Chapter 1

Risk and Expected Utility

Although it is not easy to clearly define risk, it is often defined as uncertainty regarding which outcome will occur. We adopt this definition in general. Economists are mainly concerned with risks that can be described in terms of probability distributions. Given that, risk will be identified with a random variable or its probability distribution throughout this book. This book is concerned with the economic analysis of how risks affect individual, organizational, and social behavior, and vice versa. As a basis for the analysis, we investigate the expected utility theory and the measurement of risks. The expected utility theory provides the basis for most economic and financial analyses of individual decision making under risk. We study the usefulness and weaknesses of the expected utility theory in this chapter. The issues related to risk aversion and the measurement of the riskiness of a project will be studied in the next chapter. We start with utility representation (Section 1.1) and develop the expected utility theory (Section 1.2). Section 1.3 discusses some problems with expected utility.

1.1 Utility Representation

Let us first briefly review the utility theory under certainty. An individual’s consumption of goods is determined by his or her preferences. Preferring (consumption of) good A to (consumption of) good B implies that A provides higher satisfaction than B. However, since the notion of preference is rather vague and qualitative in general, it is not easy to work with it directly. Economists have tried to transform preference into more easily manageable measures, one of which is utility. Utility is a mapping
from preferences over consumptions to real numbers which can be mathematically manipulated (see Figure 1.1). Since preference, in general, does not comply with the operation rules of real numbers, the utility theory can be only applied to a subset of possible preferences. The so-called axiomatic utility theory describes the set of preferences to which the utility theory can be applied.

Let us write $\succeq$ for the preference relation of an individual for the consumption set of goods, $S$. For $x, y$ in $S$, $x \succeq y$ implies that the individual (weakly) prefers $x$ to $y$. Strict preference and indifference will be denoted by $>$ and $\sim$, respectively. Let us first state the definition of utility representation.

**Definition 1.1 (Utility representation).** 
A function $U(.)$ from $S$ to $R$ (the real numbers) represents the preference relation $\succeq$, if $x \succeq y \iff U(x) \geq U(y)$.

Now consider the following axioms.

**Axiom U1** (Completeness). For all $x, y$ in $S$, $x \succeq y$ or $y \succeq x$ (or both).

**Axiom U2** (Transitivity). For all $x, y, z$ in $S$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.

**Axiom U3** (Continuity). For any sequence of pairs such that $\{(x^n, y^n)\}_{n=1}^{\infty}$ with $x^n \succeq y^n$ for all $n$, and $\lim x^n = x$ and $\lim y^n = y$, we have that $x \succeq y$.

Axioms U1 and U2 imply that all goods should be pairwise comparable and can be ranked based on preference. Axiom U3 requires that preference should not suddenly change. The preference is called rational if it satisfies Axioms U1 and U2.

The so-called lexicographic preference exemplifies that Axioms U1 and U2 alone are not enough for utility representation. Lexicographic preference is defined as follows: For $S = \mathbb{R}^2$, $x \succeq y$ if and only if (i) $x_1 > y_1$, or (ii) $x_2 \geq y_2$. 

![Figure 1.1 Utility representation.](image)
and \( x_1 = y_1 \). In other words, the first elements have absolute priority in preference. The second elements become important only if the first elements result in a tie. It is easy to see that lexicographic preference satisfies Axioms U1 and U2, thus is rational. However, it cannot be represented by a utility.

**Lemma 1.1**

(a) Lexicographic preference does not have a utility representation.
(b) Lexicographic preference is not continuous.

**Proof**

(a) Suppose the contrary. If a utility \( U(\cdot) \) exists, then \( U(x_1, 2) > U(x_1, 1) \) for any real number \( x_1 \). Since the set of rational numbers is dense, there exists a rational number between two different real numbers (see Figure 1.2). Let us take any rational number \( r(x_1) \) such that \( U(x_1, 2) > r(x_1) > U(x_1, 1) \). Note that for different \( x_1 \), a different \( r(x_1) \) is allocated. However, one-to-one mapping between real numbers and rational numbers is not possible, since the number of real numbers is uncountable while the number of rational numbers is countable. A contradiction.

(b) Let \( x^n = (1/n, 0) \), \( y^n = (0, 1) \). Then \( x = \lim x^n = (0, 0) \), \( y = \lim y^n = (0, 1) \). For every \( n \), \( x^n \geq y^n \). However, \( y > x \).

**Proposition 1.1 (Existence of utility).** If a preference satisfies Axioms U1–U3, then there is a continuous utility function that represents the preference.


