1

The Poisson Process

Living without memory is an unforgettable experience.

Yolande Villemaire (1949–)

It is with this citation by the Quebecois novelist and poet Yollande Villemaire that we begin this chapter on the memoryless characteristics of the exponential function. We review this distribution, which is a probability tool frequently used in the study of queues. Using properties analogous to the exponential function, we can then introduce the Poisson[1] process in this chapter, describing the moments when random events occur, such as the arrival of customers in a queue.

1.1. Review of the exponential distribution

1.1.1. *Definitions*

DEFINITION 1.1.– *A real positive random variable X follows an exponential distribution with parameter $\lambda > 0$ if:*

$$\mathbb{P}(X > t) = e^{-\lambda t},\ \forall t \in \mathbb{R}^{+} \qquad [1.1]$$

The notation $\mathbb{P}$ indicates the probability function. In equation [1.1], the expression $\mathbb{P}(X > t)$ therefore designates the probability that the random variable X has a value that is superior to a positive real number t.

Just like any real random variable, it can be characterized by a probability density function.

1 Siméon Denis Poisson (1781–1840), a French mathematician, geometrician and physician.

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Its density function is given by:

$$f_X(t) = \lambda e^{-\lambda t},\ \forall t \in \mathbb{R}^+ \qquad [1.2]$$

From the density function, we can also express the expected value [1.3] and the variance [1.4] of the random variable X following an exponential distribution with parameter λ:

$$E(X) = \int_0^{+\infty} t f_X(t)\, dt = \frac{1}{\lambda} \qquad [1.3]$$

$$\sigma^2(X) = \int_0^{+\infty} \left(t - \frac{1}{\lambda}\right)^2 f_X(t)\, dt\ = \frac{1}{\lambda^2} \qquad [1.4]$$

The exponential distribution is used in many applications in physics, biology, computer programing and, of course, telecommunications.

EXAMPLE.– In radioactivity, the lifetime of a radioactive particle can be represented by a random variable following an exponential distribution. The parameter for this distribution is the speed λ at which the particle ages (see Exercise 1.1).

It is for this reason that the distribution is also called the ageless lifespan distribution. The term "ageless" is explained in section 1.1.2.

Its average lifespan is the expected value of the random variable for the lifespan: $\frac{1}{\lambda}$.

Its half-life period is the instant $T = \ln(2)\ /\lambda$ with $\mathbb{P}(X > T) = 1/2$.

1.1.2. *The properties of an exponential distribution*

1.1.2.1. *The memoryless distribution*

PROPOSITION 1.1.– *The exponential distribution has no memory.*

In games of chance, the number of attempts needed to win does not depend on the number of attempts already made. This property is called *the memoryless property* or *the amnesia property*.

The exponential distribution also satisfies this property, which is expressed by:

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t), \qquad \forall s, t \in \mathbb{R}^+ \qquad [1.5]$$

PROOF.–

By using Bayes'[2] theorem for any random variable X, we can write for all positive, real numbers s and t:

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(X > s + t \text{ and } X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} = \frac{F_X^C(s + t)}{F_X^C(s)}$$

where $F_X^C(t) = \mathbb{P}(X > t)$ designates the complementary cumulative distribution function for the random variable X.

Equation [1.5] can thus be expressed as:

$$F_X^C(s + t) = F_X^C(s) F_X^C(t), \qquad \forall s, t \in \mathbb{R}^+ \qquad [1.6]$$

Since F_X^C is a complementary cumulative distribution function, it is monotone and bounded, and given that $F_X^C(t) \neq 1 \ \forall t \in \mathbb{R}^+$, then F_X^C is necessarily an exponential function. The constant k therefore exists such that:

$$F_X^C(t) = e^{kt}, \qquad \forall t \in \mathbb{R}^+ \qquad [1.7]$$

Now, $F_X^C < 1$, so k must be a negative constant, which we denote by $k = -\lambda$, $\lambda > 0$. We find that the complementary cumulative distribution function of a real, random variable follows the exponential distribution of the parameter λ. ■

If the lifespan X of an object follows an exponential distribution, the probability that this object will last at least $s + t$ units of time, keeping in mind that it has already lasted s units of time, is the same as the probability of lasting t units of time from its birth. In this case, we say that its lifespan is *memoryless* or *ageless*.

