
Representation of Systems: A Historical Overview

1.1. Transfer functions and matrices

1.1.1. *Transfer functions*

The first step in developing the theory of systems was taken in 1945, with the publication of [BOD 45], a reference book in which Bode summarized his own work – the notorious asymptotic diagram, the concepts of gain and phase margins, amplitude-phase relations – and the work of others, including his colleagues at Bell Telephone Laboratories. Among these was Black – credited with inventing the concept of “feedback”, in 1927 [BLA 34] – and Nyquist – who in 1932 developed the famous stability criterion that is now known by his name [NYQ 32]; Bode [BOD 45] also presented original results (such as the theorem on the integral of the sensitivity function, the full impact of which was only understood in the early 1980s). This book was complemented in 1947 by [JAM 47], whose formalism seems clearer to today’s readers and who introduced the Black–Nichols abacus in the form it is still used today.

The underlying formalism in these works is that of *transfer functions*. Let Σ be a linear time-invariant, continuous-time control system¹ governed by the differential equation:

$$D_t \cdot \mathbf{y} = N_t \cdot \mathbf{u} \quad [1.1]$$

¹ See Preface (p. x).

where $\mathbf{u}$ and $\mathbf{y}$ are, respectively, the input and the output of the system, and D_l, N_l are polynomials with indeterminate s (therefore elements of the ring $\mathbf{A} = \mathbb{R}[s]$):

$$D_l = s^n + a_1 s^{n-1} + \dots + a_n \quad [1.2]$$

$$N_l = b_0 s^{n'} + b_1 s^{n'-1} + \dots + b_{n'} \quad [1.3]$$

where $b_0 \neq 0$. A system Σ with this structure is sometimes called an *input–output system* (as we will see further on, there are also other types of representation). Since it has only one input and one output, this system is said to be *single-input–single-output* (SISO).

In equation [1.1], the indeterminate s is substituted by the differentiation operator $\partial = d/dt$; additionally, the symbols² $\mathbf{y}$ and $\mathbf{u}$ can be substituted by sufficiently regular functions $y = t \mapsto y(t)$ and $u : t \mapsto u(t)$, for example of class C^∞ . The transfer function of this system is the rational fraction $G \in \mathbb{R}(s)$ such that, if $\hat{u}$ and $\hat{y}$ denote the Laplace transforms (assuming these exist: see section A.1.1) of such functions, u and y , with the only additional hypothesis being zero initial conditions: $y^{(i)}(0) = 0$ ($i = 0, \dots, n-1$), $u^{(j)}(0) = 0$ ($j = 0, \dots, n'-1$), we have³:

$$\hat{y} = G\hat{u} \quad [1.4]$$

hence:

$$G = D_l^{-1} N_l \quad [1.5]$$

This rational fraction is assumed to be irreducible over $\mathbf{A}$ (i.e. the only common factors of D_l and N_l are units of $\mathbf{A}$, in this case, non-zero constants). The *transmission poles* and *transmission zeros* of Σ (or, through an abuse of language, of G) are the roots in $\mathbb{C}$ of D_l and N_l , respectively (by taking into account multiplicities, that is, by counting an m -order root m times).

The rational fraction $G \in \mathbb{R}(s) = \mathbb{Q}(\mathbf{A})$ (where $\mathbb{Q}(\mathbf{A})$ denotes the field of fractions of $\mathbf{A}$) is said to be *proper* if $n' \leq n$, *strictly proper* if $n' < n$ and *biproper* if $n = n'$. The difference $\delta_0 = n - n'$ is called the *relative degree* of G (in algebra, $-\delta_0$ is called the degree of G , extending the common concept of polynomial degrees to rational fractions).

² The exact significance of this substitution and these symbols is explained in section 1.6.2.

³ It is convenient to use s to denote the indeterminate of the polynomials D_l and N_l , and of the rational fraction G , as well as to denote the Laplace variable, reserving the notation ∂ (if required) for the differentiation operator d/dt .

1.1.2. Transfer matrices

(I) Transmission poles and zeros

The formalism seen earlier can be generalized for the case where the input $\mathbf{u}$ and the output $\mathbf{y}$ of the system Σ are columns with m and p entries, respectively: N_l and D_l are then polynomial matrices whose dimensions are, respectively, $p \times m$ and $p \times p$, D_l assumed to be regular, that is, invertible over the field of rational fractions $\mathbb{R}(s)$, and $G \in \mathbb{R}(s)^{p \times m}$ is a rational $p \times m$ matrix (“transfer matrix”). Such a system is said to be *MIMO* (*multi-input–multi-output*). Transmission poles and zeros were defined in this general context by McMillan in 1952 [MCM 52]: let d be a least common denominator of the elements of G , so that:

$$G = \frac{\check{N}}{d}$$

where $\check{N}$ is a polynomial matrix: $\check{N} \in \mathbf{A}^{p \times m}$. It is known ([BLS 17], section 3.4.2) that $\check{N}$ is equivalent to its normal (or Smith normal) form:

$$\Sigma = \text{diag}(e_1, \dots, e_r, 0, \dots, 0)$$

$\text{diag}(e_1, \dots, e_r, 0, \dots, 0)$ denoting, by convention, $\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ with $E = \text{diag}(e_1, \dots, e_r) \in \mathbf{A}^{r \times r}$ where $r = \text{rk}(\check{N})^4$, and $e_1, \dots, e_r \in \mathbf{A}^\times$ are the non-zero invariant factors, such that $e_1 \mid e_2 \mid \dots \mid e_r$; consequently, there exist unimodular square matrices (i.e. whose determinant is a unit of $\mathbf{A}$) U, V such that $\check{N} = U^{-1} \Sigma V$, and it is thus obtained that:

$$G = U^{-1} \text{diag}\left(\frac{e_1}{d}, \dots, \frac{e_r}{d}, 0, \dots, 0\right) V = U^{-1} \text{diag}\left(\frac{n_1}{d_1}, \dots, \frac{n_r}{d_r}, 0, \dots, 0\right) V \quad [1.6]$$

where the rational fractions n_i/d_i are irreducible over $\mathbf{A}$, $n_1 \mid n_2 \mid \dots \mid n_r$, $d_r \mid d_{r-1} \mid \dots \mid d_1$.

⁴ It must be recalled that the rank $\text{rk}(\check{N})$ is the maximum order of the non-zero minors of $\check{N}$ ([BLS 17], section 2.3.11(I)). Generally speaking, if $E \in \mathbf{A}^{q \times k}$ where $\mathbf{A}$ is an Ore domain (ibid., section 3.1.10(II)), $\text{rk}(E)$ denotes the rank of E when this matrix is considered to belong to $\mathbf{F}^{q \times k}$ where $\mathbf{F}$ is the field of fractions of $\mathbf{A}$ (ibid., section 2.3.11(III)).

DEFINITION 1.–

i) The matrix $\text{diag} \left(\frac{n_1}{d_1}, \dots, \frac{n_r}{d_r}, 0, \dots, 0 \right)$ (of dimension $p \times m$) is called the Smith–McMillan form of G over $\mathbf{A}$.

ii) The transmission poles (respectively, transmission zeros) of system Σ (or of G) are the roots in $\mathbb{C}$ of the polynomials d_i (respectively, n_i), $i = 1, \dots, r$, taking multiplicities into account.

REMARK 2.– All abbreviations used in this book are usual in the literature on systems theory.

DEFINITION 3.– The system Σ is said to be proper (respectively, strictly proper) if its transfer matrix $G = (g_{ij})$ is proper (respectively, strictly proper), that is, if the entries g_{ij} are all proper (respectively, strictly proper).

When G is proper (or even strictly proper), it is generally not true that the entries n_i/d_i of the Smith–McMillan form of G are proper.

EXAMPLE 4.– The Smith–McMillan form of:

$$\begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)^2} \end{bmatrix} \quad [1.7]$$

is:

$$\begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix}$$

REMARK 5.– It is an experimental fact that only proper systems are physically realizable. The systems to be controlled are even generally strictly proper; this is true for mechanical systems, given their inertia, as well as electric systems, electromechanical systems or thermal systems, for the same reason (see [BLS 10], Chapter 1). The exception seems to be an elementary electric circuit made up of a resistance R at the ends of which a potential difference U is applied (the system input), with the output variable being the intensity i of the current going across the resistance. However, the conventional equation $U = Ri$ (according to which the system is proper but not strictly proper) is not valid unless the propagation of the electromagnetic wave is neglected (the “true” system equation bringing into play the wave propagation equation; the non-instantaneous nature of the propagation can again be interpreted as “electric inertia”).

However, it must be noted that with inappropriate modeling, it may be found that a system has an improper transfer matrix [WIL 97].

REMARK 6.–

i) It can be seen that the system Σ having the transfer matrix [1.7] has transmission poles -1 and -2 , both with multiplicity of 2, and has transmission zero -2 with multiplicity 1. Following Rosenbrock [ROS 70], it is agreed to write:

$$\{t.p.\} = \{-1, -1, -2, -2\}, \{t.z.\} = \{-2\}$$

and $\{p.t.\}$ and $\{z.t.\}$ are said to be, respectively, the sequence of transmission poles and the sequence of transmission zeros of Σ (strictly speaking, these are not sets: see infra, notation 7).

ii) In general, if a is a non-zero polynomial that factorizes into prime factors over $\mathbb{C}[s]$ as per:

$$a = \lambda \prod_{i=1}^r (s - p_i)^{\mu_{p_i}}$$

where $\lambda \neq 0$, $r \geq 0$, $p_i \neq p_j$ if $i \neq j$ and $\mu_{p_i} > 0$, then it is possible to write that the sequence of zeros or roots of a is:

$$\mathcal{Z}\{a\} = \left\{ \underbrace{p_1, \dots, p_1}_{\mu_{p_1} \text{ terms}}, \dots, \underbrace{p_r, \dots, p_r}_{\mu_{p_r} \text{ terms}} \right\}$$

and according to Oberst et al. ([OBE 20], p. 185) that its support is:

$$\text{supp}(\mathcal{Z}\{a\}) = \{p_1, \dots, p_r\}$$

Let us proceed more axiomatically:

NOTATION 7.– Given two sets A, B , $A + B$ is their disjoint union (denoted as $\dot{\cup}$ or $\uplus$ in [BLS 17], section 1.2.6(II)). If $a \in A \cap B$, this element a is duplicated in $A + B$; for example, if $A = B = \{a\}$, considering a copy $B' = \{a'\}$ of B , then $A + B := A \cup B' = \{a, a'\}$. Since in practice, there is no need to distinguish between a and a' , it is agreed to write $A + B = \{a, a\}$; this is no longer a set but a sequence of elements (of which the order does not matter); generally speaking, $A + B$ is the concatenation of the sequences A, B . This notation, used notably by Rosenbrock [ROS 70], is convenient since conversely, $A - B$ (where B is a subsequence of A) denotes the sequence obtained by deleting from A those terms that are elements of B , hence $(A + B) - B = A$; for example, $(\{a\} + \{a\}) - \{a\} = \{a, a'\} - \{a\} = \{a'\} = \{a\}$, or by abusing the writing $(\{a\} + \{a\}) - \{a\} = \{a, a\} - \{a\} = \{a\}$. The set of elements of the sequence $\{a, a\}$ is the support of this sequence, namely, $\{a\}$.