Intuitively, Axioms U1–U3 require preference to behave like real numbers. Note that the real numbers satisfy all the axioms, once \( \succeq \) is replaced with \( \geq \).
with ≥. Since a utility is a mapping from preference to real numbers, it is not surprising that preference behaving like real numbers will have a utility representation. With the utility function, it is now convenient to analyze the individual’s decision making problem. An individual’s problem is generally expressed as a utility maximization problem with some constraints, which can be solved by mathematical optimization techniques (see Appendix).

1.2 Expected Utility Theory

We now turn to the utility representation under risk. It is important to note that we are only concerned with risks that can be identified with a random variable or its probability distribution. As in the certainty case, it will be convenient to express an individual’s problem under risk as a maximization problem that can be easily manipulated. One convenient way to do so is to apply the average concept to utility. The expected utility theory allows the expected value of utility to represent preference under risk.

While the expected utility is a simple application of statistical concepts to the utility theory, the preference of an individual does not have to comply with the statistical operations. As a result, the application of the expected utility theory can be applied to a subset of preferences.

The expected utility theory can be easily understood by considering preferences over lotteries of a gamble. Suppose that there are \( n \) possible payoffs \( (x_1, \ldots, x_n) \). A lottery \( L \) is an ordered \( n \)-tuple of probabilities for payoffs, \( L = (p_1, \ldots, p_n) \), where \( p_i \) is the probability that the payoff \( x_i \) is earned and \( \Sigma p_i = 1 \). Similar to the utility under certainty, the expected utility is a mapping from the preferences over the lotteries to real numbers which can be mathematically manipulated (see Figure 1.3).

Now consider an individual who enters into another gamble in which he earns a lottery of the original gamble. This setting is equivalent to the case in which the individual joins a compound gamble as follows. In this compound gamble, a compound lottery is defined as an ordered collection of probabilities of earning lotteries, \( L' = (L_1, \ldots, L_k; q_1, \ldots, q_k) \), where \( L_k \) is a lottery such that \( L_k = (p_{1k}, \ldots, p_{nk}) \), \( q_k \) is the probability that the lottery \( L_k \) is earned, and \( \Sigma q_k = 1 \). It is easy to see that the compound lottery can be expressed as a linear combination of lotteries: \( L' = q_1 L_1 + \ldots + q_k L_k \).

Thus a compound lottery is equivalent to a lottery \( (p'_1, \ldots, p'_n) \) where \( p'_i = \sum_k q_k p_{ik} \).

A basis lottery \( L' \) is defined as a lottery in which the \( i \)th probability is 1 and all others are zero: \( L' = (0, \ldots, p_i = 1, \ldots, 0) \), for \( 1 \leq i \leq n \). A basis lottery represents a payoff under certainty. Now, any lottery can be expressed as a compound lottery of basis lotteries: \( L = (p_1, \ldots, p_n) = p_1 L_1 + \ldots + p_n L_n \).
Let us write \( H_1 \) for the preference relation of an individual on the set of lotteries \( \Lambda \). For \( L, L' \) in \( \Lambda \), \( L \geq L' \) implies that the individual prefers (weakly) \( L \) to \( L' \). Strict preference will be denoted by \( > \). Like the utility theory under certainty, the expected utility theory requires several restrictions on preference.

**Axiom 1** (Completeness). For all \( L, L' \) in \( \Lambda \), \( L \geq L' \) or \( L' \geq L \).

**Axiom 2** (Transitivity). For all \( L, L', L'' \) in \( \Lambda \), if \( L \geq L' \) and \( L' \geq L'' \) then \( L \geq L'' \).

**Axiom 3** (Continuity). For any sequence of pairs, \( \{(L^n, L'^n)\}_{n=1}^\infty \) with \( L^n \geq L'^n \) for all \( n \), \( L = \lim L^n \) and \( L' = \lim L'^n \), we have \( L \geq L' \).

**Axiom 4** (Independence). For all \( L, L', L'' \) in \( \Lambda \) and \( 0 < \alpha < 1 \), \( L \geq L' \) if and only if \( \alpha L + (1 - \alpha)L'' \geq \alpha L' + (1 - \alpha)L'' \).

Axioms 1–3 are counterparts of Axioms U1–U3 of the utility theory under certainty. In addition, the expected utility theory requires Axiom 4. The independence axiom implies that if the individual prefers \( L \) to \( L' \), then such a preference should not be changed when each lottery is combined with another lottery. This axiom imposes a strong restriction on preference to prevent some undesirable results of the expected utility.

**Definition 1.2** (Expected utility representation). A utility function \( U : \Lambda \rightarrow R \) is called an expected utility if there is an \( n \)-tuple of real numbers \( (u_1, \ldots, u_n) \) such that, for all \( L = (p_1, \ldots, p_n) \) in \( \Lambda \), \( U(L) = p_1 u_1 + \ldots + p_n u_n \).