1.1.2.2. *The minimum of independent exponential distributions*

Let there be n independent random variables $X_1, X_2, \ldots$ and X_n following exponential distributions with the respective parameters $\lambda_1, \lambda_2, \ldots$ and λ_n. We denote the sum of these parameters with λ.

PROPOSITION 1.2.– *The random variable X, equal to the minimum of $X_1, X_2, \ldots, X_n$, follows an exponential distribution with parameter λ.*

2 Thomas Bayes (1702–1761), British mathematician and pastor of a Presbyterian Church.

PROPOSITION 1.3.– *The random variable X is equal to X_i ($i \in \{1, 2, \dots, n\}$) with a probability λ_i/λ, independently of the value of X.*

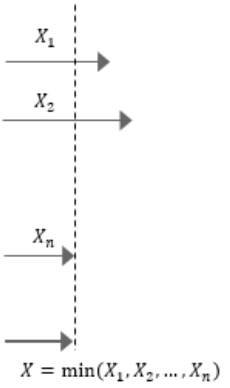


Figure 1.1. *Minimum of exponential distributions*

PROOF.–

The minimum $X = \min(X_1, X_2, \dots, X_n)$ of these random variables satisfies for any real number $t > 0$ (keeping in mind that the random variables X_i are independent of each other):

$$\begin{aligned}\mathbb{P}(X > t) &= \mathbb{P}(X_1 > t, X_2 \geq t, \dots, X_n \geq t) \\ &= \mathbb{P}(X_1 > t).\mathbb{P}(X_2 \geq t) \dots \mathbb{P}(X_n > t) \\ &= e^{-\lambda_1 t}.e^{-\lambda_2 t} \dots e^{-\lambda_n t} = e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \\ &= e^{-\lambda t}\end{aligned}$$

For X to be equal to X_i:

$$\mathbb{P}(X > t, X = X_i) = \mathbb{P}(X_i > t, X_1 \geq X_i, X_2 \geq X_i, \dots, X_n \geq X_i)$$

$$\begin{aligned}&= \int_t^{+\infty} \lambda_i e^{-\lambda_i s}.e^{-\lambda_1 s}e^{-\lambda_2 s} \dots e^{-\lambda_n s}\, ds \\ &= \int_t^{+\infty} \lambda_i e^{-(\lambda_1+\lambda_2+\cdots+\lambda_i+\cdots+\lambda_n)s}\, ds \\ &= \frac{\lambda_i e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)t}}{\lambda_1+\lambda_2+\cdots+\lambda_n} = \frac{\lambda_i}{\lambda}e^{-\lambda t} \quad \blacksquare\end{aligned}$$

The properties in Propositions 1.2 and 1.3 also apply in the case of an infinite family $\{X_n\}_{n\geq 1}$ of random variables following exponential distributions with the respective parameters from the infinite sequence $\{\lambda_n\}_{n\geq 1}$ when the sum of these parameters is finite:

$$\lambda = \sum_{n=1}^{+\infty} \lambda_n < \infty \qquad [1.8]$$

PROPOSITION 1.2b.– *The random variable* X*, equal to the minimum of* $X_1, X_2, \ldots$*, follows an exponential distribution with parameter* λ*.*

PROPOSITION 1.3b.– *The random variable* X *is equal to* X_i *(*$i \in \mathbb{N}^*$*) with a probability* λ_i/λ*, independently of the value of* X*.*

1.1.2.3. *The sum of independent exponential distributions*

Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically exponentially distributed random variables with parameter λ. We call the sum of the first *n* terms of this sequence $S_n = X_1 + X_2 + \cdots + X_n$. We posit that $S_0 = 0$.

