\begin{aligned}\vec{r}_f &= \vec{r}_i + \vec{v}\Delta t\\ &= \langle -24, 0, -15\rangle \text{ m} + \langle 8, 0, 6\rangle \text{ m/s}\cdot 2 \text{ s}\\ &= \langle -8, 0, -3\rangle \text{ m}\end{aligned}$$

Checkpoint 1.9-C-02

At time $t_1 = 12$ s, a car with mass 1300 kg is located at $\langle 94, 0, 30 \rangle$ m and has momentum $\langle 4500, 0, -3000 \rangle$ kg · m/s. The car's momentum is not changing. At time $t_2 = 17$ s, what is the position of the car?

1. $\langle 2.25 \times 10^4, 0, -1.50 \times 10^4 \rangle$ m

2. $\langle 111, 0, 18.5 \rangle$ m

3. $\langle 17.3, 0, -11.5 \rangle$ m

4. $\langle 153, 0, -9.23 \rangle$ m

1.10 Momentum at High Speeds

OBJECTIVES

After studying this section, you should be able to calculate the momentum of an object traveling at a speed near the speed of light. To do this you will:

- calculate the factor gamma (γ) for an object traveling at any speed.
- decide whether or not it is reasonable to use the approximation $\gamma \approx 1$ in a particular situation.

1.10.1 The Speed of Light

The speed of light is defined to be exactly 2.99792458×10^8 m/s. We'll typically round this to three significant figures and use the value 3×10^8 m/s. Nothing can travel faster than this speed—it is a universal speed limit. In Section 1.12 we'll discuss some interesting aspects of the speed of light, in the context of the Principle of Relativity.

1.10.2 Exact Definition of Momentum

Although most of the objects we encounter in our daily lives move at speeds that are much less than the speed of light, motion at very high speeds is not unusual. Every day many particles enter the Earth's atmosphere traveling at speeds near the speed of light. Some of these particles are protons, which react with atomic nuclei in the atmosphere to produce showers of high-speed particles that rain down on the Earth. To study high-speed particles in a controlled way, scientists use machines called particle accelerators. The largest particle accelerator currently in operation is the Large Hadron Collider at CERN, in Geneva, Switzerland (**http://cern.ch/**), where the Higgs boson was found. There are many other accelerators in the United States and other countries.

Experiments on particles moving at very high speeds, close to the speed of light c (3×10^8 m/s), show that changes in $m\vec{v}$ (the approximate momentum) are not really proportional to the strength of the interactions. As we keep applying a force to a particle near the speed of light, the speed of the particle barely increases, and it is not possible to increase a particle's speed beyond the speed of light.

Because of these experiments, we can define momentum in a precise (not approximate) way. We observe that changes in the following quantity are truly proportional to the amount of interaction:

DEFINITION OF MOMENTUM

For a particle of mass m, momentum is defined as the product of mass times velocity, multiplied by a proportionality factor gamma:

$$\vec{p} = \gamma m \vec{v}$$

The proportionality factor γ (lowercase Greek gamma) is defined as

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{|\vec{v}|}{c}\right)^2}}$$

In these equations, $\vec{p}$ represents momentum, m is the mass of the object, $\vec{v}$ is the velocity of the object, and c is the speed of light $(3 \times 10^8 \text{ m/s})$. Momentum has units of kg · m/s. The factor γ is a positive number that is always greater than or equal to one, and it has no units.

This is the full, relativistically correct definition of momentum. Albert Einstein in 1905 in his Special Theory of Relativity predicted that this would be the appropriate definition for momentum at high speeds, a prediction that has been abundantly verified in a wide range of experiments. In Chapter 6, we will see that the factor γ is also important in expressions for the energy of objects in motion.

EXAMPLE 1.10.A | Momentum of a Fast-Moving Proton

A proton (mass 1.7×10^{-27} kg) in an accelerator at CERN has a velocity of $\langle 2 \times 10^7, 1 \times 10^7, -3 \times 10^7 \rangle$ m/s.

a. What is the momentum of the proton?

b. What is the magnitude of the momentum of the proton?

Solution

a. $$|\vec{v}| = \sqrt{(2 \times 10^7)^2 + (1 \times 10^7)^2 + (-3 \times 10^7)^2} \text{ m/s}$$
$$= 3.74 \times 10^7 \text{ m/s}$$
$$\frac{|\vec{v}|}{c} = \frac{3.74 \times 10^7 \text{ m/s}}{3 \times 10^8 \text{ m/s}} = 0.125$$
$$\gamma = \frac{1}{\sqrt{1 - (0.125)^2}} = 1.00787$$
$$\vec{p} = \gamma m \vec{v}$$
$$= (1.007)(1.7 \times 10^{-27} \text{ kg})\langle 2 \times 10^7, 1 \times 10^7, -3 \times 10^7 \rangle \text{ m/s}$$
$$= \langle 3.43 \times 10^{-20}, 1.71 \times 10^{-20}, -5.14 \times 10^{-20} \rangle \text{ kg} \cdot \text{m/s}$$

b. $$|\vec{p}| = \sqrt{(3.43 \times 10^{-20})^2 + (1.17 \times 10^{-20})^2 + (-5.14 \times 10^{-20})^2}$$
$$= 6.41 \times 10^{-20} \text{ kg} \cdot \text{m/s}$$

1.10.3 Approximate Expression for Momentum

In Example 1.10.A, we found that $\gamma = 1.00787$. Because in that calculation we used only three significant figures, we could have used the approximation that $\gamma \approx 1.0$ and hardly affected our answer. Let's examine the expression for γ to see whether we can come up with a guideline for when it is reasonable to use the approximate expression

$$\vec{p} \approx 1 \cdot m\vec{v}$$

Looking at the expression for γ,

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{|\vec{v}|}{c}\right)^2}}$$

we see that it depends only on the ratio of the speed of the object to the speed of light (the object's mass doesn't appear in this expression).

If $|\vec{v}|/c$ is a very small number, then $1-(|\vec{v}|/c)^2 \approx 1-0 \approx 1$, so $\gamma \approx 1$.

APPROXIMATION FOR MOMENTUM AT LOW SPEEDS

When $|\vec{v}| \ll c$ then $\gamma \approx 1$, so

$$\vec{p} \approx 1 \cdot m\vec{v} \approx m\vec{v}$$

QUESTION When is this approximation a good one?

Some values of $(|\vec{v}|/c)$ and γ are displayed in **Table 1.4**. In these calculations, the exact value of c, 2.99792458×10^8 m/s was used. From this table you can see that even at the very high speed where $|\vec{v}|/c = 0.1$, which means that $|\vec{v}| = 3 \times 10^7$ m/s, the relativistic factor γ is only slightly different from 1.0. For large-scale objects such as a space rocket, whose speed is typically only about 1×10^4 m/s, we can ignore the factor γ, and momentum is to a good approximation $\vec{p} \approx m\vec{v}$. It is only for high-speed cosmic rays or particles produced in high-speed particle accelerators that we need to use the full relativistic definition for momentum, $\vec{p} = \gamma m\vec{v}$.

TABLE 1.4 Values of γ calculated for some speeds using the exact speed of light

$\lvert\vec{v}\rvert$ m/s	$\lvert\vec{v}\rvert/c$	γ
0	0	1.00000
3	1×10^{-8}	1.00000
300	1×10^{-6}	1.00000
3×10^6	0.01	1.00001
3×10^7	0.1	1.00504
1.49896×10^8	0.5	1.15470
2.69813×10^8	0.9	2.29416
2.96795×10^8	0.99	7.08881
2.99493×10^8	0.999	22.36627
2.99762×10^8	0.9999	70.71240
2.99792458×10^8	1	∞ Impossible!

Figure 1.10.A graphically displays the data shown in **Table 1.4**. For speeds up to about half the speed of light, γ is very nearly equal to 1, but at very high speeds, approaching the speed of light, γ increases rapidly. From examining **Figure 1.10.A**, you can see why it is not possible to exceed the speed of light. As you make a particle go faster and faster, approaching the speed of light, additional increases in the speed become increasingly difficult because a tiny increase in speed means a huge increase in γ and a corresponding huge increase in momentum, requiring huge amounts of interaction. In fact, for the speed to equal the speed of light, the momentum would have to increase to infinity! There is a cosmic speed limit in the Universe, the speed of light.

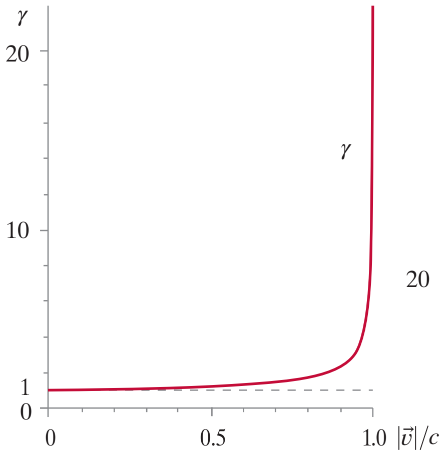


FIGURE 1.10.A γ as a function of $|\vec{v}|/c$. At low speeds, γ is approximately equal to 1. At very high speeds, $|\vec{v}| \approx c, \gamma$ increases rapidly.

The Role of Approximations and Models in Physics

QUESTION Earlier in this chapter we used the approximate expression $\vec{p} \approx m\vec{v}$ for momentum. Is this legitimate? Shouldn't we always use the exact equation?

The table of values of γ in **Table 1.4** shows that for many real-world situations in which objects are traveling at speeds small compared to the speed of light, calculations will give the same result whether we actually calculate γ or simply use the approximation that $\gamma \approx 1$. Our measurements and calculations simply aren't precise enough to make the distinction useful.

You may have heard physics described as an exact science. This is not a good description of what physics is actually about. Physics and physicists try to describe and understand what happens in the real world, which is a messy, complicated place. For example, we'll see in the next chapter that it is reasonably easy to predict the motion of a thrown ball in the highly simplified case where there is no air resistance or deflection by wind. Even if we can really get rid of air resistance by throwing the ball in a big vacuum chamber, it makes sense to simplify the analysis by neglecting the tiny but real effects of gravitational attraction by nearby mountains, the Moon, and Mars, and also to ignore the fact that the gravitational attraction changes very slightly as the ball gets closer to or farther from the center of the Earth. Moreover, to predict exactly the future motion of the ball, we would need to know exactly its initial position and velocity. This is impossible because meter sticks and clocks aren't exact.

