1

Introduction to Statically Indeterminate Structural Analysis

The teaching objectives of this chapter are as follows:

– the importance and usefulness of statically indeterminate structures;

– calculating the degree of external and internal static indeterminacy of the structures;

– analyzing kinematic static indeterminacy;

– illustrating the strengths and weaknesses of statically indeterminate structures.

In the first part, we give a general introduction to the methods of analyzing statically indeterminate structures. In this context, we describe the external, internal and kinematic static indeterminacies of the structures. In the second part, we illustrate the analysis methods for statically indeterminate structures. Lastly, we list the advantages and disadvantages of statically indeterminate structures.

1.1. Introduction

Structures are grouped into two categories: (1) statically determinate structures and (2) statically indeterminate structures. The static equations are not sufficient for analyzing statically indeterminate structures. In this case, the number of unknowns is strictly greater than the number of independent equilibrium equations.

The primary role of analysis of a statically indeterminate structure is to remove the static indeterminacy of the given structure. This removal means that we can calculate the support reactions and the internal actions when the structure is solicited by mechanical loads, or subjected to deflections, and/or undergoing a support settlement. The analysis methods for statically indeterminate structures are used here to make the number of unknowns equal to the number of equations, which allows the problem to be solved.

This book is particularly devoted to the analysis of statically indeterminate structures. To present the differences between the analysis methods for statically indeterminate structures, the problems we consider generally have a common object across the analysis methods. In this context, each chapter illustrates the theoretical foundation of the analytical method, presented in detail and accompanied by a series of numerical examples.

1.2. External static indeterminacy

The purpose of structural analysis is to determine the support reactions and the variation of internal actions in the elements of a statically indeterminate structure. The static indeterminacy of a structure can be internal, external or internal and external simultaneously. It is called externally statically indeterminate if the number of support reactions exceeds the number of independent equations. The plane structures are externally statically indeterminate if the number of support reactions is greater than 3 (Figure 1.1) and it is greater than 6 if the structure is spatial (Figure 1.2).

From this explanation, we define the degree of static indeterminacy of a system by the difference between the number of support reactions and the number of independent equations that can be constructed. The degree of external static indeterminacy f of a plane structure [1.1] or a space structure [1.2] is deduced by

$$f = r - (3 + k) \tag{1.1}$$

$$f = r - (6 + k) \tag{1.2}$$

We calculate the degree of static indeterminacy of the beam and frame (Figure 1.1).

Beam, f = 5 – 3 =2

Frame, f = 10 – (3 + 1) = 6

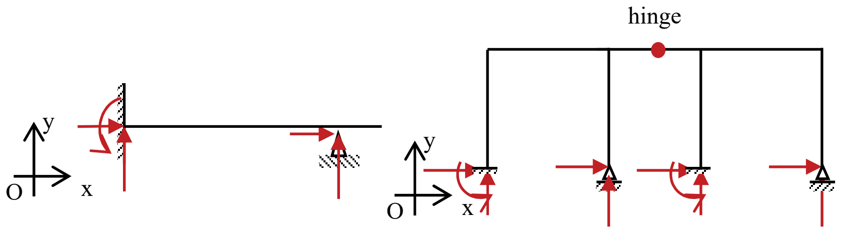


Figure 1.1. *Statically indeterminate externally of plane structures*[1]

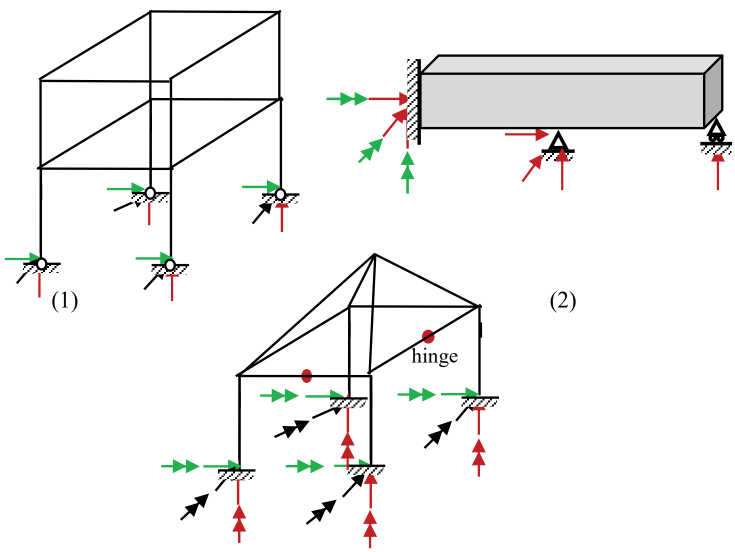


Figure 1.2. *Statically indeterminate externally of space structures*

For space structures (Figure 1.2), the degree of static indeterminacy is

Frame (1), $f = 12 - 6 = 6$

Beam (2), $f = 10 - 6 = 4$

Frame (3), $f = 24 - (6 + 12) = 6$

1 All of the figures in this chapter are available to view in full color at www.iste.co.uk/khalfallah/analysis2.zip.

The degree of static indeterminacy of trusses is calculated by using relationships [1.1] and [1.2]. Figure 1.3 presents plane and space trusses.