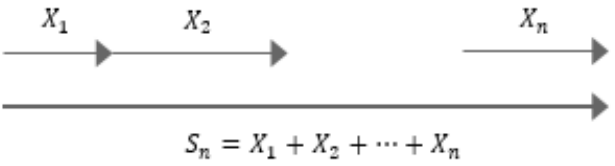


Figure 1.2. *Sum of exponential distributions*

PROPOSITION 1.4.– *The random variable* $S_n = X_1 + X_2 + \cdots + X_n$ *follows an Erlang*[3] *distribution with parameters (n, λ).*

PROOF.–

We call $N(t)$ *the integer-valued random variable equal to* n *satisfying* $S_n \leq t \leq S_{n+1}$ *for any real positive t. We first show that* $N(t)$ *follows the Poisson distribution with parameter* λt*:*

$$\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t \leq S_{n+1})$$

3 Agner Krarup Erlang (1878–1929), Danish mathematician.

$$= \int_{t_1+\cdots+t_n\le t} \left(\lambda e^{-\lambda t_1}\right)\left(\lambda e^{-\lambda t_2}\right)\ldots\left(\lambda e^{-\lambda t_n}\right).e^{-\lambda(t-t_1-\cdots-t_n)}\,dt_1\ldots dt_n$$

$$= \int_{t_1+\cdots+t_n\le t} \lambda^n e^{-\lambda(t_1+t_2+\cdots+t_n)}e^{-\lambda(t-t_1-\cdots-t_n)}\,dt_1\ldots dt_n$$

$$= \lambda^n e^{-\lambda t}\int_{t_1+\cdots+t_n\le t} dt_1\ldots dt_n = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

From this distribution, we can deduce the complementary cumulative distribution function $\mathbb{P}(S_n > t)$ for the random variable S_n:

$$\mathbb{P}(S_n > t) = \sum_{k=0}^{n-1}\mathbb{P}(N(t)=k) = e^{-\lambda t}\left(1+\lambda t+\cdots+\frac{(\lambda t)^{n-1}}{(n-1)!}\right)$$

$$= e^{-\lambda t}\sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}$$

Thus, the sum S_n follows an Erlang distribution with parameters (n,λ), which we denote by $\mathcal{E}(n,\lambda)$. ■

REMARKS.–

The Erlang distribution is a gamma distribution whose shape parameters are integers.

The density function of the Erlang distribution $\mathcal{E}(n,\lambda)$ is equal to:

$$f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!}e^{-\lambda x} \qquad [1.9]$$

1.1.2.4. *The geometric sum of exponential distributions*

Let $\{X_n\}_{n\ge1}$ be a sequence of i.i.d random variables following an exponential distribution with parameter λ. We denote by $S = X_1 + X_2 + \cdots + X_N$ the sum of the first N terms of this sequence such that N is an integer-valued random variable following a geometric distribution with parameter $p \in]0,1]$.

Thus, the random variable S is equal to:

– X_1 with a probability p;

– $X_1 + X_2$ with a probability $p(1-p)$;

– $X_1 + X_2 + \cdots + X_N$ with a probability $p(1-p)^{N-1}$ for any integer $N \in \mathbb{N}$.

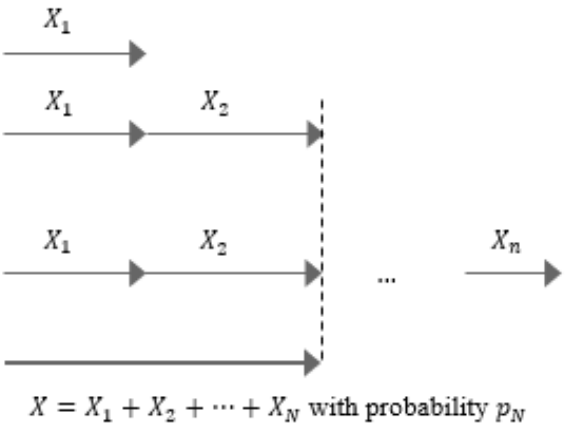


Figure 1.3. *Geometric sum of exponential distributions*

PROPOSITION 1.5.– *The random variable S follows an exponential distribution with parameter* $p\lambda$.