QUESTION If physics isn't exact, what is the point?

We can often learn a great deal by carrying out an approximate analysis of a complex situation. In some cases, the differences between an approximate analysis and a hypothetical exact analysis may be negligible. For example, it certainly makes sense to ignore the gravitational attraction of Mars when predicting the motion of a ball thrown on Earth. In other cases, a simplified model may allow us to identify the most important interactions and effects in a complex situation. A comparison of the predictions of such a model with data from real-world observations can suggest refinements to our model that would make its predictions more accurate.

A good way of describing what physics can (and can't) do is that with physics we construct, analyze, and refine simplified, idealized, approximate "models" of real-world phenomena, in the hope that such analyses will give us useful but necessarily approximate understanding of the real world.

1.10.4 Updating Position at High Speeds

For speeds near the speed of light, we cannot use the approximation $\vec{v} \approx \vec{p}/m$. Instead, we need to start with the definition of momentum $\vec{p} \approx \gamma m \vec{v}$ and solve for $\vec{v}$. The detailed derivation is given at the end of the chapter. The result is this:

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1 + \left(\frac{|\vec{p}|}{mc}\right)^2}}$$

EXAMPLE 1.10.B | Displacement of a Fast Proton

A proton with constant momentum $\langle 0, 0, 2.72 \times 10^{-19} \rangle$ kg · m/s leaves the origin at time $t = 0$. What is the location of the proton at time $t = 2$ ns? (ns = nanosecond = 10^{-9} s)

Solution

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1 + \left(\frac{|\vec{p}|}{mc}\right)^2}}$$

$$= \frac{\langle 0, 0, 2.72 \times 10^{-19} \rangle \text{ kg} \cdot \text{m/s}}{(1.7 \times 10^{-27} \text{ kg}) \sqrt{1 + \left(\frac{2.72 \times 10^{-19} \text{ kg} \cdot \text{m/s}}{(1.7 \times 10^{-27} \text{ kg})\,(3 \times 10^8 \text{ m/s})}\right)^2}}$$

$$= \langle 0, 0, 1.4 \times 10^8 \rangle \text{ m/s}$$

$$\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\,\Delta t$$

$$= \langle 0, 0, 0 \rangle \text{ m} + \langle 0, 0, 1.4 \times 10^8 \text{ m/s} \rangle (2 \times 10^{-9} \text{ s})$$

$$= \langle 0, 0, 0.28 \rangle \text{ m}$$

The proton traveled 28 cm in 2 ns.

Checkpoint 1.10-C-01

An electron (mass 9.1×10^{-31} kg) travels at a velocity of $\langle 0, 0, -2 \times 10^8 \rangle$ m/s.

a. What is the momentum of the electron?

1. $\langle 0, 0, -2.20 \times 10^{-22} \rangle$ kg · m/s
2. $\langle 0, 0, -3.28 \times 10^{-22} \rangle$ kg · m/s
3. $\langle 0, 0, -1.82 \times 10^{-22} \rangle$ kg · m/s
4. $\langle 0, 0, -2.44 \times 10^{-22} \rangle$ kg · m/s

b. What is the magnitude of the momentum of the electron?

1. 1.82×10^{-22} kg · m/s
2. 2×10^8 kg · m/s
3. 2.44×10^{-22} kg · m/s
4. 3.28×10^{-22} kg · m/s

1.11 Computational Modeling

How to run VPython code:
To run the VPython code in this section, you can either use the embedded interactive versions or go to **webvpython.org**, create a free account, and enter, run, and modify the code there.

OBJECTIVES

After studying this section, you should be able to read, interpret, and modify a computational model of an object traveling at constant velocity. In order to do this, you should be able to:

- read and interpret simple VPython code.
- create objects such as spheres and boxes and position them in a 3D coordinate system.
- write VPython instructions for simple calculations involving vector or scalar variables.
- list the information required for a loop and create a simple loop.
- modify a simple computational model to change position and velocity.

Computational modeling plays an important role not only in physics theory and experiment but in nearly all other scientific and engineering fields. Creating simple computational models based on fundamental physics principles can allow us to see more clearly how these principles govern the real-time behavior of all physical systems, including complicated ones. Appropriate computational tools can let us visualize the time evolution of the behavior of 3D physical systems and can also help us visualize vector quantities such as velocity, momentum, and other quantities we will encounter in later chapters. For these reasons, computational modeling is included as an integral part of this textbook.

QUESTION But what if I don't know how to program?

Don't worry! No prior experience is required. You will be able to learn all you need to know to build simple but powerful computational models as we go along in the course. In addition to the videos and exercises in this textbook, your instructor may assign more extensive instructional activities that will help you learn to create computational models.

1.11.1 VPython

We have chosen to use VPython to build our computational models. VPython is an extension of the widely used Python programming language. VPython supports 3D vector algebra and makes it easy for you to create navigable real-time 3D animations as a side effect of physics calculations. VPython is free and open source and runs on Windows, MacOS, and Linux. To use VPython on the web, without installing any software, go to **https://webvpython.org**.

Figure 1.11.A is a screen shot from a VPython program that models the motion of a binary star system drifting through space. You'll learn to write models such as this one.

Interactive

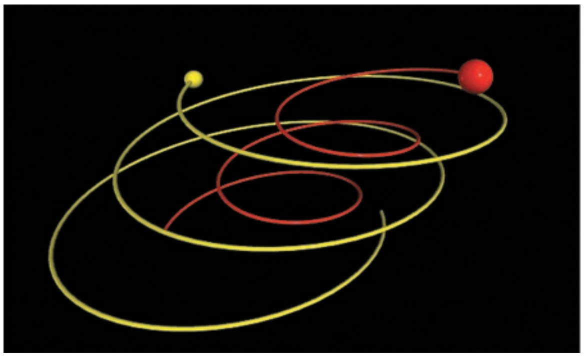


FIGURE 1.11.A Screenshot from a VPython model of a binary star system. The two stars orbit each other and also move together through space, tracing out helical trajectories.

1.11.2 A Computer Program

A computer program is a set of instructions detailing how to do a particular task. These instructions may involve doing mathematical calculations, creating graphical objects in 3D space to represent objects in a system, and doing logical tests.

Order of Execution The instructions in a program are executed in order starting with the first line and continuing on until the end. Suppose a computer executes the following four lines of VPython code (the line numbers are not part of the code, and are added for convenience in discussing the code):

```
1 a = 3
2 b = a + 5
3 a = 1
4 print(b)
```

In line 1 above, the variable `a` is created and set to `3`. In line 2, the variable `b` is created and assigned the value `a + 5`, or 8. In line 3, the variable a is assigned a new value of `1`.

QUESTION When the value of `b` is printed in line 4 above, what will be the result?

The value of `b` that is printed will be `8`. Because `a` is set to `1` after the value of `b` is assigned, this later operation does not affect the value of `b`.

Interactive

3D Cartesian Coordinate System By default the origin of the 3D Cartesian coordinate system, $\langle 0,0,0\rangle$, is in the center of the display. The orientation of the axes is the same as the one we have been using, with $+x$ to the right, $+y$ up, and $+z$ out of the plane of the display, toward you, as shown in **Figure 1.11.B**.

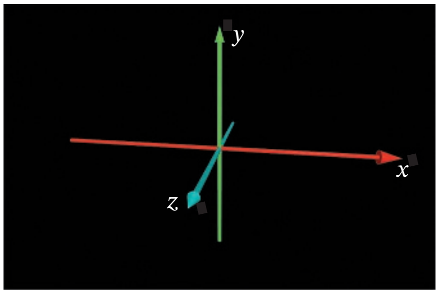


FIGURE 1.11.B Origin and x, y and z axes in the Cartesian coordinate system used in VPython.

Interactive

Video

Creating Graphical Objects The code below creates a transparent cube centered at the origin, and places red spheres at three of the corners of the cube and a yellow sphere at the origin, as illustrated in **Figure 1.11.C**.

```
box(pos=vector(0,0,0),length=2,width=2,height=2,opacity=0.4)
sphere(pos=vector(0,0,0), color=color.yellow, radius=0.2)
sphere(pos=vector(-1,-1,1), radius=0.2, color=color.red)
sphere(pos=vector(-1,-1,-1), radius=0.2, color=color.red)
sphere(pos=vector(-1,1,-1), radius=0.2, color=color.red)
```

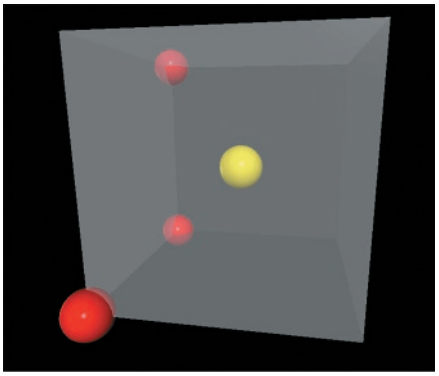


FIGURE 1.11.C A transparent cube, centered at the origin, with red spheres at three corners and a yellow sphere in the center.

Calculations in VPython The notation used in calculations in VPython will mostly look familiar. There are a few things to note:

- All multiplication must be explicit. An asterisk indicates multiplication: `a= 5 * (b+c)`
- Two asterisks denote raising to a power. x^2 is written `x**2`
- Scientific notation uses the E notation familiar from calculators: `2.5e33` or `2.5E33`
- The constant π is written `pi`
- Use `sqrt(x)` or `x**(1/2)` for square root
- Trig functions use parentheses: `sin(x)`. Angles are in radians.
- Vector operations such as addition, subtraction, scalar multiplication, etc. are supported: if `r1` and `r2` are vector variables, then `r1 + r2` adds them vectorially. `mag(r1)` gives magnitude, and `hat(r1)` gives the unit vector.

1.11.3 Loops: Repeated Operations

Video

A loop is a way of instructing a computer to repeat an operation many times. A loop requires three pieces of information:

1. Where to start
2. Where to stop
3. What operations to repeat

The code below is a simple example of a loop that does a calculation over and over.

```
1 x = 0
2 while x < 5:
3    print(x)
4    x = x + 1
```

Explanation:

1. Where to start: Line 1 creates a variable named x and assigns it a value of zero. This is the starting value for the loop.
2. Where to stop: Line 2 contains the information on where to stop. `x < 5` is a logical test, which can give a value of `True` or `False`. The loop continues as long as the value of the statement `x < 5` is `True`.