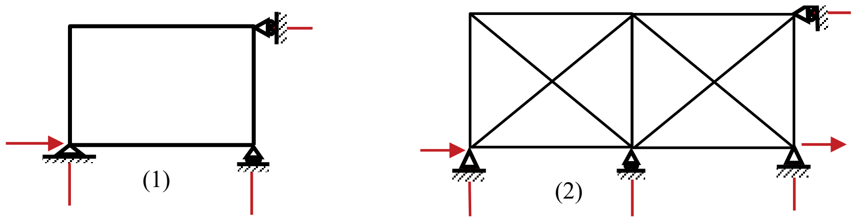


Figure 1.3. *Externally statically indeterminate plane structures*

The degree of external static indeterminacy of plane structures (Figure 1.3) is

Structure (1): f = 4 – 3 = 1

Structure (2): f = 6 – 3 = 3

In the same way, space truss structures (Figure 1.4) are the most used in the construction of large exhibition halls and sports halls, etc.

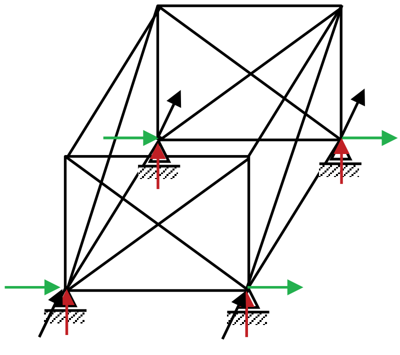


Figure 1.4. *Statically indeterminate space truss*

The degree of static indeterminacy of the structure is

f = 12 – 6 = 6

1.3. Internal static indeterminacy

In this section, we describe how to calculate the degrees of static indeterminacy of trusses, frames, beams and crossbeams.

1.3.1. *Truss structures*

Consider a truss structure with r support reactions, b bars and n joints including support joints. The number of unknowns (b + r) of the problem is the support reactions and the forces in the bars of the truss.

At each joint of the truss, it is possible to write the following equations:

$$\sum F_x = 0 \qquad [1.3a]$$

$$\sum F_y = 0 \qquad [1.3b]$$

So, the total number of independent equations is $2n$.

We define the degree of internal static indeterminacy by

$$f = (b + r) - 2n \qquad [1.4]$$

EXAMPLE 1.1.–

Calculate the degree of internal static indeterminacy of the structure (Figure 1.5).

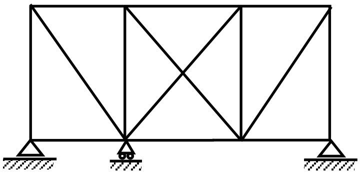


Figure 1.5. *Given truss*

Applying relationship [1.4] allows us to calculate the degree of static indeterminacy.

$r = 5$, $b = 14$, $n = 8$

$f = (14 + 5) - 2 \times 8 = 3$

The given structure is 3 times statically indeterminate internally.

In the case of a space truss, the equations of static are written as

$$\sum F_x = 0 \qquad [1.5a]$$

$$\sum F_y = 0 \qquad [1.5b]$$

$$\sum F_z = 0 \qquad [1.5c]$$

In the relationship [1.4], we can calculate the degree of internal static indeterminacy by

$$f = (b + r) - 3n \qquad [1.6]$$

EXAMPLE 1.2.–

Calculate the degree of internal static indeterminacy of the truss (Figure 1.6).

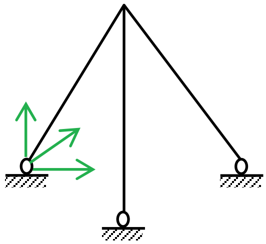


Figure 1.6. *Space truss*

$n = 4$, $b = 3$, $r = 9$

$f = (3 + 9) - 3.4 = 0$

The structure is statically determinate internally.

EXAMPLE 1.3.–

Calculate the degree of internal static indeterminacy of the structure (Figure 1.7).

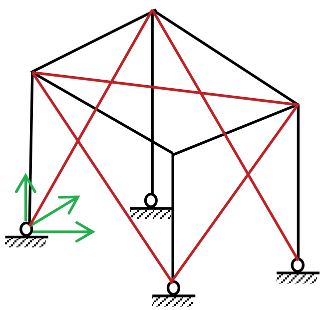


Figure 1.7. *Space truss*

$n = 8, b = 13, \; r = 12$

$f = (12+13) - 3.8 = 1$

The structure is once statically indeterminate internally.