DEFINITION 8.– Let $\mathbb{K}$ be a commutative field. Then $\mathcal{Z} : \mathbb{K}[s]^\times \rightarrow \mathbb{K}^{(\mathbb{N})}$ and its support $\text{supp}(\mathcal{Z}(\cdot)) : \mathbb{K}[s]^\times \rightarrow \mathbb{K}^{(\mathbb{N})}$ are determined by the following three conditions: (1) if $a \in \mathbb{K}[s]$ is a constant polynomial (which may be zero), then $\mathcal{Z}\{a\} = \text{supp}(\mathcal{Z}\{a\}) = \emptyset$; (2) if $a \neq 0$ is an irreducible polynomial, then $\mathcal{Z}\{a\} = \text{supp}(\mathcal{Z}\{a\})$ is the set of roots of a in an algebraic closure of $\mathbb{K}$; (3) if $a, b \in \mathbb{K}[s]^\times$, then $(+)$ denoting the concatenation: see notation 7):

$$\mathcal{Z}\{ab\} = \mathcal{Z}\{a\} + \mathcal{Z}\{b\} \quad [1.8]$$

$$\text{supp}(\mathcal{Z}\{ab\}) = \text{supp}(\mathcal{Z}\{a\}) \cup \text{supp}(\mathcal{Z}\{b\}) \quad [1.9]$$

(II) Blocking zeros

Blocking zeros (b.z.) were defined in the following way, by Ferreira and Bhattacharyya [FER 77] in 1977, based on the Smith–McMillan form of G :

DEFINITION 9.– The sequence of blocking zeros $\{b.z.\}$ over $\mathbf{A}$ is $\mathcal{Z}\{n_1\}$.

Evidently we have:

$$\{b.z.\} \subset \{t.z.\}$$

with equality in the single-input–single-output case. The following result is clear:

LEMMA 10.– We have:

$$\{b.z.\} = \{s \in \mathbb{C} : G(s) = 0\}$$

1.1.3. The discrete-time case

This formalism can also be applied to discrete-time systems: the differential equation [1.1] then becomes a recurrence equation, where the differentiation operator ∂ is replaced by the shift-forward operator $\sigma : x(t) \mapsto x(t+1)$ (this operator is often denoted by q in the literature); the Laplace variable s is replaced by the variable z of the $\mathcal{Z}$ -transform (which dates back to the middle of the 18th century, but whose application to sampled systems can largely be traced back to the 1950s, with the works of Hurewicz, Tsytkin, Lawden, Barker, Ragassini, Zadeh, etc.: see the summary by Jury and Tsytkin [JUR 71]). A discrete-time system whose transfer matrix (in the indeterminate z) is proper (respectively, strictly proper) is said to be *causal* (respectively, *strictly causal*). A causal system is also said to be *non-anticipative*.

1.2. State-space representation

1.2.1. Continuous-time state-space systems

The second step in the development of the theory of systems began in 1960, with Kalman's work [KAL 60b, KAL 60c, KAL 63]. Control systems⁵ were envisaged in a state-space representation (sometimes called a *state-space system* by abuse of language):

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} + D(\partial)\mathbf{u} \end{cases} \quad [1.10]$$

written $\{C, A, B, D\}$ (and $\{C, A, B\}$ if $D = 0$); $\mathbf{x}$ is the state (n components) and $\mathbf{u}$ and $\mathbf{y}$ have the same significance as in sections 1.1.1 and 1.1.2(I); A, B, C are matrices of real numbers, whose dimensions are $n \times n$, $n \times m$, and $p \times n$, respectively, and $D \in \mathbb{R}[s]^{p \times m}$ (often, $D \in \mathbb{R}^{p \times m}$). This formalism encompasses the preceding one: indeed, in moving from the Laplace domain, having substituted⁶ $\mathbf{u}, \mathbf{x}, \mathbf{y}$ in [1.10] with sufficiently regular functions $u : t \mapsto u(t)$, $x : t \mapsto x(t)$, $y : t \mapsto y(t)$, under the hypothesis of zero initial conditions, (namely, $x(0) = 0$ if $D \in \mathbb{R}^{p \times m}$), we obtain:

$$\hat{x}(s) = (sI_n - A)^{-1} \hat{u}(s)$$

yielding [1.4] with:

$$G(s) = C(sI_n - A)^{-1}B + D(s) \quad [1.11]$$

Reciprocally, given a control system Σ having a rational transfer matrix $G \in \mathbb{R}(s)^{p \times m}$, there exists a *non-unique* state-space representation $\{C, A, B, D\}$, for which we have relation [1.11] (see for example [BLS 10], section 7.4). The transfer matrix $G(s)$ is proper (respectively, strictly proper) if and only if $D \in \mathbb{R}^{p \times m}$ (respectively, $D = 0$). D is called the *direct term*, and by replacing G by $G - D$, it is easy to return to the case where $D = 0$.

1.2.2. Discrete-time state-space systems

A discrete-time state-space system [KAL 60b] Σ is governed by equations of the form:

$$\begin{cases} \sigma\mathbf{x} = A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} = C\mathbf{x} + D(\sigma)\mathbf{u} \end{cases} \quad [1.12]$$

⁵ See Preface (p. x).

⁶ See note 2 (p. 2).

(see section 1.1.3 for the notations), and it is thus easy to transpose what has been said above regarding continuous-time systems to discrete-time systems. This system is said to be *reversible* if $A \in \mathbb{R}^{n \times n}$ is invertible, because in this case $\mathbf{x} = A^{-1}(\sigma \mathbf{x} - B\mathbf{u})$. Again, this state-space system is denoted as $\{C, A, B, D\}$ (and $\{C, A, B\}$ if $D = 0$).

1.2.3. Controllability and observability

Using state-space representations, Kalman [KAL 60b, KAL 69, Chapter 2] could introduce the fundamental concepts of reachability, controllability, observability and constructibility, which cannot be defined using a transfer function formalism. To simplify these statements, it is assumed that $D = 0$, which changes nothing in the results.

In what follows, we place ourselves in the case of a state-space system $\{C, A, B\}$, which may be time-varying (i.e. with matrices A, B, C possibly depending on time) and it is convenient to introduce the following notations: let there be any two points x_1, x_2 in $\mathbb{R}^n$ and two times t_1, t_2 with $t_2 > t_1$ ($t_1, t_2 \in \mathbb{R}$ in the continuous-time case; $t_1, t_2 \in \mathbb{Z}$ in the discrete-time case). The semi-closed interval with origin t_1 and extremity t_2 is denoted by $[t_1, t_2[$; in the discrete-time case this is, of course, an interval of $\mathbb{Z}$ ([BLS 17], section 2.1.2(I)). Let $\omega : [t_1, t_2[\rightarrow \mathbb{R}^m$. Then, $\varphi(t; t_0, x_0; \omega)$ denotes the solution to the state equation $\dot{x} = Ax + Bu$ (continuous-time) or $x(t+1) = Ax(t) + Bu(t)$ (discrete time) at time $t \in [t_1, t_2]$ when the system input u is equal to ω and $x(t_0) = x_0$. Further, $\eta(t; t_0, x_0; \omega)$ denotes the output y at time $t \in [t_1, t_2]$.

It is assumed that ω is such that $\varphi(t; t_0, x_0; \omega)$ and $\eta(t; t_0, x_0; \omega)$ are well-defined for (almost)⁷ all $t \in [t_1, t_2]$ and all $(t_0, x_0) \in [t_1, t_2] \times \mathbb{R}^n$; in the continuous-time case, this signifies that ω is a sufficiently regular function, for example belonging to $\mathcal{L}^1([t_1, t_2]; \mathbb{R}^m)$ ([BLS 19], section 1.5.2(I)).

We write $(t_1, x_1) \rightsquigarrow (t_2, x_2)$ if there exists ω (as above) for which $\varphi(t_2; t_1, x_1; \omega) = x_2$.

DEFINITION 11.–

i) The state $x_2 \in \mathbb{R}^n$ (or the event (t_2, x_2)) is reachable on $[t_1, t_2[$ for Σ if $(t_1, 0) \rightsquigarrow (t_2, x_2)$. The state $x_1 \in \mathbb{R}^n$ (or event (t_1, x_1)) is controllable on $[t_1, t_2[$ for Σ if $(t_1, x_1) \rightsquigarrow (t_2, 0)$.

⁷ This parenthesis applies to $\eta(t; t_0, x_0; \omega)$ in the continuous-time case, “almost all” to be understood in the sense of the Lebesgue measure. The same convention applies for the rest of the text.

The state $x_1 \in \mathbb{R}^n$ (or the event (t_1, x_1)) is non-observable on $[t_1, t_2[$ for Σ if $\eta(t; t_1, x_1; 0) = 0$, for (almost) all $t \in [t_1, t_2[$. The state $x_2 \in \mathbb{R}^n$ (or event (t_2, x_2)) is non-constructible⁸ on $[t_1, t_2[$ for Σ if $\eta(t; t_2, x_2; 0) = 0$, for (almost) all $t \in [t_1, t_2[$.

ii) The state-space system Σ is completely reachable (respectively, completely controllable) on $[t_1, t_2[$ if any point in $\mathbb{R}^n$ is reachable (respectively, controllable) for Σ over this interval. This system Σ is completely observable (respectively, completely constructible) on $[t_1, t_2[$ if only the origin of $\mathbb{R}^n$ is non-observable (respectively, non-constructible) for Σ on this interval.