PROOF.–

We determine the complementary cumulative distribution function of the random variable S while simultaneously considering the random variable S_n *from Proposition 1.4*:

$$\begin{aligned}
\mathbb{P}(S > t) &= \sum_{n=1}^{+\infty} p(1-p)^{n-1}\mathbb{P}(S_n > t) \\
&= \sum_{n=1}^{+\infty} p(1-p)^{n-1} e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} \\
&= \sum_{n=1}^{+\infty} p(1-p)^{n-1} e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} \\
&= \sum_{k=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \sum_{n>k} p(1-p)^{n-1} \\
&= \sum_{k=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} (1-p)^k
\end{aligned}$$

$$= \sum_{k=0}^{+\infty} e^{-\lambda t} \frac{\left((1-p)\lambda t\right)^k}{k!}$$
$$= e^{-p\lambda t}, \quad \forall t \in \mathbb{R}^+ \qquad \blacksquare$$

1.2. Poisson process

1.2.1. *Definitions*

DEFINITION 1.2.– *A random process (or a stochastic process) is a representation of the evolution over time of a random variable.*

A random process is therefore a sequence of indexed random variables $X(t) \in \mathcal{X}$ over time $t \in \mathcal{T}$. We examine a particular case of the random process for $\mathcal{T} = \mathbb{N}^* = \{1,2,\ldots\}$: it is the punctual process for which we denote $X(t) = T_t$ for all $t \in \mathcal{T}$.

DEFINITION 1.3.– *A punctual process over $\mathbb{R}^+$ is a growing sequence $T_1, T_2, \ldots$ of real positive random variables typically representing instants when events occurs.*

EXAMPLE.– The arrival of customers in a queue is a punctual process. The instants of their arrivals, which are random variables, form a growing sequence.

The random variables $T_1, T_2, \ldots$ are called *the instants of appearance of the process*.

We denote by τ_n $(n \geq 1)$ the inter-arrival times of the process: $\tau_n = T_n - T_{n-1}$, with $T_0 = 0$.

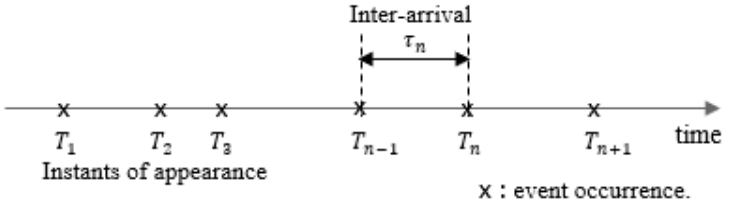


Figure 1.4. *Punctual process*

DEFINITION 1.4.– *A punctual process is called simple if the inter-arrivals are almost surely (AS) strictly positive.*

DEFINITION 1.5.– *A punctual process is called stationary if, for any* $n \geq 0$*, the random variables of inter-arrivals* $\{\tau_n\}_{n\geq 1}$ *follow the same distribution.*

In the case of a stationary punctual process, the inter-arrivals therefore have the same expected value: the average of the inter-arrivals. The inverse of this average is called the *intensity of the process,* denoted λ:

$$\lambda = \frac{1}{E(\tau_n)} = \frac{1}{E(\tau_1)} \qquad [1.10]$$

DEFINITION 1.6.– *When the times of inter-arrivals form a sequence of independent random variables following exponential distributions with parameter* $\lambda > 0$*, we say that the process is a Poisson process of intensity* λ*.*

PROPOSITION 1.6.– *A Poisson process is a simple and stationary punctual process.*

PROOF.–

This property derives directly from the definition of the Poisson process. ■

The counting measure N of a punctual process counts the number of appearances (or the number of arrivals) of the process in an interval of time (s,t):

$$N(s,t) = \sum_{n=1}^{+\infty} 1_{(s,t)}(T_n), \quad \forall s,t \in \mathbb{R}^+, s < t \qquad [1.11]$$

where $1_{(s,t)}(x)$ indicates the indicator function of the interval (s,t) at point x: it equals 1 if $x \in (s,t)$ and 0 if $x \notin (s,t)$.

We can also use the notation $N(t)$ to count the number of appearances of the process up to the instant t: $N(t) = N(0,t)$.