1.3.2. *Beam and frame structures*

Relationships [1.4] and [1.6] can be applied to frames and beams with rigid joints to calculate the degree of internal static indeterminacy. For each rigid joint, it is possible to write three equations:

$$\sum F_x = 0 \qquad [1.7a]$$

$$\sum F_y = 0 \qquad [1.7b]$$

$$\sum M_i = 0 \qquad [1.7c]$$

Note that each end of the bar on a beam or a frame has three unknowns. So we define the degree of internal static indeterminacy by

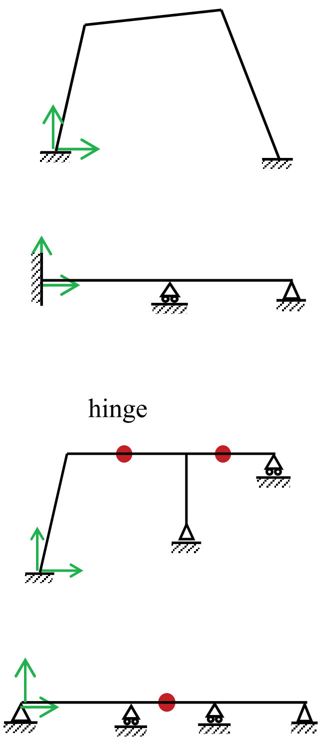
$$f = (3b + r) - 3n \qquad [1.8]$$

where n is the number of rigid joints including the support joints. If the frame or beam contains k hinges, the relationship [1.6] is written as

$$f = (3b + r) - (3n + k) \qquad [1.9]$$

EXAMPLE 1.4.–

Determine the degree of internal static indeterminacy of the structures (Figure 1.8).



b = 3, r = 6, n = 4
f = (3.3 + 6) - 3.4 = 3

b = 2, r = 6, n = 3
f = (3.2 + 6) - 3.3 = 3

$b = 4,\ r = 6,\ n = 5,\ k = 2$
$f = (3.4 + 6) – (3.5 + 2) = 1$

$b = 3,\ r = 6,\ n = 4,\ k = 1$
$f = (3.3 + 6) – (3.4 + 1) = 2$

Figure 1.8. *Static indeterminacy of frames and beams*

For space structures, we can write six equilibrium equations per joint and each bar has six unknowns. The degree of internal static indeterminacy can be deduced by

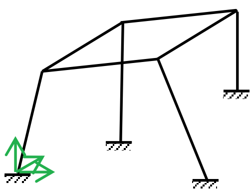
$$f = (6b + r) - 6n \qquad [1.10]$$

If the frame or beam contains k hinges, the degree of internal static indeterminacy is written as

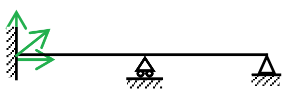
$$f=(6b+r)-(6n+k) \qquad [1.11]$$

EXAMPLE 1.5.–

Determine the degree of internal static indeterminacy of the structures (Figure 1.9).



$b = 8$, $r = 24$, $n = 8$
$f = (6.8 + 24) – 6.8 = 24$



$b = 2$, $r = 9$, $n = 3$
$f = (6.2 + 9) – 6.3 = 3$

Figure 1.9. *Static indeterminacy of space beams and frames*

1.3.3. *Crossbeams*

There is a layer of orthogonal beams linked together at the levels of the rigid joints. At each joint, we can write the following three equations:

$$\sum F_z = 0 \qquad [1.12a]$$

$$\sum M_x = 0 \qquad [1.12b]$$

$$\sum M_z = 0 \qquad [1.12c]$$

At each end of a bar, we consider a vertical force along the axis (zz), bending and torsion moments (M_{xx}, M_{yy}, M_{xy}) (Figure 1.10).

The internal forces at any section can be determined when three out of six actions of a beam element are known. Therefore, each member presents three unknowns and the degree of internal static indeterminacy is obtained by

$$f = (3b + r) - 3n \quad [1.13]$$

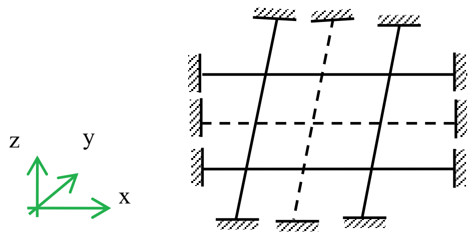


Figure 1.10. *Crossbeams*

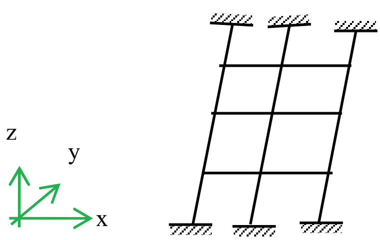
Especially when the links between the bars are joints, the degree of internal static indeterminacy becomes

$$f = (2b + r) - (3n + 2n^*) \quad [1.14]$$

where n^* is the number of articulated joints and n is the number of joint supports.