3. Operations to repeat: The colon at the end of line 2 signals the beginning of the third part of the loop: Instructions to be repeated. All instructions indented after the colon (in this case lines 3 and 4) will be repeated each time through the loop.

Note that a statement such as `x = x + 1` is not wrong in a computer language. Such a statement instructs the computer to pick up the current value of the variable *x*, add 1 to it, and replace the value of *x* with this new value. Subscripts identifying "current" and "new" values are not needed, because anything to the right of the equal sign refers to the current value, and anything to the left of the equal sign is the new value.

The output of the program is shown below:

```
0
1
2
3
4
```

The zero was printed the first time through the loop because the initial value of `x` was zero; the one was printed the second time through the loop because the value of *x* was now one, and so on.

Checkpoint 1.11-C-01

Figure 1.11.D shows four possible sequences of numbers that might be printed by the VPython loops below. Read each loop carefully and select the sequence it will produce when you run it.

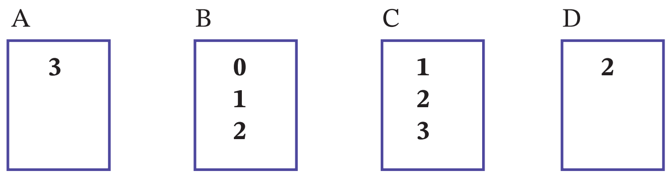


FIGURE 1.11.D Four possible outputs from the short programs below.

a. What will be the output of the VPython loop below? Before looking at the answer, write out your prediction.

```
x = 0
while x < 3:
   print(x)
   x = x + 1
```

1. Output A
2. Output B
3. Output C
4. Output D

b. The VPython loop below is similar but not identical to the one in part (a). What will be its output?

```
x = 0
while x < 3:
   x = x + 1
   print(x)
```

1. Output A
2. Output B
3. Output C
4. Output D

c. The VPython loop below is similar but not identical to the loops in parts (a) and (b). What will be its output?

```
x = 0
while x < 3:
    x = x + 1
print(x)
```

1. Output A
2. Output B
3. Output C
4. Output D

1.11.4 Modeling Motion

One of the most interesting applications of VPython is to model motion. To predict the motion of a system traveling at constant velocity, we'll use the position update equation:

$$\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\,\Delta t$$

Translated into VPython, it will look something like this:

```
mysphere.pos = mysphere.pos + vavg * deltat
```

This code assumes that you have created a sphere named `mysphere`. To refer to the position of this object, we use the built-in *position attribute* of the object and specify it by adding `.pos` to the name of the object, hence `mysphere.pos`.

Because computers are good at repeating calculations rapidly, we can use a small time step and put our calculation into a loop to be done over and over. Not only can the computer calculate new positions, it can redraw the display many times per second, so we see our model objects move smoothly (this is similar to computer-generated animations in movies).

Here's an example of how we might animate the motion of an object in a short program.

```
1 ball = sphere( pos=vector(0,-10,0))
2 velocity = vector(0, 2, 0)
3 deltat = 0.1
4 while ball.pos.y < 10:
5     rate(60)
6     ball.pos = ball.pos + (velocity * deltat)
```

Explanation of the code above:

1. Line 1 creates a sphere and names it `ball`, so it can be referred to by later instructions. The position at which the sphere is created will provide the starting value for the loop.
2. Line 2 creates a vector variable named `velocity`
3. Line 3 creates a scalar variable named `deltat`
4. Line 4 contains the stopping condition for the loop. The loop will repeat only as long as `ball.pos.y < 10`
5. Line 5 is indented, so it is executed each time through the loop. This line makes it possible to see the animation. `rate(60)` slows down the program so the loop executes only 60 times per second. The `rate()` statement also makes it possible for the computer to redraw the scene 60 times per second, using the updated position of the ball.
6. Line 6 is also indented, so it is executed each time through the loop. It specifies the calculation that is repeated inside the loop. This calculation should look familiar—it is the position update equation. Note that VPython does vector operations: the quantity (`velocity*deltat`) is a vector that is multiplied by a scalar. The resulting vector (the displacement) is added to the vector `ball.pos`.

Interactive

Either click the link to run this program and see its output or enter the code at **webvpython.org** and run it.

Video

Interactive

1.11.5 Computational Problem Solving

Creating a computational model of a physical situation allows us to ask questions about the motion of objects. As an example, we'll consider the motion of a drone near an obstacle (a hanging light).

PROBLEM **Predicting the Path of a Drone**

A programmable drone of radius 0.3 m starts at location $\langle 3, -2, -1 \rangle$ m. It is programmed to fly at constant speed directly into a box at location $\langle -4, 2, 2 \rangle$ m, taking a total of six seconds. However, there is a hanging light of radius 50 cm at location $\langle -1, 0.7, 0.5 \rangle$ m. Will the drone hit the light?

Solution
With VPython we can create a visual representation of the problem representing the model and its solution.

The following short program predicts and displays the path of the drone. Because the drone leaves a trail, we can see whether or not it hits the light. In the subsequent video the code is built up and explained step by step. You can copy this program into a file at **webvpython.org**, remove the line numbers, and run it there.

```
1 drone = sphere(pos=vector(3,-2,-1), radius=0.3,
2         color=color.cyan, make_trail = True)
3 target = box(pos=vector(-4,2,2), length=1,
4          width=1, height=1, opacity=0.4)
5 light = sphere(pos=vector(-1, 0.7, 0.5),
6         radius=0.5, color=color.yellow)
7 deltar = target.pos - drone.pos
8 totalTime = 6
9 velocity = deltar/totalTime
10 print("velocity =", velocity)
11 deltat = 0.1
12 while drone.pos.x > target.pos.x:
13    rate(100)
14    drone.pos = drone.pos + velocity*deltat
15 print("position is", drone.pos)
```

In the computational solution, Lines 1–6 create objects to represent the objects in the problem. The sphere named `drone`, created in line 1 (which is continued onto line 2), will leave a trail as it moves due to having specified `make_trail=True`.

Lines 7–9 calculate the velocity required for the drone to fly straight to the target in the prescribed time. Line 7 calculates $\Delta\vec{r}$, the position of the target relative to the drone. Because `target.pos` and `drone.pos` are both vectors (as can be seen in lines 1 and 3), `deltar` is also a vector.

Line 11 creates a scalar variable `deltat` (Δt) to be used in the position update.

Lines 12–14 are the computational loop where all the action happens. The thing that varies inside the loop is the position of the drone, which is given its starting value in line 1 where it is created.

Line 12 establishes the stop condition for the loop—the loop will continue as long as the x-component of the drone's position is larger than the x-component of the target's position. The instruction `rate(100)` allows the computer to show an animation of the path and also limits the number of repetitions of the loop to 100 per second.

Line 14 is the position update equation.

Line 15 is executed only once, after the loop has finished, so it prints the final position of the drone.

What happens? Does the drone hit the light?

When we run the program, we see that the drone's trail passes through the light, so the drone does indeed hit the light. **Figure 1.11.E** is a screenshot from the running program.

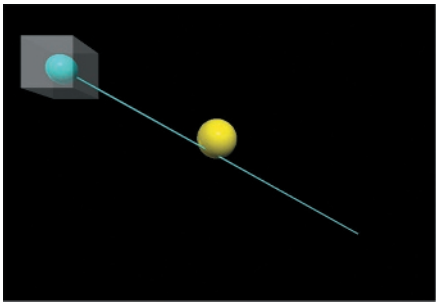


FIGURE 1.11.E A screen shot from the model of the motion of the drone.

Checkpoint 1.11-C-02

The following questions refer to the solution to the drone program.

a. In which line of code is the initial position of the drone set?

1. 1
2. 7
3. 9
4. 14

b. In which line of code is the drone's velocity used to update its position?

1. 1
2. 7
3. 9
4. 14

c. What determines when the drone will stop moving?

1. The drone will stop when it hits the light.
2. The loop stops when the time reaches the correct number.
3. The loop stops when the x-component of the drone's position is less than or equal to the x-component of the target's position.
4. The drone doesn't stop; it continues through the target and off the screen.

1.11.6 Using VPython as a Calculator

Even if you don't need to create a full computational model, you may find it useful to use VPython for calculations, especially if these calculations involve vectors. Among the advantages of doing this are:

- VPython does vector operations rapidly.
- Calculations end up more organized and easier to read and debug.
- If you realize you've entered a number incorrectly, all you need to do is change that number and re-run your small program. You don't need to go through all the calculations again by hand.

In subsequent chapters, in addition to the standard solutions, we'll show how some of the example problems can be solved using VPython as a calculator.

1.12 *The Principle of Relativity

Starred Sections:
Sections marked with an asterisk are optional. They provide additional information and context, but later sections of the textbook don't depend critically on them.

This section deals with some deep issues about the reference frame from which you observe motion. Newton's first law of motion only applies in an *inertial* reference frame, which we will discuss here in the context of the principle of relativity.

A great variety of experimental observations has led to the establishment of the following principle, first recognized by Galileo:

THE PRINCIPLE OF RELATIVITY

Physical laws work in the same way for observers in uniform motion as for observers at rest.

This principle is called the principle of relativity. (Einstein's extensions of this principle are known as special relativity and general relativity.) Phenomena observed in a room in uniform motion (for example, on a train moving with constant speed on a smooth straight track) obey the same physical laws in the same way as experiments done in a room that is not moving. According to this principle, Newton's first law of motion should be true both for an observer moving at constant velocity and for an observer at rest.

For example, suppose that you're riding in a car moving with constant velocity, and your sunglasses are lying on the dashboard. As far as you're concerned, the sunglasses aren't moving, and no interactions are required to keep them at rest on the dashboard. Someone standing at the side of the road sees the car go by, sees the sunglasses moving at a high speed in a straight line and can also see that no interactions are required to keep the sunglasses at rest on the dashboard. Both you and the bystander agree that Newton's first law of motion is obeyed: the bystander sees the sunglasses moving with constant velocity in the absence of interactions, and you see the sunglasses not moving at all (a zero constant velocity) in the absence of interactions.