REMARK 12.—

1) Intuitively, a system is completely observable on $[t_1, t_2[$ if, with control $u = 0$, the knowledge of the output y on $[t_1, t_2[$ makes it possible to determine $x(t_1)$ (or, in an equivalent manner, without assuming $u = 0$, if the knowledge of (y, u) on $[t_1, t_2[$ makes it possible to determine $x(t_1)$). In this case, $x(t_2)$ can be derived by integration of the state equation (an integration that is a simple recurrence law in discrete-time), therefore the system is completely constructible on $[t_1, t_2[$. In the continuous-time case, the integration can also be carried out backward from $x(t_2)$ to $x(t_1)$, and consequently, complete observability and complete constructibility are synonymous concepts. This is the same for complete controllability and complete reachability. On the other hand, in the discrete-time case, $x(t_1)$ can be derived from $x(t_2)$ only in the case of a reversible system. In general, in discrete-time, complete observability is thus a stronger concept than complete constructibility, and similarly, complete reachability is a stronger concept than complete controllability.

2) It is the semi-closed interval $[t_1, t_2[$ (and not the closed interval $[t_1, t_2]$) that should be considered in definition 11, and this is essential in the discrete-time case (this is unimportant in the continuous-time case, since $\{t_2\}$ has then zero measure): for reachability, the controls that can transfer $(t_1, 0)$ to (t_2, x_2) are $u(t_1), \dots, u(t_2 - 1)$. The dual concept of observability (see infra, section 2.4.4) brings into play the outputs $y(t_1), \dots, y(t_2 - 1)$.

3) In [BLS 10] (sections 10.4.2 and 10.4.4), a state-space system Σ is said to be controllable, 0-controllable, observable and 0-observable, respectively, if there exists $t_2 > t_1$ such that Σ is completely reachable, completely controllable, completely observable, completely constructible on $[t_1, t_2[$. This terminology will be retained in this book, except when it is essential to revert to the Kalmanian terminology, in section 2.4.

Definition 11 is valid for a time-varying system. Kalman showed that in the time-invariant case, this definition is equivalent to very simple algebraic conditions:

⁸ Some authors use “non-reconstructible”.

THEOREM 13.— (Kalman criterion) *The system Σ is controllable (in the specific sense of remark 12(3)) if and only if $\text{rk}(\Gamma(A, B)) = n$; it is observable if and only if $\text{rk}(\Omega(A, B)) = n$, where Γ and Ω are the controllability and observability matrices given by:*

$$\begin{aligned}\Gamma(A, B) &= \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \\ \Omega^T(C, A) &= \begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{n-1} C^T \end{bmatrix}\end{aligned}$$

DEMONSTRATION.—

1) The demonstration is given in [KAL 60b], (see also [WOL 74], theorem 3.5.3) for controllability of a continuous-time system. It is also given in section 2.3.2 in a more general context.

2) Let us study the controllability⁹ of a discrete-time system [1.12], as this case is the simplest, while also being very instructive. First of all, consider the case where $t_0 = 0$ through a shift of the origin of time. Let $\{x(k)\}_{k \geq 0}$ and $\{u(k)\}_{k \geq 0}$ denote the sequence of states starting from any initial state $x_0 \in \mathbb{R}^n$ and the sequence of control variables. Thus:

$$\begin{aligned}x(0) &= x_0 \\ x(1) &= Ax_0 + Bu(0) \\ x(2) &= A(Ax_0 + Bu(0)) + Bu(1) \\ &= A^2x_0 + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}\end{aligned}$$

and by induction, it is obtained:

$$x(k) = A^k x_0 + \underbrace{\begin{bmatrix} B & AB & \dots & A^{k-1}B \end{bmatrix}}_{\Gamma_k} \begin{bmatrix} u(k-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}$$

According to the Cayley–Hamilton theorem ([BLS 17], section 2.3.11(VII)), A^n is a linear combination (with coefficients in $\mathbb{R}$) of $I_n, A, \dots, A^{n-1}$, therefore $\text{rk}(\Gamma_k) = \text{rk}(\Gamma_n)$ for all $k \geq n$. Consequently, for all $x_1 \in \mathbb{R}^n$, there is a control sequence $\{u(k)\}_{k \geq 0}$ such that we have $x(k) = x_1$, if and only if $\text{rk}(\Gamma_n) = n$ and $k \geq n$. The Kalman criterion is thus obtained, with $\Gamma(A, B) = \Gamma_n$.

⁹ It must be remembered that the terminology used here is that defined in remark 12(3).

3) Let us now study the observability of the discrete-time system [1.12]. As the control is zero, we have $x(j+1) = Ax(j)$, hence:

$$\begin{aligned} y(0) &= Cx_0 \\ y(1) &= CAx_0 \\ &\dots \\ y(k-1) &= CA^{k-1}x_0 \end{aligned}$$

and consequently:

$$\begin{bmatrix} y^T(0) & y^T(1) & \dots & y^T(k-1) \end{bmatrix} = x_0^T \underbrace{\begin{bmatrix} C^T & A^T C^T & \dots & (A^T)^{k-1} C^T \end{bmatrix}}_{\Omega_k^T}$$

The initial state x_0 is therefore determined by the sequence $\{y(j)\}_{0 \leq j \leq k-1}$ if and only if Ω_k^T is right invertible, that is, of rank n . Using the same reasoning as in (2) above, this condition is satisfied if and only if $\text{rk}(\Omega_n^T) = n$ and $k \geq n$. ■

The result below can be derived from this demonstration (where the terminology given in remark 12(3) is used).

THEOREM 14.— *The discrete-time state system [1.12] is 0-controllable if and only if $\text{im}(A^n) \subset \text{im}(\Gamma(A, B))^{10}$, and it is 0-observable if and only if $\ker(\Omega(A, B)) \subset \ker(A^n)$.*

For short, the pair (A, B) is said to be *controllable* if and only if $\text{rk}(\Gamma(A, B)) = n$, and the pair (C, A) is said to be *observable* if and only if $\text{rk}(\Omega(A, B)) = n$; hence the *Kalman duality theorem*:

THEOREM 15.— *(C, A) is observable if and only if (A^T, C^T) is controllable.*

1.2.4. Poles of a state-space system

Consider again a state-space system Σ given by $\{C, A, B\}$.

DEFINITION 16.— *The system poles (s.p.) are the eigenvalues of A (taking into account multiplicities). The sequence of system poles is the sequence of zeros $\mathcal{Z}\{a\}$ of $a = \det(sI - A)$ (definition 8).*

¹⁰ Given a matrix $M \in \mathbb{R}^{q \times k}$, $\text{im}(M) := \{Mx : x \in \mathbb{R}^k\}$ and $\ker(M) := \{x \in \mathbb{R}^k : Mx = 0\}$.

Matrix A is similar to a diagonal table $\text{diag} (J_{\pi_i})_{1 \leq i \leq q}$ of Jordan blocks J_{π_i} . The π_i ($i = 1, \dots, q$) are the elementary divisors of A , and by embedding $\mathbb{R}$ in $\mathbb{C}$ to simplify the notation, they are of the form:

$$\pi_i = (s - \lambda_i)^{\mu(\pi_i)}$$

where $\lambda_i \in \mathbb{C}$ is an eigenvalue of A (therefore, a system pole) and $\mu(\pi_i)$ is its multiplicity, as a root of π_i (see section A.4.2(II)).

(I)

In the continuous-time case, the state equation $\dot{x} = Ax$ is written, through a change in basis, in the form of a decoupled system of equations $\dot{x}_i = J_{\pi_i} x_i$ ($i = 1, \dots, q$). Thus, up to change of basis, we have:

$$x(t) = e^{At} x_0 = \begin{bmatrix} x_1(t) \\ \vdots \\ x_q(t) \end{bmatrix} = \text{diag} [e^{J_{\pi_1} t} \ \dots \ e^{J_{\pi_q} t}] \begin{bmatrix} x_{10} \\ \vdots \\ x_{q0} \end{bmatrix}$$

To simplify the notation, the index i is omitted, and J_{π_i} is written J_{π} , $\mu(\pi_i)$ is written μ and x_{i0} is written x_0 . This thus leads to making explicit the solution:

$$x(t) = e^{J_{\pi} t} x_0$$

According to [BLS 17], section 3.4.4:

$$e^{J_{\pi} t} = e^{\lambda t} \underbrace{\begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{\mu-1}}{(\mu-1)!} \\ 0 & 1 & & \ddots & \vdots \\ \vdots & & \ddots & & \frac{t^2}{2} \\ \vdots & & & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}}_{\mu \text{ columns}} \left. \vphantom{\begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{\mu-1}}{(\mu-1)!} \\ 0 & 1 & & \ddots & \vdots \\ \vdots & & \ddots & & \frac{t^2}{2} \\ \vdots & & & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}} \right\} \mu \text{ rows} \quad [1.13]$$

(II)

In the discrete-time case, through a change of basis, the state equation $x(k+1) = Ax(k)$ takes the form of a decoupled system of equations $x_i(k+1) = J_{\pi_i} x_i(k)$ ($i = 1, \dots, q$), hence, up to change of basis:

$$x(k) = A^k x_0 = \begin{bmatrix} x_1(k) \\ \vdots \\ x_q(k) \end{bmatrix} = \text{diag} [J_{\mu_1}^k \ \dots \ J_{\mu_q}^k] \begin{bmatrix} x_{10} \\ \vdots \\ x_{q0} \end{bmatrix}$$

Using the same conventions as above, the solution:

$$x(k) = J_\pi^k x_0$$

is made explicit, based on the fact that λI_μ commutes with the nilpotent matrix $J_\pi - \lambda I_\mu$, because of the Newton binomial, and of the equality:

$$J_\pi^k = \underbrace{\begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots & \binom{k}{\mu-1}\lambda^{k-\mu+1} \\ 0 & \lambda^k & & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \binom{k}{2}\lambda^{k-2} \\ \vdots & & \ddots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & \dots & \dots & 0 & \lambda^k \end{bmatrix}}_{\mu \text{ columns}} \left. \vphantom{\begin{bmatrix} \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots & \binom{k}{\mu-1}\lambda^{k-\mu+1} \\ 0 & \lambda^k & & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \binom{k}{2}\lambda^{k-2} \\ \vdots & & \ddots & \lambda^k & \binom{k}{1}\lambda^{k-1} \\ 0 & \dots & \dots & 0 & \lambda^k \end{bmatrix}} \right\} \mu \text{ rows}$$

REMARK 17.— *It can thus be seen that the system poles, whether for continuous-time or discrete-time, make it possible to make explicit the “free response” of the system (taking into account the multiplicities $\mu(\pi_i)$, $i = 1, \dots, q$), namely, the behavior of its state as a function of time, starting from any initial condition x_0 , when the control is identically zero.*

(III)

The set of transmission poles is included in the set of system poles, with equality if and only if the system (or, for brevity $\{C, A, B\}$) is controllable and observable.