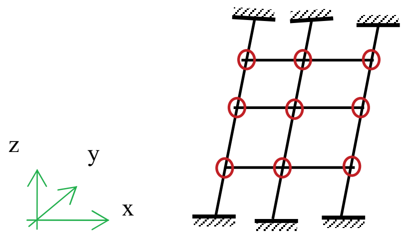
EXAMPLE 1.6.–

Determine the degree of static indeterminacy of the crossbeams (Figure 1.11).



$b = 18$, $r = 18$, $n = 15$

$f = (3.18 + 18) - 3.15 = 27$



$b = 18, r = 18, n = 6, n^* = 9$
$f = (3.18 + 18) - (3.6 + 2.9) = 36$

Figure 1.11. *Crossbeam layer*

1.4. Kinematic static indeterminacy

When a structure is stressed by one or more external actions, the joints of this structure undergo deflections. The method of displacements advocates rotations and displacements of the joints of the structure as unknowns of the problem.

The continuous beam (Figure 1.12), for example, is fixed at A and simply supported at B and C. It is solicited by a load q(x), which causes a deflection. Fixing A prevents rotation and displacement while supports B and C do not come from displacements following the vertical direction by allowing rotations ω_B and ω_C. In the case of vertical loads, there aren't longitudinal forces; for this reason, the horizontal displacements are neglected.

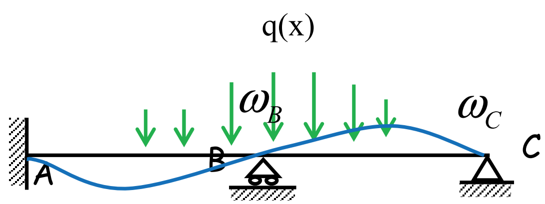
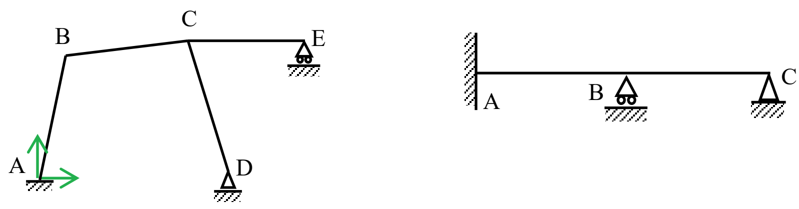


Figure 1.12. *Deflection of beam (AC)*

The unknowns of the problem are effectively independent rotations ω_B and ω_C. The number of independent deflections is called the degree of kinematic static indeterminacy or the number of active degrees of freedom. It encompasses all displacements and rotations of movable joints. The determination of the degree of kinematic static indeterminacy is briefly established in the following examples.

EXAMPLE 1.7.–

Determine the number of degrees of kinematic static indeterminacy of the structures (Figure 1.13).



(a) (b)

Figure 1.13. *Given structures*

The structure (Figure 1.13(a)) is isolated from the external environment and we deduce the vector components from the kinematic degrees of static indeterminacy.

$$\left\{q_0\right\}=\left\langle u_i \;\; v_i \;\; \omega_i \right\rangle^t \; : i = A, B, C, D \qquad [1.15]$$

Linking the structure to the supports makes it possible to quantify a few degrees of freedom (Figure 1.14). The vector grouping these degrees of freedom is called the passive degree of freedom vector, which is

$$\left\{q_p\right\}=\left\langle u_A \; v_A \; \omega_A, u_D \; v_D, v_E \right\rangle^t \qquad [1.16]$$

Note that any component of the passive degree of freedom vector is zero.

Using relationships [1.15] and [1.16], we deduce the vector of active degrees of freedom by

$$\left\{q_a\right\}=\left\langle u_B \; v_B \; \omega_B, u_C \; v_C \; \omega_C, \omega_D, u_E \; \omega_E \right\rangle^t \qquad [1.17]$$

The degrees of freedom vector of the structure (Figure 1.13(b)) is given by

$$\left\{q_0\right\}=\left\langle u_i \;\; v_i \;\; \omega_i \right\rangle^t \; : i = A, B, C$$

The passive degrees of freedom vector is given as

$$\{q_p\} = \langle u_A \; v_A \; \omega_A, v_B, u_C v_C \rangle^t$$

The active degrees of freedom vector is deduced by

$$\{q_a\} = \langle u_B \; \omega_B, \omega_C \rangle^t$$

For strusses, the forces are applied to the joints and each bar is subjected to a normal force. The joints undergo translations according to the orthogonal directions and this allows us to work out the structural deformation.