On the other hand, if the car suddenly speeds up, it moves out from under the sunglasses, which end up in your lap. To you it looks like "the sunglasses sped up in the backwards direction" without any interactions to cause this to happen, which appears to be a violation of Newton's first law of motion. The problem is that you're strapped to the car, which is an *accelerated* reference frame, and Newton's first law of motion applies only to nonaccelerated reference frames, called *inertial* reference frames. Similarly, if the car suddenly turns to the right, moving out from under the sunglasses, the sunglasses tend to keep going in their original direction, and to you it looks like "the sunglasses moved to the left" without any interactions. So, a change of speed or a change of direction of the car (your reference frame) leads you to see the sunglasses behave in a strange way.

The bystander, who is in an inertial (nonaccelerating) reference frame, doesn't see any violation of Newton's first law of motion. The bystander's reference frame is an inertial frame, and the sunglasses behave in an understandable way, tending to keep moving with unchanged speed and direction when the car changes speed or direction.

1.12.1 The Cosmic Microwave Background

The principle of relativity and Newton's first law of motion, apply only to observers who have a constant speed and direction (or zero speed) relative to the "cosmic microwave background," which provides the only backdrop and frame of reference with an absolute, universal character. It used to be that the basic reference frame was loosely called "the fixed stars", but stars and galaxies have their own individual motions within the Universe and do not constitute an adequate reference frame with respect to which to measure motion.

The cosmic microwave background is low-intensity electromagnetic radiation with wavelengths in the microwave region, which pervades the Universe, radiating in all directions. Measurements show that our galaxy is moving through this microwave radiation with a large, essentially constant velocity, toward a cluster of a large number of other galaxies. The way we detect our motion relative to the microwave background is through the Doppler shift of the frequencies of the microwave radiation, toward higher frequencies in front of us and lower frequencies behind. This is essentially the same phenomenon as that responsible for a fire engine siren sounding at a higher frequency when it is approaching us and a lower frequency when it is moving away from us.

The discovery of the cosmic microwave background provided major support for the Big Bang theory of the formation of the Universe. According to the Big Bang theory, the early Universe must have been an extremely hot mixture of charged particles and high-energy, short-wavelength electromagnetic radiation (visible light, X-rays, gamma rays, etc.). Electromagnetic radiation interacts strongly with charged particles, so light could not travel very far without interacting, making the Universe essentially opaque. Also, the Universe was so hot that electrically neutral atoms could not form without the electrons immediately being stripped away again by collisions with other fast-moving particles.

As the Universe expanded, the temperature dropped. Eventually the temperature was low enough for neutral atoms to form. The interaction of electromagnetic radiation with neutral atoms is much weaker than with individual charged particles, so the radiation was now essentially free, dissociated from the matter, and the Universe became transparent. As the Universe continued to expand (the actual space between clumps of matter got bigger), the wavelengths of the electromagnetic radiation got longer, until today this fossil radiation has wavelengths in the relatively low-energy, long-wavelength microwave portion of the electromagnetic spectrum.

1.12.2 Inertial Frames of Reference

It is an observational fact that in reference frames that are in uniform motion with respect to the cosmic microwave background, far from other objects (so that interactions are negligible), an object maintains uniform motion. Such frames are inertial frames and are reference frames in which Newton's first law of motion is valid. All of these reference frames are equally valid; the cosmic microwave background simply provides a concrete example of such a reference frame.

QUESTION Is the surface of the Earth an inertial frame?

No! The Earth is rotating on its axis, so the velocity of an object sitting on the surface of the Earth is constantly changing direction, as is a coordinate frame tied to the Earth (**Figure 1.12.A**). Moreover, the Earth is orbiting the Sun, and the Solar System itself is orbiting the center of our Milky Way Galaxy, and our galaxy is moving toward other galaxies. So, the motion of an object sitting on the Earth is actually quite complicated and definitely not uniform with respect to the cosmic microwave background.

However, for many purposes the surface of the Earth can be considered to be (approximately) an inertial frame. For example, it takes 6 hours for the rotation of the Earth on its axis to make a 90° change in the direction of the velocity of a fixed point. When a hockey puck slides across an ice rink with negligible friction, during these few seconds the puck moves in nearly a straight line at constant speed, and if the puck is struck its velocity change is much larger than the very small velocity change of the approximate inertial frame of the Earth's surface.

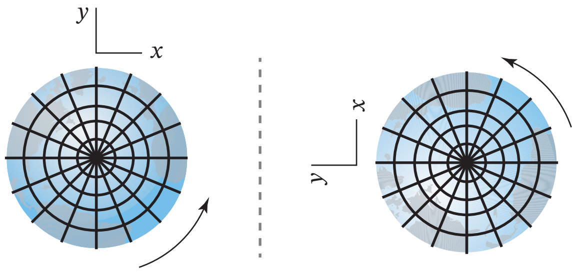


FIGURE 1.12.A Left: A particular time. Right: 6 hours later. Axes tied to the Earth rotate through 90° in a quarter of a day (6h).

Similarly, although the Earth is in orbit around the Sun, it takes 365 days to go around once, so for a period of a few days or even weeks, the Earth's orbital motion is nearly in a straight line at constant speed. For our purposes, we will consider the Earth's surface to represent an approximately inertial reference frame.

1.12.3 The Special Theory of Relativity

Einstein's Special Theory of Relativity (published in 1905) built on the basic principle of relativity introduced by Galileo but added the conjecture that the speed of a beam of light must be the same as measured by observers in different frames of reference in uniform motion with respect to each other. In **Figure 1.12.B**, observers on each spaceship measure the speed of the light c emitted by the ship at the left to be the same $\left(c = 3 \times 10^8\right)$ m/s, despite the fact that they are moving at different velocities.

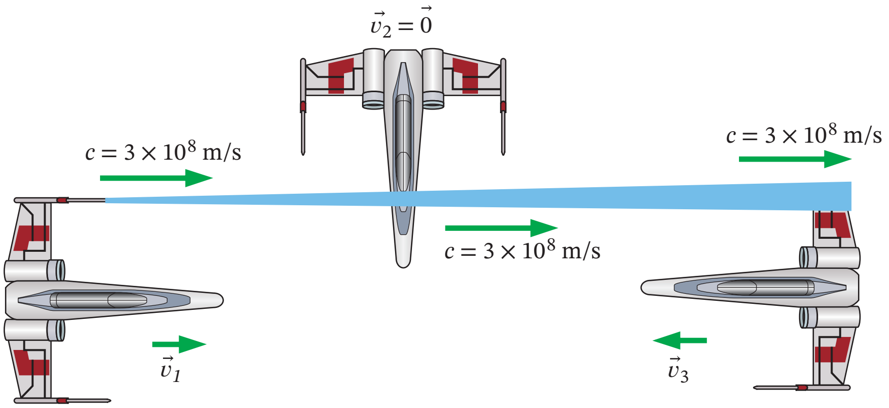


FIGURE 1.12.B Light emitted by the left spaceship is measured to have the same speed by observers in all three ships.

This additional condition seems peculiar and has far-reaching consequences. After all, the sunglasses on the dashboard of your car have different speeds relative to different observers, depending on the motion of the observer. Yet a wide range of experiments has confirmed Einstein's conjecture: all observers measure the same speed for the same beam of light, $c = 3 \times 10^8$ m/s. (The color of the light is different for the different observers, but the speed is the same.)

On the other hand, if someone on one of the spaceships throws a ball or a proton or some other piece of matter, the speed of the object will be different for observers on the other spaceships; it is only light whose speed is independent of the observer.

Einstein's theory has interesting consequences. For example, it predicts that time will be measured to run at different rates in different frames of reference. These predictions have been confirmed by many experiments. These unusual effects are large only at very high speeds (a sizable fraction of the speed of light), which is why we don't normally observe these effects in everyday life, and why we can use nonrelativistic calculations for low-speed phenomena.

However, for the Global Positioning System (GPS) to give adequate accuracy it is necessary to take Einstein's Special Theory of Relativity into account. The atomic clocks on the satellites run slower than our clocks, due to the speed of the satellites and the difference in clock rate depends on γ. Although γ for the GPS satellites orbiting the Earth is nearly 1, it differs just enough that making the approximation $\gamma \approx 1$ would make the GPS hopelessly inaccurate and useless. It is even necessary to apply corrections based on Einstein's General Theory of Relativity, which correctly predicts an additional change in clock rate due to the gravity of the Earth being weaker at the high altitude of the GPS satellites.

Checkpoint 1.12-C-01

A spaceship at rest with respect to the cosmic microwave background (ship 1) emits a beam of red light. A different spaceship (ship 2), moving at a speed of 2.5×10^8 m/s toward the first ship, detects the light. Consider the two following claims about observers on the second ship:

a. Observers on ship 2 observe that the light from ship 1 travels at 3×10^8 m/s.

b. Observers on ship 2 see that the light emitted by ship 1 is not red.

Which of these claims are true for observers on the second ship? (More than one statement may be correct.)

1. (a)
2. (b)
3. both (a) and (b)
4. neither (a) nor (b)

1.13 *Updating Position at High Speed

If $v \ll c, \vec{p} \approx m\vec{v}$ and $\vec{v} \approx \vec{p}/m$. But at high speed it is more complicated to determine the velocity from the (relativistic) momentum. Here is a way to solve for $\vec{v}$ in terms of $\vec{p}$:

$$|\vec{p}| = \frac{1}{\sqrt{1-(|\vec{v}|/c)^2}} m|\vec{v}|$$

Divide by m and square: $\dfrac{|\vec{p}|^2}{m^2} = \dfrac{|\vec{v}|^2}{1-(|\vec{v}|/c)^2}$

Multiply by $\left(1-(|\vec{v}|/c)^2\right)$: $\dfrac{|\vec{p}|^2}{m^2} - \left(\dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = |\vec{v}|^2$

Collect terms: $\left(1+\dfrac{|\vec{p}|^2}{m^2c^2}\right)|\vec{v}|^2 = \dfrac{|\vec{p}|^2}{m^2}$

$$|\vec{v}| = \frac{|\vec{p}|/m}{\sqrt{1+\left(\dfrac{|\vec{p}|}{mc}\right)^2}}$$

The expression above gives the magnitude of $\vec{v}$, in terms of the magnitude of $\vec{p}$. To get an expression for the vector $\vec{v}$, recall that any vector can be factored into its magnitude times a unit vector in the direction of the vector, so

$$\vec{p} = |\vec{p}|\hat{p} \text{ and } \vec{v} = |\vec{v}|\hat{v}$$

But because $\vec{p}$ and $\vec{v}$ are in the same direction, $\hat{v} = \hat{p}$, so

$$\vec{v} = |\vec{v}|\hat{p} = \frac{|\vec{p}|/m}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}\hat{p} = \frac{(|\vec{p}|\hat{p})/m}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}$$

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}$$

Using this velocity, we can now update the position of a particle traveling at high speed.