1.2.5. Stability of linear time-invariant systems

Consider a linear time-invariant state-space system $\{C, A, B, D\}$. Let $x(t) \in \mathbb{R}^n$ be the value of the state of this system at time t , when its control u is zero and x_0 is its state at the initial time. This initial time can be assumed, without loss of generality, to be equal to 0 through a shift of the origin of the time axis.

DEFINITION 18.— *The system, Σ , considered here, is stable in the sense of Lyapunov if, for all $\delta > 0$, there exists $\varepsilon > 0$ such that $\|x(t)\| \leq \delta$ for all $t \geq 0$ if $\|x_0\| \leq \varepsilon$. This system is exponentially stable (or asymptotically stable)¹¹ if for all $x_0 \in \mathbb{R}^n$, $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$.*

In the continuous-time case, one has $x(t) = e^{At}x_0$, and in the discrete-time case $x(t) = A^t x_0$.

¹¹ For linear time-invariant systems, asymptotic stability and exponential stability are equivalent concepts.

In the continuous-time case, we put $\mathbb{C}_s = \mathbb{C}_- := \{s \in \mathbb{C} : \Re(s) < 0\}$ and $\overline{\mathbb{C}_s} = \{s \in \mathbb{C} : \Re(s) \leq 0\}$. In the discrete-time case, we put $\mathbb{C}_s = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{C}_s} = \{z \in \mathbb{C} : |z| \leq 1\}$ ¹².

The expressions for e^{At} and A^t , reviewed in section 1.2.4, make it possible to state the following theorem:

THEOREM 19.—

1) *The system Σ is stable in the sense of Lyapunov if and only if all its poles belong to $\overline{\mathbb{C}_s}$, those belonging to the frontier $\text{Fr}(\mathbb{C}_s)$ (if any) all having multiplicity 1 (as roots of the minimal polynomial of A).*

2) *This system is exponentially stable if and only if all its poles belong to $\mathbb{C}_s$.*

A system is said to be *stable in the input–output sense* if it is proper and all its transmission poles belong to $\mathbb{C}_s$. By abuse of language, it is then said that its transfer matrix is stable.

1.3. “Geometric” approach

1.3.1. Formalism of the geometric approach

In what follows, if $\mathcal{X}, \mathcal{Y}$ are two real finite-dimensional vector spaces, $\mathfrak{L}(\mathcal{X}, \mathcal{Y})$ denotes the space of linear maps from $\mathcal{X}$ into $\mathcal{Y}$ and $\mathfrak{L}(\mathcal{X}) = \mathfrak{L}(\mathcal{X}, \mathcal{X})$.

In [1.10], where D is assumed to belong to $\mathbb{R}^{p \times m}$, the matrices A, B, C, D represent linear maps $\mathcal{A} \in \mathfrak{L}(\mathcal{X})$, $\mathcal{B} \in \mathfrak{L}(\mathcal{U}, \mathcal{X})$, $\mathcal{C} \in \mathfrak{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{D} \in \mathfrak{L}(\mathcal{U}, \mathcal{Y})$ in bases of the state-space $\mathcal{X} \cong \mathbb{R}^n$, of the input-space $\mathcal{U} \cong \mathbb{R}^m$ and the output-space $\mathcal{Y} \cong \mathbb{R}^p$. During the 1970s, under the impetus of Wonham and his collaborators, Kalmanian formalism was reformulated into the more intrinsic language of vector spaces and linear maps [WON 85]. This reformulation would have remained a simple algebraic exercise, but it led to new results, including, notably, the “internal model principle” (see [FRA 76] and [FRA 77]) – though an equivalent for this can also be found in the usual matrix formalism (see [DAV 76]).

The state-space system [1.10] can be written more intrinsically in the geometric approach:

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t) \\ y(t) = \mathcal{C}x(t) + \mathcal{D}u(t) \end{cases} \quad [1.14]$$

¹² Each time, $\mathbb{C}_s$ is the “stable part” of the complex plane and $\overline{\mathbb{C}_s}$ is the closure of this part in $\mathbb{C}$.

There is a shift to discrete-time, by replacing $\dot{x}(t)$ with $x(t+1)$, the other terms being unchanged.

The concept of pole can be defined in this formalism. The eigenvalues λ_i of the endomorphism $\mathcal{A}$ coincide with those of the matrix A , which represents it in any base of $\mathcal{X}$, and the π_i ($i = 1, \dots, q$), invariant under change of basis, only depend on the endomorphism $\mathcal{A}$ of which they are therefore the elementary divisors. The λ_i are the system poles. Let $\mathcal{X}_{\pi_i} = \ker(\lambda_i 1_{\mathcal{X}} - \mathcal{A})^{\mu_i}$ where $\mu_i = \mu(\pi_i)$; then ([BLS 10], section 13.3.4, theorem 527):

$$\mathcal{A}\mathcal{X}_{\pi_i} \subset \mathcal{X}_{\pi_i}, \mathcal{X}_{\pi_i} \cap \mathcal{X}_{\pi_j} = \{0\} \text{ if } i \neq j, \bigoplus_{i=1}^q \mathcal{X}_{\pi_i} = \mathcal{X}$$

If $\mathcal{A}_{\pi_i} : \mathcal{X}_{\pi_i} \rightarrow \mathcal{X}_{\pi_i}$ denotes the restriction of $\mathcal{A}$ to $\mathcal{X}_{\pi_i}$, $\mathcal{A}_{\pi_i}$ has π_i as its only elementary divisor. It is convenient to write the endomorphism $\mathcal{A}$ in the form of a “diagonal sum” (ibid, p. 480):

$$\mathcal{A} = \bigoplus_{i=1}^q \mathcal{A}_{\pi_i}$$

Let $e_1 \in \mathcal{X}_{\pi_i}$ be a vector such that $(\lambda_i 1_{\mathcal{X}_{\pi_i}} - \mathcal{A}_{\pi_i})^{\mu_i-1} e_1 \neq 0$. Then:

$$\{e_1, \mathcal{A}_{\pi_i} e_1, \dots, \mathcal{A}_{\pi_i}^{\mu_i-1} e_1\}$$

is a basis, said to be $\mathcal{A}_{\pi_i}$ -cyclic¹³, of $\mathcal{X}_{\pi_i}$; in this basis, the endomorphism $\mathcal{A}_{\pi_i}$ is represented by the Jordan block J_{π_i} (ibid, p. 487).

1.3.2. Reachable and non-observable subspaces

The reachable subspace $\mathcal{R}$ of $\mathcal{X}$ is the set of reachable states (definition 11(i)); it is given by:

$$\mathcal{R} = \langle \mathcal{A} | \text{im}(\mathcal{B}) \rangle := \sum_{i=1}^n \mathcal{A}^{i-1} \text{im}(\mathcal{B})$$

The non-observable subspace is the set of non-observable states, and it is given by:

$$\mathcal{N} = \bigcap_{i=1}^n \ker(\mathcal{C}\mathcal{A}^{i-1})$$

The Kalman criterion can be reformulated as follows: the system is controllable if and only if $\mathcal{R} = \mathcal{X}$, and it is observable if and only if $\mathcal{N} = 0$.

¹³ This means that this basis is generated by a single vector e_1 , with the other vectors of the basis being obtained by left-multiplying e_1 by successive powers of $\mathcal{A}_{\pi_i}$.

1.3.3. State-feedback controls, observers

(I) Pole placement by state-feedback

Consider a linear time-invariant state-space system and, to clarify ideas, assume that it is a continuous-time system given by [1.14]. With a state-feedback control:

$$u = v - \mathcal{K}x \quad [1.15]$$

($\mathcal{K} \in \mathfrak{L}(\mathcal{X}, \mathcal{U})$), where v plays the role of a command signal, the equation for the closed-loop state-space system becomes:

$$\dot{x} = (\mathcal{A} - \mathcal{B}\mathcal{K})x + \mathcal{B}v$$

In bases of $\mathcal{X}$ and $\mathcal{U}$, $\mathcal{K}$ is represented by a “gain matrix” $K \in \mathbb{R}^{n \times m}$, the “state-feedback gain matrix”.

A subset $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of the complex plane is *symmetric with respect to the real axis* if it is invariant by conjugation $\lambda \mapsto \bar{\lambda}$. Given an endomorphism $\mathcal{E}$ of $\mathcal{X}$, its spectrum $\Lambda = \sigma(\mathcal{E})$, that is, the set of its eigenvalues in $\mathbb{C}$ ([BLS 18], section 3.4.1 (II)) has this property.

The following result was obtained by Wonham [WON 67] in 1967 (the system may be a continuous-time system or discrete-time system, with the conventions given in remark 12(3)).

THEOREM 20.– *The following conditions are equivalent:*

i) *for any subset $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ of the complex plane, symmetric with respect to the real axis, there exists $\mathcal{K} \in \mathfrak{L}(\mathcal{X}, \mathcal{U})$ such that $\sigma(\mathcal{A} - \mathcal{B}\mathcal{K}) = \Lambda$ (in other words, $\mathcal{K}$ “places the set of poles of the closed-loop system at Λ ”);*

ii) *($\mathcal{A}, \mathcal{B}$) is controllable.*

If $(\mathcal{A}, \mathcal{B})$ is controllable, then Λ only determines $\mathcal{K}$ uniquely for $m = 1$. If $m > 1$, there exist many $\mathcal{K} \in \mathfrak{L}(\mathcal{X}, \mathcal{U})$ resulting in the same pole placement.

However, it must not be believed that these different control laws have the same properties, especially with respect to the robustness they confer upon the closed-loop system (see [BLS 10], section 8.1.4).

(II) Observers

For a state-space system [1.14] of which only the output y is measured (and whose control variable, u , is also known), it is useful to compute an approximation $\hat{x}(t)$ of the

state $x(t)$, whether this is to control the system or for dependability, fault-diagnosis, etc. The concept of state observer first emerged, in a stochastic context, in the form of the discrete-time Kalman filter in 1960 [KAL 60a], and then for a continuous-time filter in 1961 [KAL 61] (“Kalman-Bucy filter”). We will study deterministic observers here and their theory is chiefly covered in Luenberger’s work, in his articles between 1964–1966 [LUE 64, LUE 66] (complete state observer, reduced-order observer; see also [LUE 71]). In the following paragraphs, we will restrict ourselves to cases with complete state observers.