This counting measure also allows us to describe the punctual process:

– the process is simple if the probability that $N(s,t) \leq 1$ given that $N(s,t) \geq 1$ tends towards 1 when s tends towards t;

– the process is stationary if the random variable $N(s,t)$ has the same distribution as $N(0,t-s)$ for any interval (s,t). The intensity of the process is therefore given by the equation:

$$\lambda = E(N(0,1)) \qquad [1.12]$$

– a Poisson process of intensity $\lambda > 0$ is a punctual process such that the numbers of appearances of the process for all discontinuous intervals are independent and the number of appearances of the process for any interval (s,t) follows a Poisson distribution with parameter $\lambda(t-s)$:

$$\mathbb{P}(N(s,t)=n)=\frac{\big(\lambda(t-s)\big)^n}{n!}e^{-\lambda(t-s)},\quad \forall n\in\mathbb{N} \qquad [1.13]$$

REMARKS.–

From equation [1.12], we can say that the intensity λ of the process corresponds to the average number of appearances of the process per unit of time. This is why we also call it the arrival rate of the process.

Following from the stationariness of any process, the number of appearances in the intervals $(0,1),(1,2),\dots,(n,n+1),\dots$ are i.i.d random variables with the average λ. From the law of large numbers, we obtain:

$$\lim_{t\to\infty}\frac{1}{t}N(t)=E\big(N(0,1)\big)=\lambda \quad A.S.$$

From Proposition 1.4, the instant of the nth appearance of the Poisson process of intensity λ is a random variable following the Erlang distribution $\mathcal{E}(n,\lambda)$.

1.2.2. *Properties of the Poisson process*

1.2.2.1. *Memoryless process*

PROPOSITION 1.7.– *A Poisson process is memoryless.*

Saying that the Poisson process is memoryless means that knowledge about the process up to time t says nothing about its future evolution.

PROOF.–

We denote as $N(t)=n$ the number of appearances of the process up to time t, and we determine the probability of k appearances in an interval $(t,s+t)$:

$$\mathbb{P}(N(t,s+t)=k|N(t)=n)=\frac{\mathbb{P}(N(t,s+t)=k\ \ and\ \ N(t)=n)}{\mathbb{P}(N(t)=n)}$$

Independently of $N(t, s+t)$ and $N(t)$, we have:

$$\mathbb{P}(N(t,s+t)=k|N(t)=n)=\frac{\mathbb{P}(N(t,s+t)=k).\mathbb{P}(N(t)=n)}{\mathbb{P}(N(t)=n)}$$

Through the stationariness of the Poisson process, $\mathbb{P}(N(t,s+t)=k)=\mathbb{P}(N(s)=k)$.

Thus,

$$\mathbb{P}(N(t,s+t)=k|N(t)=n)=\mathbb{P}(N(s)=k)=\frac{(\lambda s)^k}{k!}e^{-\lambda s}$$

The obtained probability depends on neither the time t nor the number n of appearances of the process up to time t. Knowledge of the process up to time t says nothing about its future evolution; hence, its amnesia property.■

REMARK.– The memoryless property distinguishes the Poisson process among simple stationary punctual processes.

1.2.2.2. *Distribution of points in a Poisson process*

PROPOSITION 1.8.– *Given that an interval (s,t) of $\mathbb{R}^+$ contains n points of a Poisson process, these points are uniformly distributed in this interval.*

PROOF.–

For any integer $l \geq 1$, any division of the interval (s,t) in l intervals $(s,t_1),(t_1,t_2),\ldots,(t_{l-1},t)$ and all integers $n_1,n_2,\ldots,n_l$ with a sum equal to n, we have:

$$\mathbb{P}(N(s,t_1)=n_1,\ldots,N(t_{l-1},t)=n_l \mid N(s,t)=n)$$
$$=\frac{\mathbb{P}(N(s,t_1)=n_1,\ldots,N(t_{l-1},t)=n_l)}{\mathbb{P}(N(s,t)=n)}$$

$$=\frac{e^{-\lambda(t_1-s)}\frac{\left(\lambda(t_1-s)\right)^{n_1}}{n_1!}\ldots e^{-\lambda(t-t_{l-1})}\frac{\left(\lambda(t-t_{l-1})\right)^{n_l}}{n_l!}}{e^{-\lambda(t-s)}\frac{\left(\lambda(t-s)\right)^{n}}{n!}}$$

$$=\binom{n}{n_1,\ldots,n_l}\left(\frac{t_1-s}{t-s}\right)^{n_1}\ldots\left(\frac{t-t_{l-1}}{t-s}\right)^{n_l}$$