THE RELATIVISTIC POSITION UPDATE EQUATION

$$\vec{r}_f = \vec{r}_i + \frac{1}{\sqrt{1+\left(\frac{|\vec{p}|}{mc}\right)^2}}\left(\frac{\vec{p}}{m}\right)\Delta t$$

for small Δt.

Note that at low speeds $|\vec{p}| \approx m|\vec{v}|$, and the denominator becomes

$$\sqrt{1+\left(\frac{|\vec{v}|}{c}\right)^2} \approx 1$$

so the equation becomes the familiar $\vec{r}_f = \vec{r}_i + (\vec{p}/m)\Delta t$.

Summary

Key Ideas

- Fundamental physics principles apply to all kinds of matter, from galaxies to subatomic particles.
- Change is an indication of an interaction.
- Position and motion in 3D space can be described precisely by vectors.
- The momentum of an object depends on both mass and velocity.

Interactions

One indicator of an interaction is change of velocity (change of direction and/or change of speed).

Newton's first law of motion

Every body persists in its state of rest or of moving with constant speed in a constant direction, except to the extent that it is compelled to change that state by forces acting on it.

Vectors

A 3D vector is a quantity with magnitude and a direction, which can be expressed as a triple $\langle x, y, z\rangle$. A vector is indicated by an arrow: $\vec{r}$.

A scalar is a single number.

Legal mathematical operations involving vectors include:

- Adding one vector to another vector
- Subtracting one vector from another vector
- Multiplying or dividing a vector by a scalar
- Finding the magnitude of a vector
- Taking the derivative of a vector

Operations that are *not* legal with vectors include:

- A vector cannot be added to a scalar.
- A vector cannot be set equal to a scalar.
- A vector cannot appear in the denominator (you can't divide by a vector).

A unit vector $\hat{r} = \vec{r}/|\vec{r}|$ has magnitude 1.
A vector can be factored using a unit vector: $\vec{p} = |\vec{p}|\hat{p}$.
Direction cosines: $\hat{r} = \langle \cos\theta_x, \cos\theta_y, \cos\theta_z\rangle$

The symbol Δ

The symbol Δ (delta) means "change of": $\Delta t = t_f - t_i$, $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$.
Δ always means "final minus initial."

Velocity and change of position

Definition of average velocity:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Velocity is a vector. $\vec{r}$ is the position of an object (a vector). t is the time. Average velocity is equal to the change in position divided by the time elapsed. SI units of velocity are meters per second (m/s).

The position update equation:

$$\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\,\Delta t$$

The final position (vector) is the vector sum of the initial position plus the product of the average velocity and the elapsed time.

Definition of instantaneous velocity:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity is the limiting value of the average velocity as the time elapsed becomes very small.

Velocity in terms of momentum:

$$\vec{v} = \frac{\vec{p}/m}{\sqrt{1 + \left(\frac{|\vec{p}|}{mc}\right)^2}} \text{ or } \vec{v} \approx \vec{p}/m \text{ at low speeds}$$

Acceleration

Acceleration is the time rate of change of velocity: $\vec{a} = d\vec{v}/dt$.

Momentum

Definition of momentum:

$$\vec{p} = \gamma m\vec{v}$$

$$\text{where } \gamma = \frac{1}{\sqrt{1 - (|\vec{v}|/c)^2}} \quad \text{(lowercase Greek gamma)}$$

Momentum (a vector) is the product of the relativistic factor gamma (a scalar), mass, and velocity.

Combined into one equation: $\vec{p} = \frac{1}{\sqrt{1 - (|\vec{v}|/c)^2}} m\vec{v}$.

Approximation for momentum at low speeds:

$$\vec{p} \approx m\vec{v} \text{ at speeds such that } |\vec{v}| \ll c$$

Useful numbers:

- Radius of a typical atom $\approx 1 \times 10^{-10}$ meter
- Radius of a proton or neutron $\approx 1 \times 10^{-15}$ meter
- Speed of light: 3×10^8 m/s

Questions

1.1-Q-01 On a periodic table, for each element there is given the atomic number (the number of protons or electrons in an atom) and the atomic mass, which is essentially the number of nucleons, protons plus neutrons, in the nucleus, averaged over the various isotopes of the element, which differ in the number of neutrons. Make a graph of the number of neutrons vs. the number of protons in the elements. You needn't graph every element, just enough to see the trend. What do you observe about the data? (This reflects the need for more neutrons in proton-rich nuclei in order to prevent the electric repulsion of the protons of each other from destroying the nucleus.)

1.2-Q-01 Moving objects left the traces labeled $A - F$ in **Figure 1.15.A**. The dots were deposited at equal time intervals (for example, one dot each second). In each case, the object starts from the square. Which trajectories show evidence that the moving object was interacting with another object somewhere? If there is evidence of an interaction, what is the evidence?

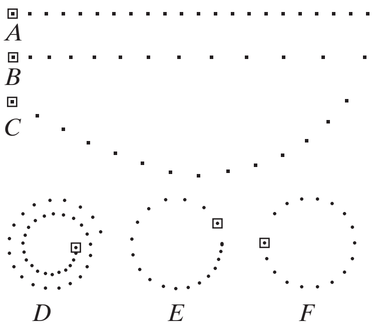


FIGURE 1.15.A Traces left by a moving object. Dots mark equal time intervals.

1.2-Q-02 A car moves along a straight road. It moves at a speed of 50 km/h for 4 minutes, then during 4 minutes it gradually speeds up to 100 km/h, continues at this speed for 4 minutes, then during 4 minutes gradually slows to a stop. Make a sketch like the figures in Section 1.2, marking dots for the position along the road every minute.

1.3-Q-01 Why do we use a spaceship in outer space, far from other objects, to illustrate Newton's first law? Why not a car or a train? (More than one of the following statements may be correct.)

1. A car or train touches other objects and interacts with them.
2. A car or train can't travel fast enough.
3. The spaceship has negligible interactions with other objects.
4. A car or train interacts gravitationally with the Earth.
5. A spaceship can never experience a gravitational force.

1.3-Q-02 Which of the following observers might observe something that appears to violate Newton's first law of motion? Explain why.

1. A person standing still on a street corner
2. A person riding on a roller coaster
3. A passenger on a starship traveling at 0.75c toward the nearby star Alpha Centauri
4. An airplane pilot doing aerobatic loops
5. A hockey player coasting across the ice

1.3-Q-03 Place a ball on a book and walk with the book in uniform motion. Note that you don't really have to do anything to the ball to keep the ball moving with constant velocity (relative to the ground) or to keep the ball at rest (relative to you). Then stop suddenly or abruptly change your direction or speed. What does Newton's first law of motion predict for the motion of the ball (assuming that the interaction between the ball and the book is small)? Does the ball behave as predicted? It may help to take the point of view of a friend who is standing still watching you.

1.3-Q-04 A spaceship far from all other objects uses its thrusters to attain a speed of 1×10^4 m/s. The crew then shuts off the power. According to Newton's first law, what will happen to the motion of the spaceship from then on?

1.4-Q-01 Which of the following are vectors?

1. $\vec{r}/2$
2. $|\vec{r}|/2$
3. $\langle r_x, r_y, r_z \rangle$
4. $5\vec{r}$

1.9-Q-01 Which of the following statements about the velocity and momentum of an object are correct?

1. The momentum of an object is always in the same direction as its velocity.
2. The momentum of an object can be either in the same direction as its velocity or in the opposite direction.
3. The momentum of an object is perpendicular to its velocity.
4. The direction of an object's momentum is not related to the direction of its velocity.
5. The direction of an object's momentum is tangent to its path.

1.10-Q-01 Answer the following questions about the factor γ (gamma) in the full relativistic equation for momentum:

a. Is γ a scalar or a vector quantity?

b. What is the minimum possible value of γ?

c. Does γ reach its minimum value when an object's speed is high or low?

d. Is there a maximum possible value for γ?

e. Does γ become large when an object's speed is high or low?

f. Does the approximation $\gamma \approx 1$ apply when an object's speed is low or when it is high?

1.10-Q-02 In which of these situations is it reasonable to use the approximate equation for the momentum of an object, instead of the full relativistically correct equation?

1. A car traveling on an interstate highway
2. A commercial jet airliner flying between New York and Seattle
3. A neutron traveling at 2700 meters per second
4. A proton in outer space traveling at 2×10^8 m/s
5. An electron in a television tube traveling 3×10^6 m/s

Problems

The difficulty of a problem is represented by the number of dots preceding the problem number.

Section 1.4

•1.4-P-01 What is the sum of the two vectors $\langle 8, 12, -7 \rangle$ m and $\langle -4, 0, 6 \rangle$ m?

•1.4-P-02 Figure 1.16.A shows several arrows representing vectors in the xy plane.

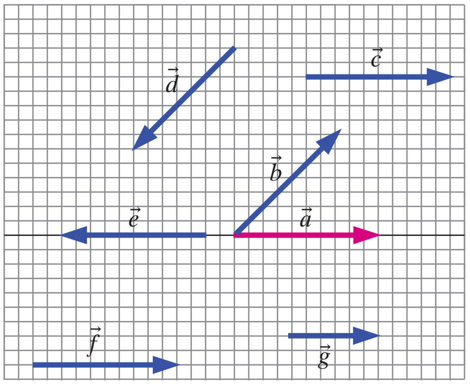


FIGURE 1.16.A Arrows representing vectors in the xy plane.

a. Which vectors have magnitudes equal to the magnitude of $\vec{a}$?

b. Which vectors are equal to $\vec{a}$?