EXAMPLE 1.8.–

Determine the active and passive degrees of freedom of the truss structure (Figure 1.14).

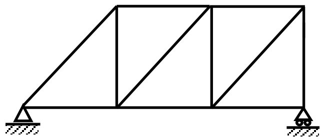


Figure 1.14. *Plane truss*

We isolate the structure from its supports (Figure 1.15).

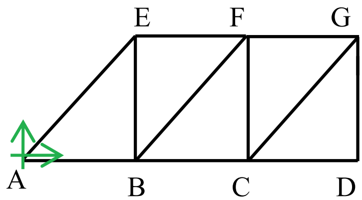


Figure 1.15. *Free-body structure*

Each joint of the structure (Figure 1.15) can experience displacements along the axes (*xx*) and (*yy*).

$$\{q_0\} = \langle u_i \;\; v_i \rangle^t \; : i = A, B, C, D, E, F, G$$

The passive degrees of freedom vector is given as

$$\{q_p\} = \langle u_A \ v_A, v_D \rangle^t$$

Hence, the active degrees of freedom vector is given as

$$\{q_a\} = \langle u_B \ v_B, u_c \ v_c, u_D, u_E \ v_E, u_F \ v_F, u_G \ v_G \rangle^t$$

EXAMPLE 1.9.–

The truss structure (Figure 1.16) is three-dimensional. Each joint moves in the three orthogonal directions. So each joint has 3 degrees of freedom.

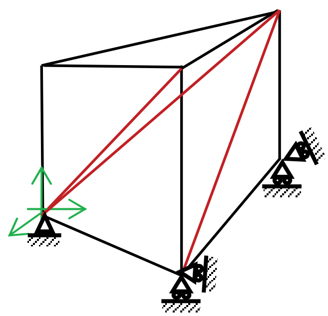


Figure 1.16. *Space truss*

Each joint of the free-body structure (Figure 1.17) can undergo three displacements along the orthogonal axes. The degrees of freedom vector of the structure is given as

$$\{q_0\} = \langle u_i \ v_i \ w_i \rangle^t \ : i = A, B, C, D, E, F, G \qquad [1.18]$$

The passive degrees of freedom vector is given as

$$\{q_p\} = \langle u_A \ v_A \ w_A, v_B \ w_B, u_C \ w_C \rangle^t$$

Hence, the active degrees of freedom vector can be deduced as

$$\left\{q_a\right\} = \left\langle u_B, v_c, u_D\ v_D\ w_D, u_E\ v_E\ w_E, u_F\ v_F\ w_F \right\rangle^t$$

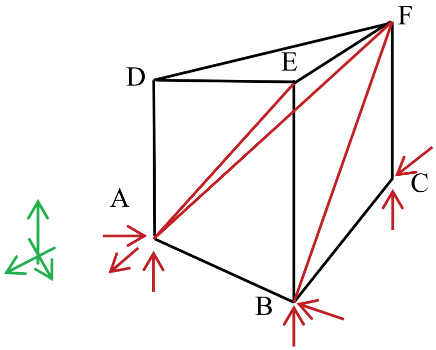


Figure 1.17. *Free-body structure*

1.5. Statically indeterminate structural analysis methods

Since the mid-1800s, several methods have been developed to analyze statically indeterminate structures. The static equations are insufficient to analyze this type of structure. In order to do so, it is necessary to use a statically indeterminate structural analysis method, regrouped in two categories: (1) the method of forces or flexibility and (2) the method of displacements or rigidity.

In the method of forces, the number of external actions must be eliminated, making the fundamental (statically determinate) structure equal to the degree of system static indeterminacy. The obtained structure is called a free-body structure, which undergoes inconsistent deflections at the site of the eliminated actions. To correct these deflections, we must apply unit actions instead of ignored forces. For example, in the method of forces, the solution is obtained by constructing a system of equations whose number is equal to the degree of static indeterminacy. In this case, the unknowns of the problem are the redundant actions and the final solution is obtained by a superposition of the effects of the eliminated actions.

In the displacement method, the fictitious supports must be introduced to prevent the joints from having a freedom to move. Then, we can calculate their corresponding support reactions. This allows us to calculate the displacements at the level of the fictitious supports, which create actions at the ends of each bar. Lastly, we use the superposition principle to calculate the final actions at the ends of each bar.

The unknowns of this method are displacements. In this case, the number of fictitious supports added must be equal to the number of possible displacements of the structure. The bases and principles of the methods for analyzing statically indeterminate structures are described in detail in Chapters 2–6.

1.6. Superposition principle

When the deflections of a structure are proportional to the actions which generate them, the superposition principle can be applied. Note that all actions applied to point i generate deflections at point j; force F_i creates a displacement δ_{ji} at point j (moment M_i generates a rotation ω_{ji} at point j). In particular, force F_i (moment M_i) creates a displacement δ_{ii} (a rotation ω_{ii}) at the point of application i.