The probability that any one of the l intervals contains one of the n points of the Poisson process is therefore proportional to its length: each point is distributed uniformly across the interval (s,t), independently of the other points.■

1.2.2.3. *Superposition of the Poisson process*

PROPOSITION 1.9. (Palm[4] theorem).– *The superposition of K independent Poisson processes of intensity $\lambda_1, \dots, \lambda_K$ is a Poisson process of intensity $\lambda = \lambda_1 + \cdots + \lambda_K$.*

PROOF.–

We denote by $N_1, N_2, \dots, N_K$ the counting measures of K initial Poisson processes, and we determine the counting measure N of the superposed process:

$$\mathbb{P}(N(s,t) = n) = \sum_{\substack{n_1,\dots,n_K \\ n_1+\cdots+n_K=n}} \mathbb{P}(N_1(s,t) = n_1, \dots, N_K(s,t) = n_K)$$

The initial Poisson processes are independent. Since a Poisson process is simple, the initial Poisson processes AS have nothing in common. Thus, we can write:

$$\mathbb{P}(N(s,t) = n) = \sum_{\substack{n_1,\dots,n_K \\ n_1+\cdots+n_K=n}} \mathbb{P}(N_1(s,t) = n_1) \dots \mathbb{P}(N_K(s,t) = n_K)$$

$$= \sum_{\substack{n_1,\dots,n_K \\ n_1+\cdots+n_K=n}} \mathbb{P}(N_1(s,t) = n_1) \dots \mathbb{P}(N_K(s,t) = n_K)$$

$$= \sum_{\substack{n_1,\dots,n_K \\ n_1+\cdots+n_K=n}} \frac{\big(\lambda_1(t-s)\big)^{n_1}}{n_1!} e^{-\lambda_1(t-s)} \dots \frac{\big(\lambda_K(t-s)\big)^{n_K}}{n_K!} e^{-\lambda_K(t-s)}$$

$$= \sum_{\substack{n_1,\dots,n_K \\ n_1+\cdots+n_K=n}} \binom{n}{n_1, \dots, n_K} \big(\lambda_1(t-s)\big)^{n_1} \dots \big(\lambda_K(t-s)\big)^{n_K} e^{-\lambda(t-s)}$$

$$= \frac{\big(\lambda(t-s)\big)^n}{n!} e^{-\lambda(t-s)}$$

We thus find the counting measure of a Poisson process of intensity λ. ■

4 Conny Palm de Conrad (1907–1951), Swedish engineer, electrician and statistician.

Figure 1.5 is a representation of this superposition of two Poisson processes into one process.

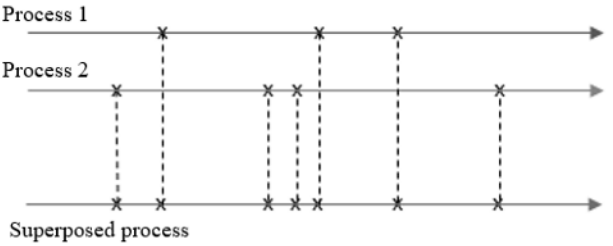


Figure 1.5. *Superposition of Poisson processes*

PROPOSITION 1.10.– *The probability that any one point of a superposed process belongs to one of the K initial processes is proportional to the intensity of this process.*

PROOF.–

$$\begin{aligned}&\mathbb{P}(N_i(s,t)=1 \mid N(s,t)=1)\\&=\frac{\mathbb{P}(N_i(s,t)=1, N_1(s,t)=0,\ldots,N_K(s,t)=0)}{\mathbb{P}(N(s,t)=1)}\\&=\frac{\lambda_i(t-s)e^{-\lambda(t-s)}}{\lambda(t-s)e^{-\lambda(t-s)}}=\frac{\lambda_i}{\lambda}\ \blacksquare\end{aligned} \qquad [1.14]$$

1.2.2.4. *Subdivision of a Poisson process*

Let N be the counting measure of a Poisson process of intensity λ and $p_1, \ldots, p_K$ K strictly positive real numbers with a sum equal to 1.