Note that the utility of a basis lottery \( L_i \) is \( u_i ; U(L_i) = u_i \). Since any lottery can be expressed as a linear combination of basis lotteries, the expected utility transforms a linear combination of basis lotteries to a linear combination of
utilities of basis lotteries: \( U(L) = U(p_1L^1 + \ldots + p_nL^n) = \sum p_i U(L^i) \). Note that the expected utility is expressed as an expected value of utilities over outcomes. Indeed, the expected utility is conventionally expressed as \( EU(W) \), where \( W \) is a random variable, usually representing wealth: \( EU(W) = \sum p_i U(W_i) \), where \( U(.) \) is called a von Neumann–Morgenstern utility. The following lemma shows that a utility is an expected utility if and only if it is linear.

**Lemma 1.2 (Linearity).** A utility \( U \) is an expected utility if and only if it is linear: \( U(a_1L_1 + \ldots + a_kL_k) = a_1U(L_1) + \ldots + a_kU(L_k) \), where \( \sum_{k=1}^{K} a_k = 1 \), and \( a_k \geq 0 \) for all \( k \).

**Proof.** (\( \Rightarrow \)) Since a lottery can be expressed as a compound lottery of basis lotteries \( \{L^i\} \), we may put \( L = (p_1, \ldots, p_n) = \sum p_i L^i \). Under linearity, \( U(L) = \sum p_i U(L^i) = \sum p_i u_i \).

(\( \Leftarrow \)) Consider a linear combination of two lotteries \( L \) and \( L' \), \( aL + bL' \), where \( a + b = 1 \). It suffices to show that \( U(aL + bL') = aU(L) + bU(L') \) for an expected utility \( U \). Using basis lotteries, we can express each lottery as a linear combination of basis lotteries: \( L = \sum p_i L^i \) and \( L' = \sum q_i L'^i \). Therefore, \( aL + bL' = \sum (ap_i + bq_i)L^i \). Then we have \( U(aL + bL') = \sum (ap_i + bq_i)u_i = a\sum p_i u_i + b\sum q_i u_i = aU(L) + bU(L') \).

The linearity property of the expected utility is not surprising, once it is observed that the expectation operator is linear, and that the expected utility is expressed as an expected value of utilities. Now, we are ready to show that there is an expected utility for a preference that satisfies Axioms 1–4.

**Proposition 1.2 (Existence of expected utility).** Suppose that preference \( \succeq \) satisfies Axioms 1–4. Then there exists an expected utility \( U(.) \) representing the preference. In other words, for \( L = (p_1, \ldots, p_n) \) and \( L = (p'_1, \ldots, p'_n) \), \( L \succeq L' \) if and only if \( U(L) \geq U(L') \).

**Proof.** Let us consider only the bounded case in which there exist lotteries \( T \) and \( D \) such that \( T \succeq L \succeq D \) for every lottery \( L \) (see Figure 1.4). Our task is to express the lottery \( L \) as \( aT + (1 - a)D \), where \( a \) is a number. The number \( a \) will be assigned as the utility of the lottery. The proof is composed of four steps.

First, note that if \( L > L' \) and \( a \in (0, 1) \), then \( L > aL + (1 - a)L' > L' \). For, from the independence axiom (Axiom 4), \( L = aL + (1 - a)L > aL + (1 - a)L' > aL' + (1 - a)L' = L' \).
Second, \( bT + (1 - b)D > aT + (1 - a)D \) if and only if \( b > a \), for \( a, b \in [0, 1] \). For the “if” part (\( \iff \)), note that for \( b > a \), \( bT + (1 - b)D = cT + (1 - c)[aT + (1 - a)D] \), where \( c = (b - a)/(1 - a) \). Since \( T > aT + (1 - a)D \), \( cT + (1 - c)[aT + (1 - a)D] > c[aT + (1 - a)D] + (1 - c)[aT + (1 - a)D] = aT + (1 - a)D \). For the “only if” part (\( \Rightarrow \)), we need to show that \( aT + (1 - a)D \geq bT + (1 - b)D \) for \( a \geq b \). If \( a = b \), the result holds clearly. If \( a > b \), then the result is obtained by exchanging the roles of \( a \) and \( b \) in the “if” part of the proof.

Third, for any \( L \), there exists a unique \( a(L) \in [0, 1] \) such that \( a(L)T + (1 - a(L))D \sim L \) (see Figure 1.4). The existence of such \( a(L) \) follows from the second step and the continuity axiom, and the uniqueness from the second step.