We first apply force F_i to beam (ABCD). This force generates a deflection field of the beam including displacements δ_{ii} and δ_{ji}, at joints i and j, respectively. In the same way, we apply force F_j to point j that generates displacements δ_{ij} and δ_{jj}, respectively, at joints i and j (Figure 1.18). This approach is clarified in Figure 1.18. The relationship between the applied force and the resulting displacement is given as

$$\delta_{ii} = f_{ii} F_i \qquad [1.19]$$

where f_{ii} is the displacement at point i when a unit force is applied at the same point in the direction of force F_i. We apply force F_j at point j that generates a displacement δ_{ij} at point i.

$$\delta_{ij} = f_{ij} F_j \qquad [1.20]$$

If both forces are simultaneously applied, the displacement δ_i at point i is given as

$$\delta_i = f_{ii} F_i + f_{ij} F_j \qquad [1.21]$$

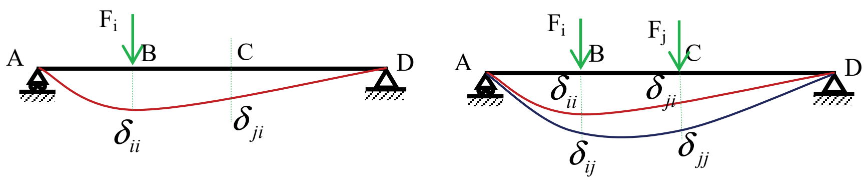


Figure 1.18. *Superposition of displacements and forces*

The relationship [1.21] can be generalized for several simultaneously applied forces. The total displacement at point i can be evaluated by

$$\delta_i = f_{i1} F_1 + f_{i2} F_2 + f_{i3} F_3 + \ldots\ldots\ldots + f_{in} F_n \qquad [1.22]$$

In matrix form, relationship [1.22] is written as

$$\delta_i = \left[f_{ij} \right] \left\{ F_j \right\} \qquad [1.23]$$

In general, all actions A_i can be a support reaction, a bending moment, a shear stress or a normal force; due to the effect of several external actions, they can be obtained by a superposition of effects.

$$A_i = \left[A_{uij} \right] \left\{ F_j \right\} \qquad [1.24]$$

where A_{uij} is the intensity of the action at point i when a unit action is applied at joint j.

The relationship [1.24] evaluates the magnitude of the action of a statically indeterminate structure by using the principle of the superposition of effects of this action of statically determinate systems due to the effect of unit actions (see Chapter 3 for more details).

1.7. Advantages and disadvantages of statically indeterminate structures

Obviously, statically indeterminate structures have advantages and disadvantages, which are shown in the following section.

1.7.1. Advantages of statically indeterminate structures

Some of the advantages of statically indeterminate structures are as follows:

– *Rigidity*

Statically indeterminate structures are known for their rigidities (small deflections) compared to statically determinate structures (Figure 1.19).

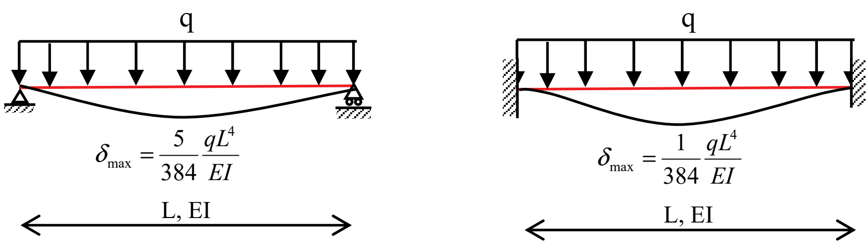


Figure 1.19. *Maximum displacements*

Moreover, we can see that the maximum displacement of a statically indeterminate structure is $\frac{1}{5}$ compared to a statically determinate structure.

– *Stress*

Maximum normal stresses in statically indeterminate structures are smaller than those generated in statically determinate structures by keeping the same geometry and applied loading (Figure 1.20). This figure shows that the maximum moment of the statically indeterminate beam is weaker than that of the statically determinate beam.

– *Distribution of actions*

Statically indeterminate structures have a great ability to distribute actions such as the slope-deflection method (Chapter 4) and the moment-distribution method (Chapter 5). This property has not been considered during the analysis of statically determinate structures.