•1.4-P-03 What is the magnitude of the vector $\vec{v}$, where $\vec{v} = \langle 8 \times 10^6, 0, -2 \times 10^7 \rangle$ m/s?

•1.4-P-04 In Figure 1.16.B, three vectors are represented by arrows in the xy plane. Each square in the grid represents one meter. For each vector, write out the components of the vector, and calculate the magnitude of the vector.

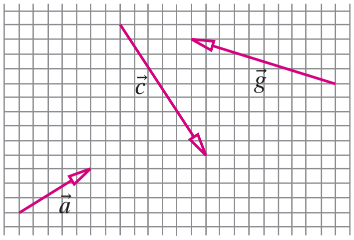


FIGURE 1.16.B Vectors in the xy plane. Each square represents one meter.

•1.4-P-05 The following questions refer to the vectors depicted by arrows in Figure 1.16.C. Each square in the grid represents one meter.

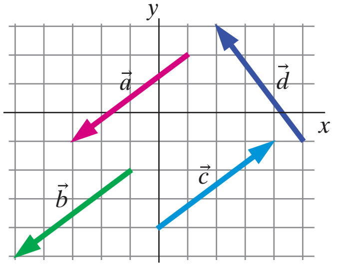


FIGURE 1.16.C Vectors in the xy plane. Each square represents one meter.

a. What are the components of the vector $\vec{a}$? (Note that because the vector lies in the xy plane, its z component is zero.)

b. What are the components of the vector $\vec{b}$?

c. Is this statement true or false? $\vec{a} = \vec{b}$

d. What are the components of the vector $\vec{c}$?

e. Is this statement true or false? $\vec{c} = -\vec{a}$

f. What are the components of the vector $\vec{d}$?

g. Is this statement true or false? $\vec{d} = -\vec{c}$

•1.4-P-06 On a piece of graph paper, draw arrows representing the following vectors. Make sure the tip and tail of each arrow you draw are clearly distinguishable.

a. Placing the tail of the vector at $\langle 5, 2, 0\rangle$, draw an arrow representing the vector $\vec{p} = \langle -7, 3, 0\rangle$. Label it $\vec{p}$.

b. Placing the tail of the vector at $\langle -5, 8, 0\rangle$, draw an arrow representing the vector $-\vec{p}$. Label it $-\vec{p}$.

•1.4-P-07 What is the result of multiplying the vector $\vec{a}$ by the scalar f, where $\vec{a} = \langle 0.02, -1.7, 30.0\rangle$ and $f = 2.0$?

•1.4-P-08 a. In **Figure 1.16.D**, what are the components of the vector $\vec{d}$?

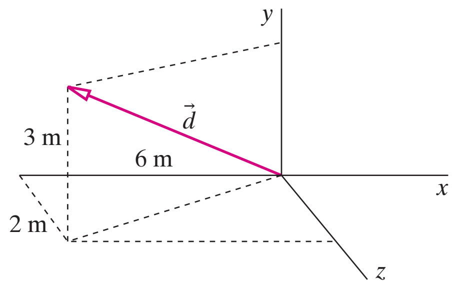


FIGURE 1.16.D The vector $\vec{d}$.

b. If $\vec{e} = -\vec{d}$, what are the components of $\vec{e}$?

c. If the tail of vector $\vec{d}$ were moved to location $\langle -5, -2, 4\rangle$ m, where would the tip of the vector be located?

d. If the tail of vector $-\vec{d}$ were placed at location $\langle -1, -1, -1\rangle$ m, where would the tip of the vector be located?

•1.4-P-09 What is the unit vector in the direction of $\langle 2, 2, 2\rangle$? What is the unit vector in the direction of $\langle 3, 3, 3\rangle$?

•1.4-P-10 a. On a piece of graph paper, draw the vector $\vec{f} = \langle -2, 4, 0\rangle$, putting the tail of the vector at $\langle -3, 0, 0\rangle$. Label the vector $\vec{f}$.

b. Calculate the vector $2\vec{f}$, and draw this vector on the graph, putting its tail at $\langle -3, -3, 0\rangle$, so you can compare it to the original vector. Label the vector $2\vec{f}$.

c. How does the magnitude of $2\vec{f}$ compare to the magnitude of $\vec{f}$?

d. How does the direction of $2\vec{f}$ compare to the direction of $\vec{f}$?

e. Calculate the vector $\vec{f}/2$, and draw this vector on the graph, putting its tail at $\langle -3, -6, 0\rangle$, so you can compare it to the other vectors. Label the vector $\vec{f}/2$.

f. How does the magnitude of $\vec{f}/2$ compare to the magnitude of $\vec{f}$?

g. How does the direction of $\vec{f}/2$ compare to the direction of $\vec{f}$?

h. Does multiplying a vector by a scalar change the magnitude of the vector?

i. The vector $a(\vec{f})$ has a magnitude three times as great as that of $\vec{f}$, and its direction is opposite to the direction of $\vec{f}$. What is the value of the scalar factor a?

•1.4-P-11 Write the vector $\vec{a} = \langle 400, 200, -100\rangle$ m/s^2 as the product $|\vec{a}| \cdot \hat{a}$.

•1.4-P-12 a. On a piece of graph paper, draw the vector $\vec{g} = \langle 4, 7, 0\rangle$ m. Put the tail of the vector at the origin.

b. Calculate the magnitude of $\vec{g}$.

c. Calculate $\hat{g}$, the unit vector pointing in the direction of $\vec{g}$.

d. On the graph, draw $\hat{g}$. Put the tail of the vector at $\langle 1, 0, 0\rangle$ m so you can compare $\hat{g}$ and $\vec{g}$.

e. Calculate the product of the magnitude $|\vec{g}|$ times the unit vector $\hat{g}$, $(|\vec{g}|)(\hat{g})$.

•1.4-P-13 A proton is located at $\langle 3 \times 10^{-10}, -3 \times 10^{-10}, 8 \times 10^{-10}\rangle$ m.

a. What is $\vec{r}$, the vector from the origin to the location of the proton?

b. What is $|\vec{r}|$?

c. What is $\hat{r}$, the unit vector in the direction of $\vec{r}$?

•1.4-P-14 In **Figure 1.16.E**, the vector $\vec{r}_1$ points to the location of object 1 and $\vec{r}_2$ points to the location of object 2. Both vectors lie in the xy plane.

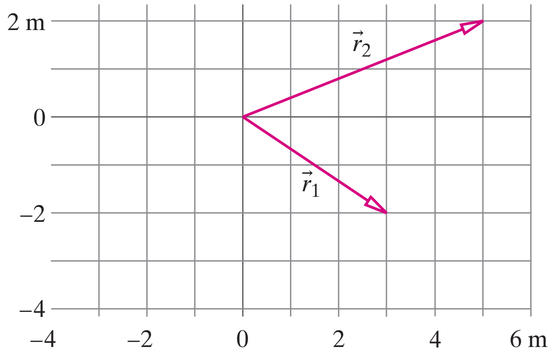


FIGURE 1.16.E The two vectors $\vec{r}_1$ and $\vec{r}_2$ lie in the xy plane.

a. Calculate the position of object 2 relative to object 1, as a relative position vector.

b. Calculate the position of object 1 relative to object 2, as a relative position vector.

•1.4-P-15 a. What is the vector whose tail is at $\langle 9.5, 7, 0\rangle$ m and whose head is at $\langle 4, -13, 0\rangle$ m?

b. What is the magnitude of this vector?

•1.4-P-16 A man is standing on the roof of a building with his head at the position $\langle 12, 30, 13\rangle$ m. He sees the top of a tree, which is at the position $\langle -25, 35, 43\rangle$ m.

a. What is the relative position vector that points from the man's head to the top of the tree?

b. What is the distance from the man's head to the top of the tree?

•1.4-P-17 A star is located at $\langle 6 \times 10^{10}, 8 \times 10^{10}, 6 \times 10^{10}\rangle$ m. A planet is located at $\langle -4 \times 10^{10}, -9 \times 10^{10}, 6 \times 10^{10}\rangle$ m.

a. What is the vector pointing from the star to the planet?

b. What is the vector pointing from the planet to the star?

•1.4-P-18 A planet is located at $\langle -1\times 10^{10}, 8\times 10^{10}, -3\times 10^{10}\rangle$ m. A star is located at $\langle 6\times 10^{10}, -5\times 10^{10}, 1\times 10^{10}\rangle$ m.

a. What is $\vec{r}$, the vector from the star to the planet?

b. What is the magnitude of $\vec{r}$?

c. What is $\hat{r}$, the unit vector (vector with magnitude 1) in the direction of $\vec{r}$?

•1.4-P-19 A proton is located at $\langle x_p, y_p, z_p\rangle$. An electron is located at $\langle x_e, y_e, z_e\rangle$.

a. What is the vector pointing from the electron to the proton?

b. What is the vector pointing from the proton to the electron?

••1.4-P-20 A cube is 3 cm on a side, with one corner at the origin.

a. What is the unit vector pointing from the origin to the diagonally opposite corner at location $\langle 3, 3, 3\rangle$ cm?

b. What is the angle from this diagonal to one of the adjacent edges of the cube?

Section 1.6

•1.6-P-01 A "slow" neutron produced in a nuclear reactor travels from location $\langle 0.2, -0.05, 0.1\rangle$ m to location $\langle -0.202, 0.054, 0.098\rangle$ m in 2 microseconds ($1\ \mu s = 1\times 10^{-6}$ s).

a. What is the average velocity of the neutron?

b. What is the average speed of the neutron?

•1.6-P-02 The position of a baseball relative to home plate changes from $\langle 15, 8, -3\rangle$ m to $\langle 20, 6, -1\rangle$ m in 0.1 s. Write the average velocity of the baseball during this time interval as a vector.

•1.6-P-03 You jog at a steady speed of 2 m/s. You start from the location $\langle 0, 0, 0\rangle$ and for the first 200 s your direction is given by the unit vector $\langle 1, 0, 0\rangle$. Next you jog for 300 s in the direction given by the unit vector $\langle \cos 45°, 0, \cos 45°\rangle$. Finally, you jog for 150 s in the direction given by the unit vector $\langle \cos 60°, 0, \cos 30°\rangle$.

a. Now what is your position?

b. What was your average velocity?