Fourth, let us assign \( U(L) = a(L) \). Now, it suffices to show this utility is linear, due to Lemma 1.2. For this, note that \( L \sim U(L)T + (1 - U(L))D \) and \( L' \sim U(L')T + (1 - U(L'))D \). Thus, \( aL + (1 - a)L' \sim [aU(L) + (1 - a)U(L')]T + [1 - aU(L) - (1 - a)U(L')]D \). By our definition of \( U \), \( U(aL + (1 - a)L') = aU(L) + (1 - a)U(L') \).

Even if a preference has a utility representation, it is not unique. In fact, there are infinitely many utilities for the same preference. However, different utilities are related to each other due to the common properties that they share. Note that a utility under certainty only requires a higher number to be assigned to a more preferred consumption. Therefore, two utility functions are compatible if they assign a higher number to a more preferred consumption. This property is called **ordinality**: the order of preference is preserved across utility functions. The ordinality property also implies...
that the rank of differences in utility values may be changed. For example, for \( x \gtrless y \gtrless z \), two different utility functions, \( U \) and \( V \), can exist such that \( U(x) = 1000 > U(y) = 2 > U(z) = 1; V(x) = 10 > V(y) = 9 > V(z) = 5 \). Note that the order of utility values is preserved in both utilities. However, the rank of differences between utility values is not preserved: \( U(x) - U(y) > U(y) - U(z) \), but \( V(x) - V(y) < V(y) - V(z) \). It is known that two utilities for a preference have the relation \( U(x) = f(V(x)) \), where \( f(.) \) is an increasing function. The ordinality property can be easily derived from this relation.

On the other hand, the expected utility has the property of cardinality, implying that the rank of the utility differences as well as the order is preserved. This cardinality property is obtained from the following relation between expected utilities.

**Proposition 1.3** (Cardinality of expected utility). Suppose that \( U \) is an expected utility of a preference. Then, another utility \( V \) is an expected utility of the preference, if and only if \( V \) is an affine transformation of \( U: V(L) = sU(L) + t, s > 0 \) for all \( L \).

**Proof.** \((\Leftrightarrow)\) For \( a + b = 1, V(aL + bL') = sU(aL + bL') + t = s[aU(L) + bU(L')] + t = a[sU(L) + t] + b[sU(L') + t] = aV(L) + bV(L'). \) Note that the second equality comes from the fact that \( U \) is an expected utility. The linearity of \( V \) implies that \( V \) is an expected utility.

\((\Rightarrow)\) We will find numbers \( s \) and \( t \) satisfying the relation. Let \( T \) and \( D \) be such that \( T \gtrless L \gtrsim D \) for all \( L \). Define \( a \) by \( U(L) = aU(T) + (1 - a)U(D) \). Thus \( a = [U(L) - U(D)]/[U(T) - U(D)] \). Also \( V(L) = V(aT + (1 - a)D) = aV(T) + (1 - a)V(D) = a(V(T) - V(D)) + V(D) \).

Now, by substituting \( a \) into this equation and defining \( s = [V(T) - V(D)]/[U(T) - U(D)] > 0 \) and \( t = V(D) - U(D)[V(T) - V(D)]/[U(T) - U(D)] \), we have \( V(L) = sU(L) + t \).

Suppose that \( L \gtrsim L' \gtrsim L'' \). For two expected utilities \( U \) and \( V \), \( V(L) - V(L') = s[U(L) - U(L')] \). Thus, we should have \( V(L) - V(L') > V(L') - V(L'') \) if and only if \( U(L) - U(L') > U(L') - U(L'') \). Unlike the utility under certainty, the expected utility preserves the rank of utility differences.