PROPOSITION 1.11. (Raikov[5] theorem).– *The K punctual processes obtained by attributing each point of the Poisson process to the* k*th punctual process of probability* p_k *are independent Poisson processes of respective intensities* $\lambda p_1, \ldots, \lambda p_K$.

5 Dmitrii Abramovich Raikov (1905), Russian professor with a PhD in mathematical physics.

PROOF.–

For any interval (s,t), the counting measures $N_1,\dots,N_K$ of K obtained punctual processes satisfy:

$$\begin{aligned}&\mathbb{P}(N_1(s,t)=n_1,\dots,N_K(s,t)=n_K)\\&=\binom{n}{n_1,\dots,n_K}p_1^{n_1}\dots p_K^{n_K}.\frac{\lambda^n(t-s)^n}{n!}e^{-\lambda(t-s)}\\&=\frac{\left(\lambda p_1(t-s)\right)^{n_1}}{n_1!}e^{-\lambda p_1(t-s)}\dots\frac{\left(\lambda p_K(t-s)\right)^{n_K}}{n_K!}e^{-\lambda p_K(t-s)}\end{aligned}\qquad[1.15]$$

with $n=n_1+\cdots+n_K$.■

Figure 1.6 represents this subdivision of the Poisson process.

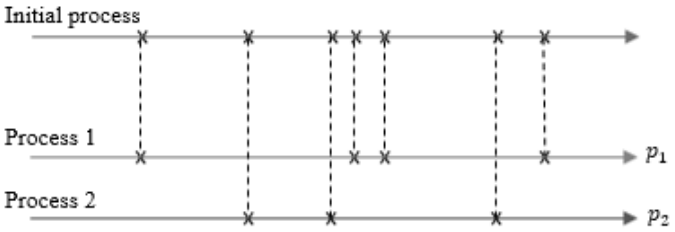


Figure 1.6. *Subdivision of a Poisson process*

1.3. Exercises

EXERCISE 1.1.– *Radioactive particle*

The lifetime of a radioactive particle follows an exponential distribution with parameter λ.

1) What is its lifetime?

2) Calculate the probability that the particle will still exist after its expected lifetime.

3) What is its expected lifetime given that the particle still exists after its expected lifetime?

EXERCISE 1.2.– *Discretization of exponential distribution*

Let X be a random exponential variable with parameter λ.

1) Show that the ceiling function of X follows a geometric distribution which we can determine.

2) Show that the floor function of X follows a geometric distribution which we can determine.

3) More generally, what is the distribution of an integer-valued random variable's ceiling function of X/τ for any real number > 0?

EXERCISE 1.3.– *Synchronization*

Let X_1 and X_2 be two independent exponentially distributed random variables with respective parameters λ_1 and λ_2.

Show that the variables X_1 and X_2 are equal with a probability of zero.

Directions: we can use $\mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$ for all events A_1 and A_2.

EXERCISE 1.4.– *"Very" random variable*

Let X be a real positive random variable with a probability density function $f_X(x)$. The entropy of the random variable X is equal to:

$$h(X) = -\int_0^{+\infty} f_X(t).\ln\big(f_X(t)\big)\,dt$$

1) Calculate the entropy of the random variable X_e following an exponential distribution with parameter λ.

2) Using the Gibbs[6] inequality for all probability density functions $f(x)$ and $g(x)$ over $\mathbb{R}^+$: $\int_0^{+\infty} f(t).\ln\big(f(t)/g(t)\big)\,dt \geq 0$, with an inequality if, and only if, $f = g$, show that, for any random variable X with a probability density $f_X(x)$ and an average $1/\lambda$, we get:

$$h(X) \leq 1 - \ln\lambda$$

and that we have inequality if, and only if, X is a random variable following an exponential distribution with parameter λ.