•1.6-P-04 The position of a golf ball relative to the tee changes from $\langle 50, 20, 30\rangle$ m to $\langle 53, 18, 31\rangle$ m in 0.1 second. As a vector, write the velocity of the golf ball during this short time interval.

••1.6-P-05 The crew of a stationary spacecraft observe an asteroid whose mass is 4×10^{17} kg. Taking the location of the spacecraft as the origin, the asteroid is observed to be at location $\langle -3\times 10^3, -4\times 10^3, 8\times 10^3\rangle$ m at a time 18.4 s after lunchtime. At a time 21.4 s after lunchtime, the asteroid is observed to be at location $\langle -1.4\times 10^3, -6.2\times 10^3, 9.7\times 10^3\rangle$ m. Assuming that the velocity of the asteroid does not change during this time interval, calculate the vector velocity $\vec{v}$ of the asteroid.

••1.6-P-06 A spacecraft traveling at a velocity of $\langle -20, -90, 40\rangle$ m/s is observed to be at a location $\langle 200, 300, -500\rangle$ m relative to an origin located on a nearby asteroid. At a later time, the spacecraft is at location $\langle -380, -2310, 660\rangle$ m.

a. How long did it take the spacecraft to travel between these locations?

b. How far did the spacecraft travel?

c. What is the speed of the spacecraft?

d. What is the unit vector in the direction of the spacecraft's velocity?

••1.6-P-07 Table 1.5 shows the positions at three different times for a bee in flight (a bee's top speed is about 7 m/s).

TABLE 1.5 Positions of a bee at different times

Time	Position
6.3 s	$\langle -3.5, 9.4, 0\rangle$ m
6.8 s	$\langle -1.3, 6.2, 0\rangle$ m
7.3 s	$\langle 0.5, 1.7, 0\rangle$ m

a. Between 6.3 s and 6.8 s, what was the bee's average velocity? Be careful with signs.

b. Between 6.3 s and 7.3 s, what was the bee's average velocity?

c. Of the two average velocities you calculated, which is the best estimate of the bee's instantaneous velocity at time 6.3 s?

d. Using the best information available, what was the displacement of the bee during the time interval from 6.3 s to 6.33 s?

Section 1.7

•1.7-P-01 An insect whose average velocity is $\langle -6, 9.5, -5\rangle$ m/s arrives at location $\langle -9, 11, -5\rangle$ m at time $t = 3.4$ s. What was the position of the insect 2 seconds earlier?

•1.7-P-02 At time $t_1 = 12$ s, a car is located at $\langle 84, 78, 24\rangle$ m and has velocity $\langle 4, 0, -3\rangle$ m/s. At time $t_2 = 18$ s, what is the position of the car? (The velocity is constant in magnitude and direction during this time interval.)

•1.7-P-03 An electron passes location $\langle 0.02, 0.04, -0.06\rangle$ m, and 2 μs later is detected at location $\langle 0.02, 1.84, -0.86\rangle$ m (1 microsecond is 1×10^{-6} s).

a. What is the average velocity of the electron?

b. If the electron continues to travel at this average velocity, where will it be in another 5 μs?

•1.7-P-04 After World War II, the U.S. Air Force carried out experiments on the amount of acceleration a human can survive. These experiments, led by John Stapp, were the first to use crash dummies as well as human subjects. Stapp himself was a test subject and became an effective advocate for automobile safety belts. In one of the experiments, Stapp rode a rocket sled that decelerated from 140 m/s (about 310 mi/h) to 70 m/s in just 0.6 s.

a. What was the absolute value of the (negative) average acceleration?

b. The acceleration of a falling object if air resistance is negligible is 9.8 m/s/s, called "one g." What was the absolute value of the average acceleration in g's? (Stapp eventually survived a test at 46 g's!)

••1.7-P-05 At a certain instant, a ball passes location $\langle 7, 21, -17\rangle$ m. In the next 3 s, the ball's average velocity is $\langle -11, 42, 11\rangle$ m/s. At the end of this 3 s time interval, what is the height y of the ball?

••1.7-P-06 You throw a ball. Assume that the origin is on the ground, with the $+y$ axis pointing upward. Just after the ball leaves your hand its position is $\langle 0.06, 1.03, 0\rangle$ m. The average velocity of the ball over the next 0.7 s is $\langle 17, 4, 6\rangle$ m/s. At time 0.7 s after the ball leaves your hand, what is the height of the ball above the ground?

••1.7-P-07 **Figure 1.16.F** shows the trajectory of a ball traveling through the air, affected by both gravity and air resistance. **Table 1.6** gives the positions of the ball at several successive times.

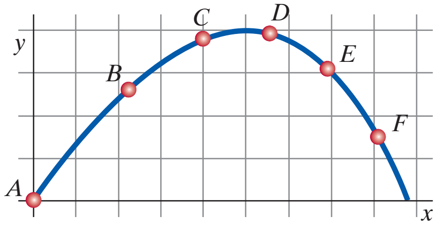


FIGURE 1.16.F Positions of a ball at 1-s intervals.

TABLE 1.6 **Successive positions of the ball**

Location	$t(s)$	Position (m)
A	0.0	$\langle 0, 0, 0\rangle$
B	1.0	$\langle 22.3, 26.1, 0\rangle$
C	2.0	$\langle 40.1, 38.1, 0\rangle$

a. What is the average velocity of the ball as it travels between location *A* and location *B*?

b. If the ball continued to travel at the same average velocity during the next second, where would it be at the end of that second? (That is, where would it be at time $t = 2$ s?)

c. How does your prediction from part (b) compare to the actual position of the ball at $t = 2$ s (location *C*)? If the predicted and observed locations of the ball are different, explain why.

••1.7-P-08 At 6 s after 3:00, a butterfly is observed leaving a flower whose location is $\langle 6, -3, 10\rangle$ m relative to an origin on top of a nearby tree. The butterfly flies until 10 s after 3:00, when it alights on a different flower whose location is $\langle 6.8, -4.2, 11.2\rangle$ m relative to the same origin. What was the location of the butterfly at a time 8.5 s after 3:00? What assumption did you have to make in calculating this location?

Section 1.8

•1.8-P-01 A toy battery-powered car can travel to the right or left along the x-axis, where the $+x$ direction is defined as "to the right." **Figure 1.16.G** shows various x vs. t graphs for the car.

Select the graph that matches each of these descriptions of the motion. There can be more than one correct answer.

a. The car is at rest and remains at rest.

b. The car is at rest, then suddenly moves with a constant velocity to the right.

c. The car travels with a constant velocity to the left, then suddenly stops and remains at rest.

d. The car travels with a constant velocity to the right.

e. The car travels with a constant velocity to the left.

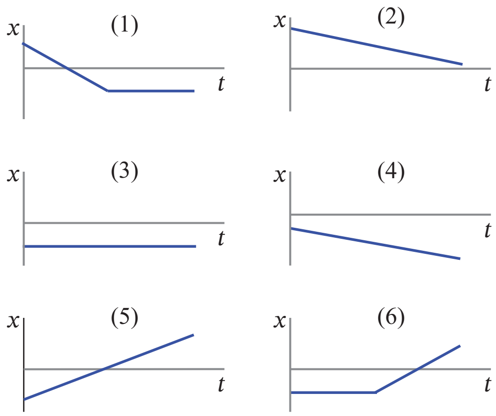
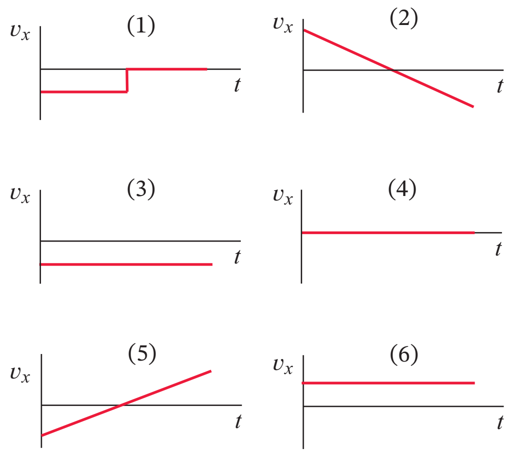


FIGURE 1.16.G Possible graphs of x vs. t.

•1.8-P-02 A toy battery-powered car can travel to the right or left along the x-axis, where the $+x$ direction is defined "to the right." **Figure 1.16.H** shows various v_x vs. t graphs for the car.



(7) None of the Above

FIGURE 1.16.H Possible graphs of v_x vs. t.

Select the graph that matches each of the descriptions of the motion. If no graph matches the description, choose (7) None of the Above.

a. The car is at rest and remains at rest.

b. The car travels with a constant velocity to the left, then suddenly stops and remains at rest.

c. The car travels with a constant velocity to the right.

d. The car travels with a constant velocity to the left.

e. The car travels with a constant velocity to the left, then suddenly changes direction and travels with a constant velocity to the right.

•1.8-P-03 **Figure 1.16.I** shows the x vs. t graph for a toy car.

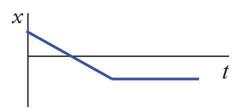


FIGURE 1.16.I x vs. t for a toy car.

Which of the v_x vs. t graphs in **Figure 1.16.H** is consistent with this x vs. t graph?

•1.8-P-04 **Figure 1.16.J** shows the v_x vs. t graph for a toy car.

FIGURE 1.16.J v_x vs. t for a toy car.

Which of the x vs. t graphs in **Figure 1.16.G** is consistent with the v_x vs. t graph? (There may be more than one.)

•1.8-P-05 Which of the x vs. t graphs in **Figure 1.16.G** describe the motion of an object for which there is *no* net interaction with its surroundings during the entire time interval depicted in the graph? (There may be more than one.)

•1.8-P-06 Which of the v_x vs. t graphs in **Figure 1.16.H** describe the motion of an object for which there is *no* net interaction with its surroundings during the entire time interval depicted in the graph? (There may be more than one.)

Section 1.9

•1.9-P-01 A baseball has a mass of 0.155 kg. A professional pitcher throws a baseball 90 mi/h, which is 40 m/s. What is the magnitude of the momentum of the pitched baseball?

•1.9-P-02 A hockey puck with a mass of 0.4 kg has a velocity of $\langle 38, 0, -27\rangle$ m/s. What is the magnitude of its momentum, $|\vec{p}|$?