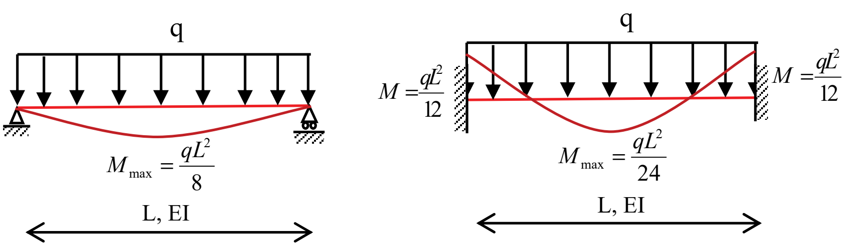


Figure 1.20. *Maximum bending moment*

The advantage of the distribution of actions is observed when a portion of a structure undergoes a localization of stresses or damage due to an earthquake, an impact force, a crack, an explosion or another force. In addition, statically indeterminate structures contain several structural elements and supports that make the structure more rigid, which statically determinate structures do not have.

Furthermore, if a bar or support deteriorates, the statically indeterminate structure will not necessarily be damaged and the load can be distributed to adjacent elements. The destruction or elimination, for example, of the interior support of statically indeterminate and statically determinate beams, damages the statically determinate beam, while the statically indeterminate beam becomes a statically determinate beam (Figure 1.21). In this case, the damage is an elimination or destruction of the interior support.

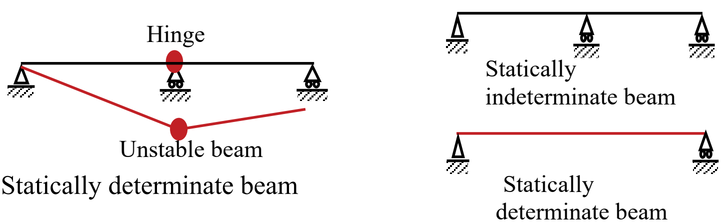


Figure 1.21. *Destruction of the interior support*

1.7.2. *Disadvantages of statically indeterminate structures*

The disadvantages of statically indeterminate structures compared to statically determinate structures are as follows:

– *Support settlements*

A support settlement does not generate stresses in statically determinate structures, but it does in statically indeterminate structures. For this reason, it is necessary to take this effect into account during the design phase.

To clarify this phenomenon, consider a support settlement of a statically determinate beam with a hinge and another statically indeterminate beam (see Figure 1.22). The members of the statically determinate beam undergo the movement of the rigid body because of the presence of the hinge, whereas the support settlement of a statically indeterminate beam generates additional moments to those of the mechanical loads at the ends of each bar (Chapters 4 and 5).

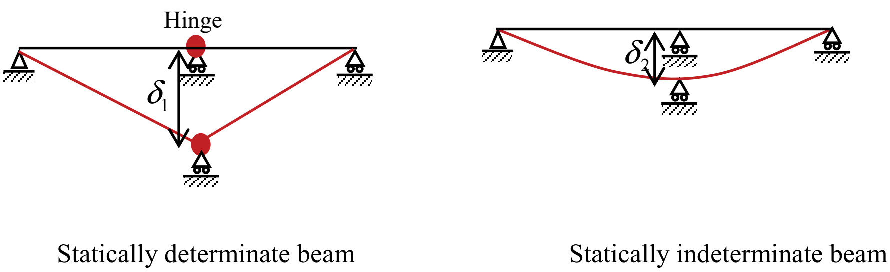


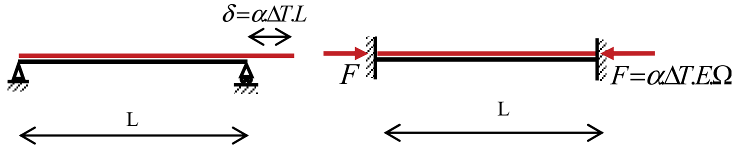
Figure 1.22. *Support settlement*

– *Temperature change and manufacturing errors*

Identical to the effect of the support settlement, the effect of temperature or a manufacturing error does not cause any additional stress in statically determinate structures but can generate significant stresses in statically indeterminate structures.

When a statically determinate beam is subjected to a temperature change ΔT, it lengthens by $\delta = \alpha.\Delta T.L$ (α coefficient of thermal expansion) and no stress can develop inside the beam because of the freedom of movement given by the hinge (Figure 1.23).

However, the statically indeterminate beam is impeded at its ends vis-à-vis the axial deflection. The temperature effect creates an axial force of intensity equal to $\alpha.\Delta T.E.\Omega$. The effect of a manufacturing error is very similar to that of the temperature change.



Statically determinate structure

Statically indeterminate structure

Figure 1.23. *Temperature effect*

1.8. Conclusion

In this chapter, we first presented in detail the external, internal and kinematic static indeterminacy of statically indeterminate structures. The static indeterminacy of a structure plays a very important role in choosing the appropriate method of analysis.

In the second part, we illustrated statically indeterminate structural analysis methods that are classified in two categories: (1) the method of forces or flexibility and (2) the method of displacements or rigidity.