Consider again the continuous-time state-space system [1.14], with the discrete-time case being similar. A complete state observer provides an estimated state $\hat{x}(t)$ governed by a differential equation of the form:

$$\partial \hat{x} = \mathcal{A}x + \mathcal{B}u + \mathcal{L}(y - \mathcal{C}x - \mathcal{D}u) \quad [1.16]$$

where $\mathcal{L} \in \mathfrak{L}(\mathcal{Y}, \mathcal{X})$. In bases of $\mathcal{Y}$ and $\mathcal{X}$, $\mathcal{L}$ is represented by a “gain matrix” $L \in \mathbb{R}^{p \times m}$, the “observer gain matrix”. Let $\tilde{x}$ be the observation error, defined by $\tilde{x}(t) = x(t) - \hat{x}(t)$. By subtracting [1.16] from the state equation, we obtain:

$$\partial \tilde{x} = (\mathcal{A} - \mathcal{L}\mathcal{C}) \tilde{x} \quad [1.17]$$

and the observer poles are thus the eigenvalues of $\mathcal{A} - \mathcal{L}\mathcal{C}$. The eigenvalues of this endomorphism are identical to those of its transpose, consequently the observer poles are the eigenvalues of ${}^t\mathcal{A} - {}^t\mathcal{C} {}^t\mathcal{L}$. The result that follows, valid for a continuous- or discrete-time system (using the conventions in remark 12(3)), is therefore a consequence of the Kalman duality theorem (theorem 15) and theorem 20:

THEOREM 21.— *The following conditions are equivalent:*

i) *for any set $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ in the complex plane, symmetric with respect to the real axis, there exists $\mathcal{L} \in \mathfrak{L}(\mathcal{Y}, \mathcal{X})$ such that $\sigma(\mathcal{A} - \mathcal{L}\mathcal{C}) = \Lambda$ (in other words, $\mathcal{L}$ “places the set of poles of the observer at Λ ”);*

ii) *$(\mathcal{C}, \mathcal{A})$ is observable.*

1.3.4. Canonical Kalman decomposition, stabilizability and detectability

The formalism of the geometric approach is useful in characterizing the *canonical Kalman decomposition*, initially obtained by Kalman in [KAL 63].

(I) Decomposition along controllability–stabilizability

Let $\mathcal{X}$ be the state-space, $\mathcal{R}$ be the reachability subspace and $\mathcal{X}_c$ be a complementary subspace of $\mathcal{R}$ in $\mathcal{X}$. Let $\mathcal{B}_c$ be a basis of $\mathcal{R}$ and $\mathcal{B}_c$ be a basis of $\mathcal{X}_c$.

In the ordered basis $\mathcal{B}_c \cup \mathcal{B}_{\bar{c}}$ of $\mathcal{X}$ and a basis of $\mathcal{U}$, the linear maps $\mathcal{A}$ and $\mathcal{B}$ are represented by matrices A and B of the form:

$$A = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix}, \quad B = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

where (A_c, B_c) is controllable. Since $\mathcal{AR} \subset \mathcal{R}$, $\mathcal{A}$ induces an endomorphism $\mathcal{A}_{\bar{c}} : \mathcal{X}/\mathcal{R} \rightarrow \mathcal{X}/\mathcal{R}$ of which $A_{\bar{c}}$ is a representative matrix ([BLS 17], section 2.2.3(I)).

Consider the state-feedback control [1.15] and let $K = \begin{bmatrix} K_c & K_{\bar{c}} \end{bmatrix}$ be the matrix that represents $\mathcal{K}$ in the basis $\mathcal{B}_c \cup \mathcal{B}_{\bar{c}}$ and a basis of $\mathcal{U}$. We have:

$$A - BK = \begin{bmatrix} A_c - B_c K_c & A_{12} - B_c K_{\bar{c}} \\ 0 & A_{\bar{c}} \end{bmatrix}$$

and $\sigma(A - BK) = \sigma(A_c - B_c K_c) \cup \sigma(A_{\bar{c}})$. The pair (A, B) is controllable if and only if the matrix $A_{\bar{c}}$ is empty. According to theorem 20, it is possible to choose K such that $\sigma(A_c - B_c K_c)$ is included in $\mathbb{C}_s$. On the other hand, if (A, B) is not controllable, since the state-feedback control has no action on $\sigma(A_{\bar{c}})$, it is possible to exponentially stabilize the system with this control variable if and only if $\sigma(A_{\bar{c}})$ is included in $\mathbb{C}_s$. This observation leads to the concept of stabilizability defined in lemma-definition 22, equivalent to that given by Wonham in [WON 67] in 1967 and reiterated by Hautus [HAU 70] in 1970:

Let $\alpha \in \mathbb{R}[s]$ be the minimal polynomial of $\mathcal{A}$. It is possible to factorize α as follows:

$$\alpha = \alpha^- \alpha^+$$

where the roots of α^- all belong to $\mathbb{C}_s$ and where the roots of α^+ all belong to the complement of this set in $\mathbb{C}$. The endomorphism of $\mathcal{X}$ obtained by substituting $\mathcal{A}$ for the indeterminate in polynomial α^+ (respectively, α^-) is written $\alpha^+(\mathcal{A})$ (respectively, $\alpha^-(\mathcal{A})$). Writing $\mathcal{X}^+(\mathcal{A}) := \ker \alpha^+(\mathcal{A})$ and $\mathcal{X}^-(\mathcal{A}) := \ker \alpha^-(\mathcal{A})$, we have:

$$\mathcal{X} = \mathcal{X}^+(\mathcal{A}) \oplus \mathcal{X}^-(\mathcal{A})$$

The subspaces $\mathcal{X}^+(\mathcal{A})$ and $\mathcal{X}^-(\mathcal{A})$ are stable through $\mathcal{A}$, and the restriction $\mathcal{A}|_{\mathcal{X}^+(\mathcal{A})}$ (respectively, $\mathcal{A}|_{\mathcal{X}^-(\mathcal{A})}$) is an endomorphism of $\mathcal{X}^+(\mathcal{A})$ (respectively, $\mathcal{X}^-(\mathcal{A})$) for which the minimal polynomial is α^+ (respectively, α^-); $\mathcal{X}^+(\mathcal{A})$ and $\mathcal{X}^-(\mathcal{A})$ are called the *unstable subspace* of $\mathcal{A}$ and the *stable subspace* of $\mathcal{A}$, respectively.

LEMMA-DEFINITION 22.– 1) *The following conditions are equivalent:*

- i) $\mathcal{X}^+(A) \subset \mathcal{R}$;
- ii) $\sigma(\mathcal{A}_{\bar{e}}) \subset \mathbb{C}_s$.

2) *The system (or pair) (A, B) is stabilizable if one of the above equivalent conditions is satisfied.*

DEMONSTRATION.– 1) Point (i) clearly signifies that the unstable modes are controllable ([WON 85], theorem 2.3), and (ii) the non-controllable modes are stable. These propositions are equivalent, each one being the contraposition of the other. ■

(II) Decomposition along observability–detectability

Let $\mathcal{N} \subset \mathcal{X}$ be the non-observable subspace. The orthogonal of $\mathcal{N}$ is the subspace:

$$\mathcal{N}^\perp = \langle {}^t\mathcal{A} | \text{im } ({}^t\mathcal{C}) \rangle \quad [1.18]$$

of the dual $\mathcal{X}^*$ of $\mathcal{X}$ ([BLS 18], section 3.10.2(II))¹⁴. Since ${}^t\mathcal{A}\mathcal{N}^\perp \subset \mathcal{N}^\perp$, the endomorphism ${}^t\mathcal{A}$ of $\mathcal{X}^*$ induces an endomorphism ${}^t\mathcal{A}_{\bar{o}} : \mathcal{X}^*/\mathcal{N}^\perp \rightarrow \mathcal{X}^*/\mathcal{N}^\perp$.

Let $\mathcal{X}_{\bar{o}}^*$ be a complement of $\mathcal{N}^\perp$ in $\mathcal{X}^*$, $\mathcal{B}_o^*$ be a basis of $\mathcal{N}^\perp$ and $\mathcal{B}_{\bar{o}}^*$ be a basis of $\mathcal{X}_{\bar{o}}^*$. In the ordered basis $\mathcal{B}^* = \mathcal{B}_o^* \cup \mathcal{B}_{\bar{o}}^*$ of $\mathcal{X}^*$ and a basis $\mathcal{B}_{\mathcal{Y}}^*$ of $\mathcal{Y}^*$, the linear maps ${}^t\mathcal{A}$ and ${}^t\mathcal{C}$ are represented by matrices of the form:

$$A^T = \begin{bmatrix} A_o^T & A_{21}^T \\ 0 & A_{\bar{o}}^T \end{bmatrix}, \quad C^T = \begin{bmatrix} C_o^T \\ 0 \end{bmatrix}$$

where the pair (A_o^T, C_o^T) is controllable and where A_o^T is a representative matrix of ${}^t\mathcal{A}_{\bar{o}}$. Thus, we have in the bases $\mathcal{B}$ and $\mathcal{B}_{\mathcal{Y}}$, duals of $\mathcal{B}^*$ and $\mathcal{B}_{\mathcal{Y}}^*$:

$$A = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix}, \quad C = \begin{bmatrix} C_o & 0 \end{bmatrix}$$

where the pair (C_o, A_o) is observable.

Consider the complete state observer [1.16] and the error equation [1.17]. In the bases $\mathcal{B}, \mathcal{B}_{\mathcal{Y}}$, the linear map $\mathcal{L} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ is represented by a matrix $L = \begin{bmatrix} L_o \\ L_{\bar{o}} \end{bmatrix}$ and:

$$A - LC = \begin{bmatrix} A_o - L_o C_o & 0 \\ A_{21} - L_{\bar{o}} C_o & A_{\bar{o}} \end{bmatrix}$$

¹⁴ Following common usage, A^T denotes the transpose of the matrix A and ${}^t\mathcal{A}$ denotes the transpose of the linear map $\mathcal{A}$.

Consequently, there exists an exponentially stable observer if and only if $\sigma(\mathcal{A}_{\bar{c}})$ is included in $\mathbb{C}_s$, which leads to the concept of detectability in lemma-definition 23, equivalent to that given by Wonham [WON 67] in 1967 (nonetheless, the term “detectability” was only introduced by Hautus [HAU 70] in 1970); it is also valid for a discrete-time system.

LEMMA-DEFINITION 23.– 1) *The following conditions are equivalent:*

- i) $\mathcal{N} \subset \mathcal{X}^-(\mathcal{A})$;
- ii) $\sigma(\mathcal{A}_{\bar{o}}) \subset \mathbb{C}_s$.