### 1.3 Problems with the Expected Utility

The expected utility theory allows us to express a utility as an expected value of utilities over possible outcomes. However, this convenience is obtained by restricting the set of preferences to which the theory can be applied. Several problems have been pointed out in literature (see Schoemaker, 1982).
One example is the so-called Allais paradox (Allais and Hagen, 1979). To illustrate, consider the following gamble. The rewards for the first, second, and third prizes are $2,500,000, $500,000, and $0, respectively. Now, consider the lotteries $L_1 = (0, 1, 0)$, $L'_1 = (0.10, 0.84, 0.06)$, $L_2 = (0, 0.16, 0.84)$, $L'_2 = (0.10, 0, 0.90)$, where each element represents the probability of earning each prize in order. It seems to be acceptable that an individual prefers $L_1$ to $L'_1$, and $L'_2$ to $L_2$. However, this preference violates the independence axiom. For this, let us add the lotteries $(0, 0.08, 0.92) - (0, 0.92, 0.08)$ to both sides of $L_1 > L'_1$. If the independent axiom holds, then we will have $(0, 1, 0) + (0, 0.08, 0.92) - (0, 0.92, 0.08) > (0.10, 0.84, 0.06) + (0, 0.08, 0.92) - (0, 0.92, 0.08) \Rightarrow (0, 0.16, 0.84) > (0.10, 0, 0.90) \Rightarrow L_2 > L'_2$. Therefore, $L_1 > L'_1$ and $L'_2 > L_2$ violate the independence axiom, which leads to the nonexistence of an expected utility representation. Indeed, if $U$ is an expected utility for this preference, then $L_1 > L'_1$ implies $U(500,000) > 0.10U(2,500,000) + 0.84U(500,000) + 0.06U(0)$. Adding $0.84U(0) - 0.84U(500,000)$ to both sides leads to $0.16U(500,000) + 0.84U(0) > 0.10U(2,500,000) + 0.90U(0)$. This implies $L_2 > L'_2$.

Another problem regarding the expected utility is the treatment of ambiguity, as pointed out by the Ellsberg paradox. Suppose that there are 90 balls in a box. Each ball has one of three colors, red, white, and black. It is known that 30 balls are red, but the proportions of white and black balls are not known. Now, consider the following gambles.

**Gamble 1.** You choose one color and pick up one ball. You will receive $1m if the colors match, or $0 otherwise.

**Gamble 2.** You choose two colors and pick up one ball. You will receive $1m if the ball has one of the colors you choose, or $0 otherwise.

It is often the case that people choose red in gamble 1 and choose white and black in gamble 2. This result is interpreted as people disliking the ambiguity, or the uncertainty of probability. The red color has a sure probability of 1/3 in gamble 1, and the white and the black colors have a sure probability of 2/3 in gamble 2. However, this preference cannot be captured by the expected utility, since it is linear in probability. Since the rational expected value of the probability of white (or black) is 1/3 as for red, choosing any color(s) should provide the same expected utility. Dislike of ambiguity may be important in the insurance context when people have little or no information regarding risks. Examples of such risks will include new types of catastrophe risks or terrorism risks. Incorporating this ambiguity into utility may require nonlinearity in the probability. Diverse nonexpected utility models have been suggested. A general form
of utility is the one which transforms both probabilities and wealth. For example, general utility can be expressed as $\Sigma \pi_i V(W_i)$, where $\pi_i$ is the transformation obtained from the distribution function of wealth (see Quiggin, 1982). In Yaari’s dual theory, $V(W_i) = W_i$ (Yaari, 1987). In the prospect theory, $V(W_i) = W_i - R_i$, where $R_i$ is a reference point (Kahneman and Tversky, 1979). In general, the function is nonlinear in the probability due to the transformation of distribution $\pi_i$. Interested readers are referred to Machina (1987, 2000) and Gollier (2000). Given the problems with the expected utility, we should be cautious in interpreting the results of analyses with the expected utility. Nevertheless, the expected utility is the most widely accepted analytic tool in the insurance and finance areas.

Our discussions in this book will also be based on the expected utility framework. In this book, utilities under risk are von Neumann–Morgenstern utilities, unless stated otherwise. In addition, utility is usually expressed as a function of wealth in literature, which is also adopted throughout this book. Since higher wealth is preferred to lower wealth, $U(W)$ is increasing in $W$.

### 1.4 Conclusion

Risk is generally identified with uncertainty of outcomes. Expected utility is an important tool in dealing with decision making under risk. Expected utility is a natural extension of utility under risk. Expected utility, however, is based on strict assumptions regarding preference. Therefore, it is possible that some decision making is not explained by the expected utility, if the preference violates the assumptions. The violation of the assumptions of the expected utility theory is often described as irrational. Diverse non-expected utility theories such as the prospect theory have been developed to explain such irrational behavior, which forms the basis for behavioral economics.

Risk is generally described by the probability distribution in economics. However, some risks (for example, catastrophe risks) may not have a well-established probability distribution. The probability distribution itself may be exposed to uncertainty, in that the realized probability distribution is not known (Knight, 1921). Under the expected utility theory, uncertainty in the probability distribution is not a problem, since it does not affect the expected utility due to linearity. However, if the probabilities are transformed in the calculation of utility as in nonexpected utility theories, then the uncertainty in probability will affect the utility. The uncertainty in the probability distribution will then become important in understanding decision making under uncertainty. Moreover, in highly uncertain cases
even the probability distribution may not be known (see Gomory, 1995). These cases will be difficult to analyze, since proper analytic tools have not yet been developed.

BIBLIOGRAPHY