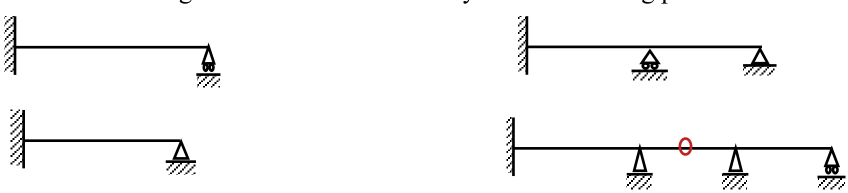
As it is important and commonly used in the analysis of statically indeterminate structures, the principle of superposition of effects is described.

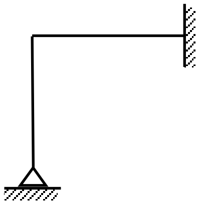
Finally, at the end of this chapter, we have illustrated the advantages and disadvantages of statically indeterminate structures compared to statically determinate structures. In this, several parameters are cited showing their impacts on statically indeterminate and statically determinate structures.

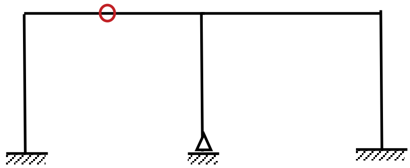
1.9. Problems

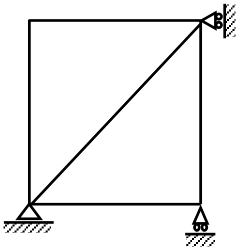
Exercise 1

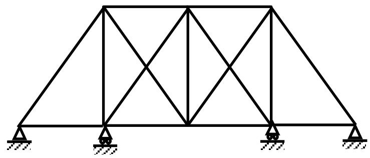
Determine the degree of static indeterminacy of the following plane structures:





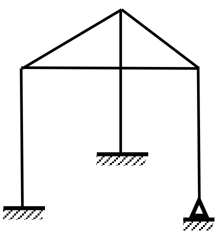


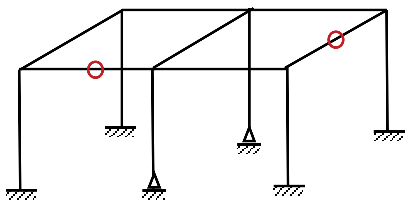


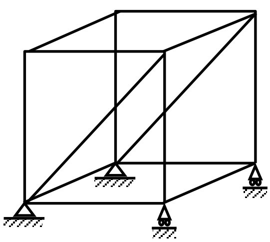


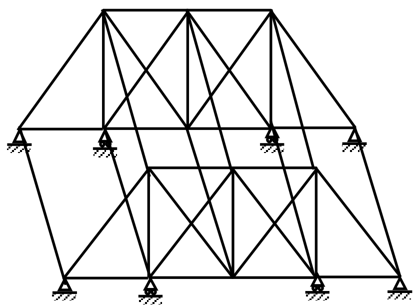
Exercise 2

Determine the degree of static indeterminacy of the following plane structures:



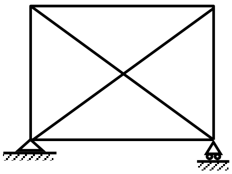
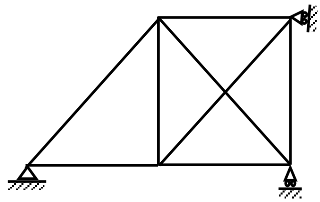
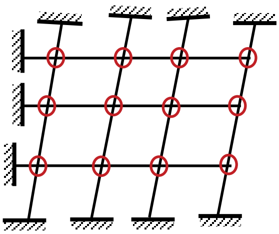
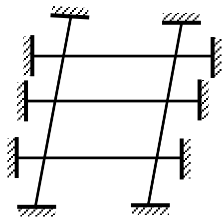
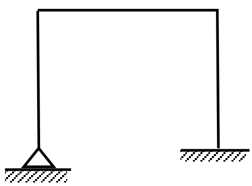
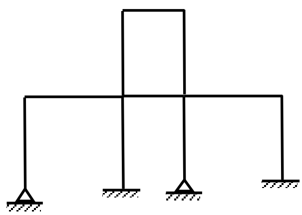






Exercise 3

Determine the degrees of internal and external static indeterminacy of the following plane structures:

Exercise 4

Determine the degree of internal and external static indeterminacy of the space structures given in Exercise 2.

Exercise 5

Give the vector of the degrees of freedom of the free-body structure, the vector of the passive degrees of freedom and the vector of the active degrees of freedom of the structures in Exercise 1.

Exercise 6

Give the vector of the degrees of freedom of the free-body structure, the vector of the passive degrees of freedom and the vector of the active degrees of freedom of the structures in Exercise 2.