2) *The system (or the pair) $(\mathcal{C}, \mathcal{A})$ is detectable if one of the above equivalent conditions is satisfied.*

DEMONSTRATION.– 1) Point (i) can also be written as $\mathcal{N}^\perp \supset \mathcal{X}^-(\mathcal{A})^\perp$; equivalently, $\mathcal{X}^-(\mathcal{A})^\perp = \mathcal{X}^+({}^t\mathcal{A})$, therefore (i) is equivalent to (ii) according to lemma-definition 22. ■

By construction, $(\mathcal{C}, \mathcal{A})$ is detectable if and only if $({}^t\mathcal{A}, {}^t\mathcal{C})$ is stabilizable.

(III) General decomposition

According to Wonham ([WON 85], Chapter 3, exercise 3.6), let $\mathcal{X}_{c\bar{o}}$, $\mathcal{X}_{\bar{c}o}$, $\mathcal{X}_{\bar{c}\bar{o}}$, $\mathcal{X}_{co}$ be subspaces of $\mathcal{X}$ such that:

$$\mathcal{X}_{c\bar{o}} = \mathcal{N} \cap \mathcal{R} \quad (\mathcal{N} + \mathcal{R}) \oplus \mathcal{X}_{\bar{c}o} = \mathcal{X}$$

$$\mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{\bar{c}\bar{o}} = \mathcal{N} \quad \mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{co} = \mathcal{R}$$

We have:

$$\mathcal{N} + \mathcal{R} = (\mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{\bar{c}\bar{o}}) + (\mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{co}) = \mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{\bar{c}\bar{o}} \oplus \mathcal{X}_{co}$$

and consequently $\mathcal{X}_{c\bar{o}} \oplus \mathcal{X}_{\bar{c}\bar{o}} \oplus \mathcal{X}_{co} \oplus \mathcal{X}_{\bar{c}o} = \mathcal{X}$ (the signs $c, \bar{c}, o, \bar{o}$, placed as indices, signify, respectively: controllable, non-controllable, observable, non-observable). The subspaces $\mathcal{R}, \mathcal{N}$, their sum and their intersection are unique, while the other subspaces are only unique up to isomorphism.

By choosing an ordered basis $\mathcal{B}_{c\bar{o}} \cup \mathcal{B}_{co} \cup \mathcal{B}_{\bar{c}\bar{o}} \cup \mathcal{B}_{\bar{c}o}$, in $\mathcal{X}$, where $\mathcal{B}_{c\bar{o}}, \mathcal{B}_{co}, \mathcal{B}_{\bar{c}\bar{o}}, \mathcal{B}_{\bar{c}o}$ are bases of $\mathcal{X}_{c\bar{o}}, \mathcal{X}_{co}, \mathcal{X}_{\bar{c}\bar{o}}, \mathcal{X}_{\bar{c}o}$, we obtain the *canonical Kalman decomposition* ([KAL 63], theorem 5): the linear maps $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are represented by matrices (each * denoting any submatrix of the appropriate dimension):

$$\left\{ \begin{array}{l} \tilde{A} = \begin{bmatrix} A_{c\bar{o}} & * & * & * \\ 0 & A_{co} & 0 & * \\ 0 & 0 & A_{\bar{c}o} & * \\ 0 & 0 & 0 & A_{\bar{c}o} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \\ 0 \end{bmatrix} \\ \tilde{C} = \begin{bmatrix} 0 & C_{co} & 0 & C_{\bar{c}o} \end{bmatrix} \end{array} \right. \quad [1.19]$$

where:

$$\left(\underbrace{\begin{bmatrix} A_{c\bar{o}} & * \\ 0 & A_{co} \end{bmatrix}}_{A_c}, \underbrace{\begin{bmatrix} B_{c\bar{o}} \\ B_{co} \end{bmatrix}}_{B_c} \right)$$

is controllable, where:

$$\left(\underbrace{\begin{bmatrix} C_{co} & C_{\bar{c}o} \end{bmatrix}}_{C_o}, \underbrace{\begin{bmatrix} A_{co} & * \\ 0 & A_{\bar{c}o} \end{bmatrix}}_{A_o} \right)$$

is observable and where:

$$\{A_{co}, B_{co}, C_{co}\}$$

is controllable and observable. We thus have the equalities:

$$G(s) = \tilde{C} \left(sI_n - \tilde{A} \right)^{-1} \tilde{B} + D = C_{co} (sI - A_{co})^{-1} B_{co} + D \quad [1.20]$$

and the transmission poles are the eigenvalues of A_{co} .

1.4. Polynomial matrix description

1.4.1. PBH test (Hautus criterion)

Necessary and sufficient conditions for controllability and observability were independently obtained by Popov, Belevitch and Hautus in the late 1960s [BEE 68, HAU 69, POP 73] (“PBH test” or “Hautus criterion”):

THEOREM 24.–

i) The pair (A, B) is controllable if and only if:

$$\text{rk} \left(\begin{bmatrix} sI_n - A & B \end{bmatrix} \right) = n, \quad \forall s \in \mathbb{C} \quad [1.21]$$

ii) The pair (C, A) is observable if and only if:

$$\text{rk} \left(\begin{bmatrix} sI_n - A^T & C^T \end{bmatrix} \right) = n, \quad \forall s \in \mathbb{C} \quad [1.22]$$

DEMONSTRATION.– It is sufficient to demonstrate (i). It must first be noted that $\text{rk}(sI_n - A) < n$ if and only if s is an eigenvalue of A .

Assume that there exists a left eigenvector $\eta^T \in \mathbb{C}^{1 \times n}$, $\eta \neq 0$, associated with the eigenvalue λ , such that $\eta^T B = 0$. We then have $\eta^T A^k B = \lambda^k AB = 0$ ($k = 0, 1, \dots, n-1$), therefore $\text{rk}(\Gamma(A, B)) < n$.

Conversely, assume $\text{rk}(\Gamma(A, B)) < n$. There thus exists a non-zero row vector $\xi^T \in \mathbb{C}^{1 \times n}$ such that:

$$\xi^T A^k B = 0 \quad (k = 0, \dots, n-1) \quad [1.23]$$

Let $\psi \in \mathbb{C}[s]$ be a non-zero polynomial of minimal degree such that $\xi^T \psi(A) = 0$. Such a polynomial exists, with $1 \leq d^\circ(\psi) \leq n$, according to the Cayley–Hamilton theorem ([BLS 17], section 2.3.11(VII)). If λ is a root of ψ , it is possible to write $\psi(s) = (s - \lambda)\varphi(s)$ with $0 \leq d^\circ(\varphi) \leq n-1$. By writing $\eta^T = \xi^T \varphi(A)$, we have $\eta \neq 0$ and $\eta^T (A - \lambda I_n) = \xi^T \varphi(A) (A - \lambda I_n) = \xi^T \psi(A) = 0$. According to [1.23], we get $\eta^T B = 0$, and, consequently, $\text{rk}(\begin{bmatrix} \lambda I_n - A & B \end{bmatrix}) < n$. ■

If $\text{rk}(\begin{bmatrix} sI_n - A & B \end{bmatrix}) < n$ or $\text{rk}(\begin{bmatrix} sI_n - A^T & C^T \end{bmatrix}) < n$, s is necessarily a system pole and is called a *non-controllable pole* in the first case, and a *non-observable pole* in the second.

These poles, which are never transmission poles (sometimes, by abuse of language of language, it is said that “they are cancelled” when calculating the transfer function or matrix), had already been discovered by Kalman (under a different name) using his canonical decomposition ([KAL 63], theorem 5).

1.4.2. Rosenbrock representation

(I) The Rosenbrock formalism

Conditions [1.21] and [1.22] bring in polynomial matrices. This formalism was generalized in 1970 by Rosenbrock [ROS 70], who envisaged a representation of a system Σ in the form:

$$\begin{cases} P.\xi = Q.u \\ y = T.\xi + U.u \end{cases} \quad [1.24]$$

where ξ is a column with l entries, called the *pseudo-state* (or *partial state*), and P, Q, T, U are matrices with dimensions $l \times l, l \times m, p \times l$ and $p \times m$, respectively, with $\det(P) \neq 0$ (the *Rosenbrock representation* [1.24] will sometimes be referred to as a *Rosenbrock system*, through an abuse of language).

The entries of these matrices belong to $\mathbf{A} = \mathbb{R}[s]$ (this is why [1.24] is still called a “polynomial matrix description of system Σ ”¹⁵); in [1.24], the indeterminate s is substituted by the differentiation operator ∂ for continuous-time systems, and by the shift-forward operator σ for discrete-time systems. We write:

$$\Sigma \equiv [P, Q, T, U] \quad [1.25]$$

The state-space representation [1.10] is a specific case of the Rosenbrock representation [1.24], obtained when $\xi = x$, $P(s) = sI_n - A$, $Q(s) = B$, $T(s) = C$, $U(s) = D(s)$; therefore this state-space system is symbolized by:

$$\Sigma \equiv [sI_n - A, B, C, D]$$

Similarly, [1.1] is a particular case of [1.24], obtained when $\xi = y$, $Q = N_l$, $P = D_l$, $T = I_p$, $U = 0$ and is therefore symbolized by:

$$\Sigma \equiv [D_l, N_l, I_p, 0]$$

Rosenbrock defined the following “system-matrix” (often referred to later as the “Rosenbrock matrix”) for the system $[P, Q, T, U]$:

$$\mathfrak{R} = \begin{bmatrix} P & -Q \\ T & U \end{bmatrix}$$

so that [1.24] is written as

$$\mathfrak{R} \begin{bmatrix} \xi \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} \quad [1.26]$$

¹⁵ In the literature, we often find the abbreviation PMD for *polynomial matrix description*.

(II) Decoupling zeros and invariant zeros

Using the concept of Smith zero (see section A.4.2(III)), the following definition is taken from [MAC 76, ROS 70, ROS 74, ROS 77]:

DEFINITION 25.–

i) The input-decoupling zeros (*i.d.z.*)¹⁶ of Σ are the Smith zeros of $\begin{bmatrix} P & -Q \end{bmatrix}$.

The output-decoupling zeros (*o.d.z.*) of Σ are the Smith zeros of $\begin{bmatrix} P \\ T \end{bmatrix}$.

The invariant zeros (*i.z.*) of Σ are the Smith zeros of $\mathfrak{R}$.

The input–output-decoupling zeros (*i.o.d.z.*) of Σ are those *o.d.z.* of Σ that disappear when the *i.d.z.* are eliminated, or, equivalently, they are those *i.d.z.* that disappear when the *o.d.z.* are eliminated.