6 Willard Gibbs (1839–1903), American physicist and mathematician, professor of mathematical physics.

3) Deduce from this that the exponential distribution is the distribution of maximum entropy among all distributions with the same average and density over $\mathbb{R}^+$.

EXERCISE 1.5.– *Call expiry*

A personal call and a professional call have durations following exponential distributions with respective averages of 30 seconds and one minute.

Supposing these two calls (a personal call and a professional call) begin at the same instant:

1) After how long can we expect a call to be completed?

2) What is the probability that the professional call will finish first?

Supposing that, at any instant t, we count n personal calls and m professional calls taking place at the same time:

3) What is the probability that a professional call will finish first?

EXERCISE 1.6.– *Inter-arrivals*

In a call center, we suppose that the inter-arrivals of calls follow an exponential distribution with parameter λ. Office hours begin at 8 am.

1) Calculate the probability that no calls will come in during the first half-hour of working hours.

2) What is the probability that no calls will come in during the second half-hour of working hours, given that no calls came in during the first half-hour?

3) Calculate the probability that the second call will come in during the first half-hour of work.

4) Calculate the probability that the nth call will come in after instant t.

EXERCISE 1.7.– *ATM*

Customers approach an ATM following a Poisson process. The rate of arrival of customers is one every 15 minutes.

1) Calculate the probability that no customer will approach for one hour.

2) Calculate the probability that n successive customers will arrive in the interval of one minute.

3) If a customer arrives at the ATM, how much time has passed on average since the arrival of the previous customer?

EXERCISE 1.8.– *Bus*

To reach point B from point A, we can take one of two bus routes: line 1 or line 2. The buses for these lines arrive following Poisson processes of respective intensities λ_1 and λ_2 at stop A.

1) What is the average delay time for the next bus at stop A?

2) What is the probability that the bus for line 2 will arrive first at stop A?

3) If we prefer the bus for line 1, how long has passed since the last bus for this route passed?

Now, we only consider the buses for line 1.

4) What is the average delay time for the next bus?

5) From the perspective of the user showing up at stop A at an arbitrary instant, show that the average length of time between two buses is the double of the average effective length of time between two buses $1/\lambda_1$.

EXERCISE 1.9.– *Gas station*

In a gas station, the pumps are spread out in two different rows. At this station, we count an average of 15 cars arriving in 10 minutes; these arrivals are presumed to follow a Poisson process. We estimate that the car arriving uniformly and randomly picks which row it will join.

1) Calculate the probability that no cars will choose the first row in the station for one hour.

2) What is the average number of cars arriving in the second row in the space of one hour?

3) What is the probability that more than one minute will pass between two arrivals in the second row?

EXERCISE 1.10.– *Interlacing*

Let there be two Poisson processes of respective intensities λ_1 and λ_2.

Calculate the average number of appearances of one between two appearances of the other.

EXERCISE 1.11.– *Discretization of the Poisson process*

Let there be a growing sequence of random variables $\{t_n\}_{n\in\mathbb{N}^*}$ with integer values. In the case where the random variables $d_n = t_n - t_{n-1}$are i.i.d following geometric distributions with parameter $p \in (0,1)$, we speak of a punctual process over $\mathbb{N}$ called the Bernoulli[7] process.

For this process, the number of arrivals X_k at instant k, for any integer $k \geq 1$, follows a Bernoulli distribution with the average p defined by: $\mathbb{P}(X_k = 1) = p$, $\mathbb{P}(X_k = 0) = 1 - p$. The intensity of this simple and stationary process is equal to $p = E(X_i)$.

1) For any natural integer $s < t$, calculate its counting measure in an interval (s, t), by analogy with the Poisson process.

2) Deduce that the Bernoulli process of intensity $p = \lambda\tau$ (fixed, strictly positive λ and a sufficiently small time step τ) tends towards a Poisson process of intensity λ when τ approaches zero.

7 Daniel Bernoulli (1700–1782), Swiss doctor, physicist and mathematician.