•1.9-P-03 What is the magnitude (in kg · m/s) of the momentum of a 1000-kg airplane traveling at a speed of 500 mi/h? (Note that you need to convert speed to meters per second.)

•1.9-P-04 A baseball has a mass of about 155 g. What is the magnitude of the momentum of a baseball thrown at a speed of 100 miles per hour? (Note that you need to convert mass to kilograms and speed to meters per second. See the inside back cover of the textbook for conversion factors.)

•1.9-P-05 If a particle has momentum $\vec{p} = \langle 4, -5, 2\rangle$ kg · m/s, what is the magnitude $|\vec{p}|$ of its momentum?

•1.9-P-06 An object with mass 1.6 kg has momentum $\langle 0, 0, 4\rangle$kg · m/s.

a. What is the magnitude of the momentum?

b. What is the unit vector corresponding to the momentum?

c. What is the speed of the object?

•1.9-P-07 A tennis ball of mass m traveling with velocity $\langle v_x, 0, 0\rangle$ hits a wall and rebounds with velocity $\langle -v_x, 0, 0\rangle$.

a. What was the change in momentum of the tennis ball?

b. What was the change in the magnitude of the momentum of the tennis ball?

•1.9-P-08 A basketball has a mass of 570 g. Heading straight downward, in the $-y$ direction, it hits the floor with a speed of 5 m/s and rebounds straight up with nearly the same speed. What was the momentum change $\Delta\vec{p}$?

••1.9-P-09 A basketball has a mass of 570 g. Moving to the right and heading downward at an angle of 30° to the vertical, it hits the floor with a speed of 5 m/s and bounces up with nearly the same speed, again moving to the right at an angle of 30° to the vertical. What was the momentum change $\Delta\vec{p}$?

••1.9-P-10 The first stage of the giant Saturn V rocket reached a speed of 2300 m/s at 170 s after liftoff.

a. What was the average acceleration in m/s/s?

b. The acceleration of a falling object if air resistance is negligible is 9.8 m/s/s, called "one g." What was the average acceleration in g's?

••1.9-P-11 A 50-kg child is riding on a carousel (merry-go-round) at a constant speed of 5 m/s. What is the magnitude of the change in the child's momentum $|\Delta\vec{p}|$ in going all the way around (360°)? In going halfway around (180°)? It is very helpful to draw a diagram and to do the vector subtraction graphically.

••1.9-P-12 **Figure 1.16.K** shows a portion of the trajectory of a ball traveling through the air. Arrows indicate its momentum at several locations.

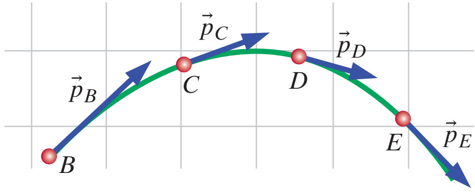


FIGURE 1.16.K Momentum of a ball at different locations in its trajectory.

At various locations, the ball's momentum is:

$$\vec{p}_B = \langle 3.03, 2.83, 0\rangle \text{ kg} \cdot \text{m/s}$$
$$\vec{p}_C = \langle 2.55, 0.97, 0\rangle \text{ kg} \cdot \text{m/s}$$
$$\vec{p}_D = \langle 2.24, -0.57, 0\rangle \text{ kg} \cdot \text{m/s}$$
$$\vec{p}_E = \langle 1.97, -1.93, 0\rangle \text{ kg} \cdot \text{m/s}$$
$$\vec{p}_F = \langle 1.68, -3.04, 0\rangle \text{ kg} \cdot \text{m/s}$$

a. Calculate the change in the ball's momentum between each pair of adjacent locations.

b. On a copy of the diagram, draw arrows representing each $\Delta\vec{p}$ you calculated in part (a).

c. Between which two locations is the magnitude of the change in momentum greatest?

•1.9-P-13 What is the velocity of a 3-kg object when its momentum is $\langle 60, 150, -30\rangle$ kg · m/s?

•1.9-P-14 A 1500-kg car located at $\langle 300, 0, 0\rangle$ m has a momentum of $\langle 45000, 0, 0\rangle$ kg · m/s. What is its location 10 s later?

••1.9-P-15 An ice hockey puck of mass 170 g enters the goal with a momentum of $\langle 0, 0, -6.3\rangle$ kg · m/s, crossing the goal line at location $\langle 0, 0, -26\rangle$ m relative to an origin in the center of the rink. The puck had been hit by a player 0.4 s before reaching the goal. What was the location of the puck when it was hit by the player, assuming negligible friction between the puck and the ice? (Note that the ice surface lies in the xz plane.)

••1.9-P-16 A space probe of mass 400 kg drifts past location $\langle 0, 3\times 10^4, -6\times 10^4\rangle$ m with momentum $\langle 6\times 10^3, 0, -3.6\times 10^3\rangle$ kg · m/s. Assuming the momentum of the probe does not change, what will be its position 2 minutes later?

Section 1.10

•1.10-P-01 A proton (mass 1.7×10^{-27} kg) in an accelerator attains a speed of $0.88c$. What is the magnitude of the momentum of the proton?

•1.10-P-02 An electron (mass 9×10^{-31} kg) with a speed of $0.95c$ is emitted by a supernova, where c is the speed of light. What is the magnitude of the momentum of this electron?

•1.10-P-03 A cosmic-ray proton (mass 1.7×10^{-27} kg) coming from outer space hits the upper atmosphere with a speed $0.9999c$, where c is the speed of light. What is the magnitude of the momentum of this proton? Note that $|\vec{v}|/c = 0.9999$.

•1.10-P-04 A proton (mass 1.7×10^{-27} kg) in a particle accelerator is traveling at a speed of $0.99c$.

a. If you use the approximate nonrelativistic equation for the magnitude of momentum of the proton, what answer do you get?

b. What is the magnitude of the correct relativistic momentum of the proton?

c. The approximate value (the answer to part a) is significantly too low. What is the ratio of magnitudes you calculated (correct/approximate)?

•1.10-P-05 When the speed of a particle is close to the speed of light, the factor γ, the ratio of the correct relativistic momentum $\gamma m\vec{v}$ to the approximate nonrelativistic momentum $m\vec{v}$, is quite large. Such speeds are attained in particle accelerators, and at these speeds the approximate nonrelativistic equation for momentum is a very poor approximation. Calculate γ for the case where $|\vec{v}|/c = 0.9996$.

•1.10-P-06 An electron (mass 9×10^{-31} kg) travels at speed $|\vec{v}| = 0.996c$, where $c = 3 \times 10^8$ m/s is the speed of light. The electron travels in the direction given by the unit vector $\hat{v} = \langle 0.655, -0.492, -0.573 \rangle$. The mass of an electron is 9×10^{-31} kg.

a. What is the value of γ? You can simplify the calculation if you notice that $(|\vec{v}|/c)^2 = (0.996)^2$.

b. What is the speed of the electron?

c. What is the magnitude of the electron's momentum?

d. What is the vector momentum of the electron? Remember that any vector can be "factored" into its magnitude times its unit vector, so that $\vec{v} = |\vec{v}|\hat{v}$.

•1.10-P-07 If $|\vec{p}|/m$ is $0.85c$, what is $|\vec{v}|$ in terms of c?

Section 1.11

These problems are intended to introduce you to using a computer to model matter, interactions, and motion in 3D. Computational modeling is something you learn by doing, so jump in even if you don't have any prior experience. You can go to **https://webvpython.org** to do these exercises.

•1.11-P-01 Start with the code from Section 1.11.2 that creates the display shown in **Figure 1.11.C**. Complete it by adding spheres at the remaining corners of the cube.

•1.11-P-02 Write a VPython program that represents the x, y, and z axes by three boxes (rectangular solids) of different colors. The display from one possible solution is shown in **Figure 1.16.L**.

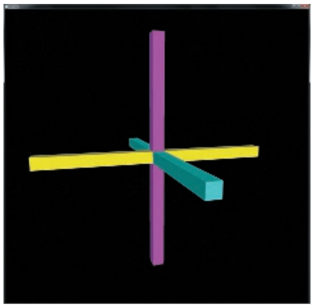


FIGURE 1.16.L Use boxes to represent the x, y, and z axes.

••1.11-P-03 Write a VPython program that uses three `while` loops to create a display in which each of the axes is represented by a linear array of boxes, with spaces between the boxes. Use loops to generate each array. **Figure 1.16.M** shows a possible example.

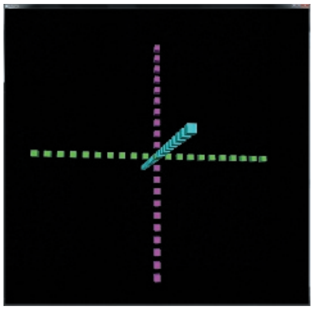


FIGURE 1.16.M Use arrays of boxes to represent the x, y, and z axes.

••1.11-P-04 a. An important skill is being able to read and understand an existing program in order to be able to make useful modifications. Before running the program below, study the program carefully line by line, then answer the following questions:

1. What is the initial velocity of the particle?
2. Is the particle initially located in front of the box or behind it?
3. In which line of code is the position of the particle updated?
4. What is the value of the time step Δt?
5. Will the particle bounce off of the red box or travel through it?

b. Now run the program, and see if your answers were correct.

c. Modify the program to start the particle from an initial position on the $+x$ axis, to the right of and in front of the red box. Give the particle a velocity that will make it travel to the left, along the x axis, passing in front of the box.

```
box(pos=vector(0,0,-1), size=(5,5,0.5),
  color=color.red, opacity = 0.4)
particle = sphere(pos=vector(-5,0,-5),
   radius=0.3, color=color.cyan,
     make_trail=True)
v = vector(0.5,0,0.5)
```

```
delta_t = 0.05
t = 0
while t < 20:
    rate(100)
    particle.pos = particle.pos + v * delta_t
    t = t + delta_t
```

•1.11-P-05 Start with the six-line program from Section 1.11.4 that moves a ball in the $+y$ direction. Modify the code so the ball moves in the $-x$ direction.

••1.11-P-06 Start with the program in Section 1.11.5 that predicts the path of a drone. Modify the program so the drone does not hit the light. (Hint: you could divide the path into two segments.)