Using the conventions from remark 6, notation 7 and definition 8, the sequence of hidden modes (*h.m.*) of Σ is given by:

$$\{h.m.\} = \{i.d.z.\} + \{o.d.z.\} - \{i.o.d.z.\}$$

The sequence of system zeros (*s.z.*) is given by:

$$\{s.z.\} = \{t.z.\} + \{h.m.\} \quad [1.27]$$

ii) The system poles (*s.p.*) are the Smith zeros of P .

The *i.d.z.* and *o.d.z.* are also called the non-controllable poles and the non-observable poles of Σ , respectively.

As a result of theorem 24, we have the following corollary:

COROLLARY 26.– A state-space system is controllable (respectively, observable) if and only if it has no input-decoupling zero (respectively, output-decoupling zero).

The transfer matrix of [1.24] is:

$$G = TP^{-1}Q + U \quad [1.28]$$

and definition 1 remains valid for the transmission poles and zeros. When $P(s) = sI_n - A$, the Smith zeros of P are the eigenvalues of A , and consequently,

¹⁶ See remark 2.

definition 25(ii) coincides with definition 16. Using the notations in section 1.3(II), the *i.d.z.*, *o.d.z.*, *i.o.d.z.* are the eigenvalues of $A_{\bar{e}} = \begin{bmatrix} A_{\bar{c}\bar{o}} & * \\ 0 & A_{\bar{e}o} \end{bmatrix}$, $A_{\bar{o}} = \begin{bmatrix} A_{c\bar{o}} & * \\ 0 & A_{\bar{c}o} \end{bmatrix}$ and $A_{\bar{c}\bar{o}}$ respectively, hence the following corollary:

COROLLARY 27.— *A state-space system is stabilizable (respectively, detectable) if and only if all its input-decoupling zeros (respectively, output-decoupling zeros), if any, belong to $\mathbb{C}_s$.*

On the other hand, the hidden modes are the eigenvalues of:

$$\begin{bmatrix} A_{c\bar{o}} & * & * \\ 0 & A_{\bar{c}\bar{o}} & * \\ 0 & 0 & A_{\bar{c}o} \end{bmatrix}$$

The term “invariant zeros” can be justified¹⁷ by the following result:

THEOREM 28.— *The invariant zeros of a state-space system are invariant by state feedback.*

DEMONSTRATION.— The Rosenbrock matrix of the state-space system [1.10] is:

$$\mathfrak{R}(s) = \begin{bmatrix} sI - A & -B \\ C & D(s) \end{bmatrix}$$

The Rosenbrock matrix of this system fed by the feedback control $\mathbf{u} = \mathbf{v} - K\mathbf{x}$ is:

$$\mathfrak{R}_{cl}(s) = \begin{bmatrix} sI - A + BK & -B \\ C - D(s)K & D(s) \end{bmatrix}$$

The matrices $\mathfrak{R}$ and $\mathfrak{R}_{cl}$ are right-equivalent over the ring $\mathbb{R}[s]$, as we move from $\mathfrak{R}_c$ to $\mathfrak{R}$ by carrying out elementary column operations ([BLS 17], sections 2.3.11(II) and 3.1.3(II)). They therefore have the same invariant factors and, consequently, the same Smith zeros. ■

REMARK 29.— *The transmission zeros are not invariant by state-feedback, as can be seen in the following example: the state-space system given by equations $\dot{x} = u$, $y = x + u$, has transfer function $\frac{s+1}{s}$. With the state-feedback control $u = -x + v$, the equation for the closed-loop system is $\dot{x} = -x + v$, $y = v$, and its transfer function is 1.*

¹⁷ However, this is not the initial reason for this terminology (see [MAC 76, MAC 77]).

(III) Rosenbrock relations

Rosenbrock [ROS 70, ROS 74, ROS 77] obtained the relations:

$$\{s.p.\} = \{t.p.\} + \{h.m.\} \quad [1.29]$$

$$\{t.z.\} \subset \{i.z.\} \subset \{s.z.\} \quad [1.30]$$

the inclusions of [1.30] becoming equalities if $\{h.m.\} = \emptyset$; if G is right-invertible, then:

$$\{t.z.\} + \{i.d.z.\} \subset \{i.z.\} \quad [1.31]$$

and if G is *invertible*, then:

$$\{t.z.\} + \{h.m.\} = \{i.z.\} \quad [1.32]$$

For any matrix $M \in \mathbf{A}^{q \times s}$, the minor formed of rows $\{i_1, \dots, i_k\}$ and columns $\{j_1, \dots, j_k\}$ of M is denoted by $M_{j_1, \dots, j_k}^{i_1, \dots, i_k}$, where $1 \leq i_1 < \dots < i_k \leq q$ and $1 \leq j_1 < \dots < j_k \leq s$. Assuming $\mathfrak{R}$ is of rank $l + r$ ($r \leq \min\{m, p\}$), consider the non-zero minors of order $l + r$ of the form:

$$\mathfrak{R}_{1,2,\dots,l,l+j_1,\dots,l+j_r}^{1,2,\dots,l,l+i_1,\dots,l+i_r}$$

and let ϕ be a gcd of all these minors. Rosenbrock [ROS 74] demonstrated the relation:

$$\{s.z.\} = \{\text{zeros of } \phi\} \quad [1.33]$$

where the multiplicities are taken into account, as in remark 6. A generalization of [1.30] was obtained in [BLS 97] (see [3.22] in Chapter 3, further on in this book; see also [3.21], which is a new result).

1.5. The behavioral approach

1.5.1. Controllability without control variables

If in the representation [1.24] of $\Sigma \equiv [P, Q, T, U]$ we write $\mathbf{w} = [\boldsymbol{\xi}^T, \mathbf{u}^T, \mathbf{y}^T]^T$, this system of equations is expressed in the form:

$$R.\mathbf{w} = 0 \quad [1.34]$$

where:

$$R = \begin{bmatrix} P & -Q & 0 \\ T & U & -I_p \end{bmatrix} \quad [1.35]$$

We have $R \in \mathbf{A}^{q \times k}$ ($\mathbf{A} = \mathbb{R}[s]$, $q = l + p$, $k = l + m + p$); considering the continuous-time case, in [1.34], the operator ∂ replaced the indeterminate s . It is clear that the Smith zeros of R are identical to the Smith zeros of $\begin{bmatrix} P & -Q \end{bmatrix}$. In [1.34], the control (or output) variables can no longer be distinguished from the other variables. Hence, Willems remarks [WIL 89] (surprising for a number of automation engineers) that controllability is a property of the system which does not depend on the choice of control variables (thus, from this point of view, the system under consideration is not a control system as defined in the Preface). It was seen in section 1.3 that a state-space representation using four matrices A, B, C, D is not intrinsic. The question that also arises is how to make intrinsic a polynomial matrix representation, such as the one in [1.34]. Willems [WIL 86a, WIL 86b, WIL 87, WIL 89, WIL 91] associated with [1.34] the behavior of Σ :

$$\mathfrak{B} = \{w \in W^k : R.w = 0\} \quad [1.36]$$

where W is a function space that is an $\mathbf{A}$ -module, to clarify ideas a subspace of the $\mathbf{A}$ -module $\mathcal{D}'(\mathbb{R})$ of distributions on the real line ([BLS 18], section 4.4.1), with the most typical example being $\mathcal{E}(\mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{C})$. In the case of a discrete-time system, W is an $\mathbf{A}$ -submodule of $\mathbb{R}^{\mathbb{Z}}$. We then write:

$$\mathfrak{B} = \ker_W (R\bullet) \quad [1.37]$$

where $(R\bullet)$ is the left-multiplication by the operator R . An element of $\mathfrak{B}$ is called a *trajectory* of Σ . Willems's definition of controllability of system Σ is given below ([POL 98, WIL 89], definition 5.2.2).

DEFINITION 30.— *Let $\mathfrak{B} \subset W^k$ be the behavior of Σ . This system is controllable if for any two trajectories $w^1, w^2 \in \mathfrak{B}$, there exist $t_1 > 0$ and a trajectory $w \in \mathfrak{B}$ such that*

$$w(t) = \begin{cases} w^1(t), & t < 0 \\ w^2(t - t_1), & t > t_1 \end{cases} \quad [1.38]$$

On the other hand, we have the notion *à la Willems* of an autonomous system ([WIL 89], [POL 98], definition 3.2.1):

DEFINITION 31.– Let $\mathfrak{B} \subset W^k$ be the behavior of Σ . This system is autonomous¹⁸ if for any two trajectories $w^1, w^2 \in \mathfrak{B}$, necessarily $w^1 = w^2$ whenever $w^1(t) = w^2(t)$ for all $t \leq 0$.

By abuse of language $\mathfrak{B}$ (instead of Σ) may be said to be controllable or autonomous à la Willems.

It must be checked that definition 30 is consistent with the Kalmanian definition of controllability (section 1.2(I)).

LEMMA 32.– Let $W = \mathcal{E}(\mathbb{R})$ and consider the system with equation $e(\partial)w = 0$ ($w \in W$) where $e \in \mathbf{A}$ is a non-zero polynomial. This system is controllable (à la Willems) if and only if e is a unit of $\mathbf{A}$.

DEMONSTRATION.– If e is a unit of $\mathbf{A}$, then $e(\partial)w = 0 \implies w = 0$ and $\mathfrak{B} = 0$. Thus, the system is controllable in the sense of definition 30. If $n = d^\circ(e) > 0$, then every trajectory w is determined by the initial conditions $w(0), \dot{w}(0), \dots, w^{(n-1)}(0)$ ([BLS 19], section 1.5.1). Thus, if $w^1, w^2 \in \mathfrak{B}$ do not have the same initial conditions, there does not exist a trajectory w satisfying [1.38], therefore the system is not controllable. ■

We will agree with the following definition:

DEFINITION 33.– Let W be an $\mathbf{A}$ -submodule of $\mathcal{D}'(\mathbb{R})$. This module is said to be Willemsian¹⁹ if given any two elements $w_1, w_2 \in W$, there exist two non-empty open subsets U_1, U_2 of $\mathbb{R}$ such that $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ (where $\bar{U}_i$ is the closure of U_i in $\mathbb{R}$), and an element $w \in W$ such that $w|_{U_i} = w_i|_{U_i}$ ($i = 1, 2$).

Note that the above definition is still valid if $\mathbb{R}$ is replaced by a smooth connected manifold.

LEMMA 34.– The $\mathbf{A}$ -modules $\mathcal{E}(\mathbb{R}), \mathcal{D}'(\mathbb{R}), \mathcal{E}'(\mathbb{R}), \mathcal{D}(\mathbb{R})$ (where $\mathcal{E}'(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ are, respectively, the space of compactly supported distributions and the space of compactly supported infinitely differentiable functions on the real line) are Willemsian. Any subspace of the space $\mathcal{A}(\mathbb{R})$ of analytic functions in $\mathbb{R}$ ([BLS 18], section 4.3.2), such that $\mathcal{A}(\mathbb{R})$, or the space W_0 of exponential polynomials ([BLS 17], section 3.3.3(I)) is not Willemsian.

18 The notion of autonomous system, as specified here, must not be confused with that of an autonomous differential equation $\dot{x} = f(x)$ ($f(x)$ independent of t).

19 This notion should not be confused with that of “Willems submodule relative to W ” (see [PIL 98], definition 1).

DEMONSTRATION.—

1) Let us demonstrate that $\mathcal{E}(\mathbb{R})$ is Willemsian. Let $w_1, w_2 \in \mathcal{E}(\mathbb{R})$. According to Whitney's theorem ([BLS 18], section 4.4.1(I)) there exists a non-empty open subset U_1 of $\mathbb{R}$ and a closed subset V_2 whose interior contains $\bar{U}_1$, as well as a “bump function” $\psi \in \mathcal{E}(\mathbb{R})$ such that $\psi(t) = 0$ for all $t \in U_1$ and $\psi(t) = 1$ for all $t \in \mathbb{R} - V_2$. Let $U_2 = \mathbb{R} - V_2$ and $w \in \mathcal{E}(\mathbb{R})$ be the function defined to be $w = (1 - \psi)w_1 + \psi w_2$. Then $\bar{U}_1 \cap \bar{U}_2 = \emptyset$ and $w|_{U_i} = w_i|_{U_i}$ ($i = 1, 2$).

2) The spaces $\mathcal{D}'(\mathbb{R}), \mathcal{E}'(\mathbb{R}), \mathcal{D}(\mathbb{R})$ are $\mathcal{E}(\mathbb{R})$ -modules. If W is one of them and $w_1, w_2 \in W$, take U_1, U_2 and $\psi \in \mathcal{E}(\mathbb{R})$ as above; then $w = (1 - \psi)w_1 + \psi w_2 \in W$ and $w|_{U_i} = w_i|_{U_i}$ ($i = 1, 2$).

3) The space $\mathcal{A}(\mathbb{R})$ is not an $\mathcal{E}(\mathbb{R})$ -module. If U_1, U_2 are two disjoint, non-empty open subsets of $\mathbb{R}$ and $w_1, w_2 \in \mathcal{A}(\mathbb{R})$ are such that $w_1|_{U_1} = w_2|_{U_2}$, then $w_1 = w_2$ according to the analytic continuation theorem (ibid., section 4.3.2(I)). ■

LEMMA 35.— *Let us assume $W = \mathcal{E}(\mathbb{R})$.*

1) *The following conditions are equivalent:*

- i) *the system with behavior [1.36] is controllable;*
- ii) *the non-zero invariant factors of R are all units of $\mathbb{R}[s]$.*

2) *For a system defined by a Rosenbrock representation [1.24], the above conditions are equivalent to:*

- iii) *the system has no input-decoupling zeros.*

DEMONSTRATION.— We have (ii) $\Leftrightarrow$ (iii) if the system is defined by [1.24].

Let $\Sigma = \text{diag}(e_1, \dots, e_r, 0, \dots, 0)$ be the normal form of R , where $e_i \neq 0$ ($i = 1, \dots, r$) and $e_1 \mid e_2 \mid \dots \mid e_r$ ([BLS 17], section 3.4.2(I)); we then have $r = \text{rk}(R)$. There exist polynomial matrices U, V , invertible over $\mathbb{R}[s]$, such that $URV^{-1} = \Sigma$. Consequently, $R(\partial)w = 0 \Leftrightarrow \Sigma(\partial)w' = 0$ where $w' = V(\partial)w$. It follows immediately that condition [1.38] is satisfied if and only if it is satisfied with w^1, w^2, w replaced by $V(\partial)w^1, V(\partial)w^2, V(\partial)w$ respectively. Therefore, (i) $\Rightarrow$ (ii) according to lemma 32.

Let us demonstrate that, conversely, (ii) $\Rightarrow$ (i). This is clear if $r = k$. If $r < k$, the equality $\Sigma(\partial)w' = 0$ brings into play, apart from the equations $e_i(\partial)w'_i = 0$ ($i = 1, \dots, r$), $k - r$ components $w'_{r+1}, \dots, w'_k$ of w' , which are not subject to any condition. Let $t_1 > 0$ and, for all $i \in \{r+1, \dots, k\}$, let there be any two functions $w_i^{(1)}, w_i^{(2)} \in \mathcal{E}(\mathbb{R})$; since $\mathcal{E}(\mathbb{R})$ is Willemsian (lemma 34), there exists $w'_i \in \mathcal{E}(\mathbb{R})$ such that $w'_i(t) = w_i^{(1)}(t)$ for all $t < 0$ and $w'_i(t) = w_i^{(2)}(t - t_1)$ for all $t > t_1$, therefore (ii) $\Rightarrow$ (i). ■

THEOREM 36.— *Let $W = \mathcal{E}(\mathbb{R})$. The state-space system [1.10] is controllable à la Willems (definition 30) if and only if it is completely reachable in the sense of Kalman (definition 11(ii)).*

DEMONSTRATION.— This is a consequence of theorem 24 and lemma 35. ■

With respect to autonomy (definition 31), we have the following result:

LEMMA 37.— *Let $W = \mathcal{E}(\mathbb{R})$. The system with behavior [1.36] is autonomous à la Willems if and only if R is right-regular (i.e. $r = k$).*

DEMONSTRATION.— Using the notations from the demonstration of lemma 35, we have:

$$\mathfrak{B} = V^{-1}(\partial) \ker_W(\Sigma \bullet) \cong V^{-1}(\partial) (\ker_W(\mathfrak{E} \bullet) \oplus W^{k-r})$$

where $\mathfrak{E} = \text{diag}(e_1, \dots, e_r)$. The system with behavior $\ker_W(\mathfrak{E} \bullet)$ is autonomous, while the system with behavior W^{k-r} is autonomous if and only if $k = r$. ■

REMARK 38.—

1) Conditions (i) and (ii) from lemma 35 are still equivalent if W is the space of distributions $\mathcal{D}'(\mathbb{R})$. Indeed, $\mathcal{D}'(\mathbb{R}) \supset \mathcal{E}(\mathbb{R})$, therefore the statement in lemma 32 remains valid with $W = \mathcal{D}'(\mathbb{R})$. Further, $\mathcal{D}'(\mathbb{R})$ is Willemsian (lemma 34).

2) On the other hand, conditions (i) and (ii) in lemma 35 are no longer equivalent if W is a subspace of $\mathcal{A}(\mathbb{R})$, the space of analytic functions in $\mathbb{R}$, for example $\mathcal{A}(\mathbb{R})$ or W_0 . Indeed, such a space is not Willemsian. The analytic continuation principle also implies that, in this case, any system is autonomous à la Willems.

3) Similarly, if $W = \mathcal{D}(\mathbb{R})$, consider the system determined by the operator $R = s$ identified with ∂ ; R has a unique invariant factor, which is $e(s) = s$ and which is not invertible over $\mathbb{R}[s]$. However, it follows from $\partial w = 0$ that $w = 0$ if $w \in \mathcal{D}(\mathbb{R})$, therefore $\ker_W(R \bullet) = 0$. Consequently, the system is controllable à la Willems (definition 30), although e is not a unit of $\mathbf{A}$. Lemma 32 is therefore not valid in this case.

4) With $W = \mathcal{E}(\mathbb{R})$ or $W = \mathcal{D}'(\mathbb{R})$, a system is both autonomous and controllable à la Willems if and only if $\mathfrak{B} = 0$ (where $\mathfrak{B}$ is its behavior) according to lemmas 35 and 37.

1.5.2. Observability in the behavioral approach

In behavioral formalism, the concept of observability was defined by Willems [WIL 89], and the conception of observers in this context has been addressed in [VAL 99] (see also the more complete developments in [BLU 09, BLU 10, FUH 08]).

Let $W = \mathcal{E}(\mathbb{R})$, $\mathbf{A} = \mathbb{R}[s]$, with s identified with ∂ . Let $\mathfrak{B} = \ker_W(R\bullet)$ where $R = \begin{bmatrix} R' & -R'' \end{bmatrix}$, $R' \in \mathbf{A}^{q \times k'}$, $R'' \in \mathbf{A}^{q \times k''}$, $k' + k'' = k$, such that:

$$\mathfrak{B} = \left\{ \begin{bmatrix} w' \\ w'' \end{bmatrix} \in W^k : R'w' = R''w'' \right\} \quad [1.39]$$

Since the elements of $\mathfrak{B}$ are trajectories of Σ , the w' and w'' such that $\begin{bmatrix} w' \\ w'' \end{bmatrix} \in \mathfrak{B}$ are *subtrajectories* of this system.

DEFINITION 39.—The subtrajectories w' are said to be observable from the subtrajectories w'' if every subtrajectory $w' \in W^{k'}$ can be uniquely determined from any subtrajectory $w'' \in W^{k''}$ such that $\begin{bmatrix} w' \\ w'' \end{bmatrix} \in \mathfrak{B}$.

We have the following result ([WIL 89], proposition 4.4):

THEOREM 40.— w' is observable from w'' if and only if R' is left-invertible over $\mathbf{A}$.

DEMONSTRATION.—To say w' is observable from w'' is equivalent to saying that for all $w', \tilde{w}' \in W^{k'}$, if there exists $w'' \in W^{k''}$ such that:

$$\begin{cases} R'w' = R''w'' \\ R'\tilde{w}' = R''w'' \end{cases} \Rightarrow w' = \tilde{w}'$$

Or again, for all $w' \in W^{k'}$, $R'w' = 0 \Rightarrow w' = 0$. Using the reasoning from the demonstration of lemma 35, this condition is satisfied if and only if R' admits the Smith normal form $\begin{bmatrix} I_{k'} \\ 0 \end{bmatrix}$; in other words, R' is left-invertible over $\mathbf{A}$. ■

Consider the system defined by the Rosenbrock representation [1.24]. Let ξ, u, y , be C^∞ functions and such that:

$$R \begin{bmatrix} \xi \\ u \\ y \end{bmatrix} = 0 \quad [1.40]$$

where R is given by [1.35]. Relation [1.40] can be written in the form:

$$\begin{bmatrix} P \\ T \end{bmatrix} \xi = \begin{bmatrix} Q & 0 \\ -U & I_p \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$

COROLLARY 41.— ξ is observable from $\begin{bmatrix} u \\ y \end{bmatrix}$ if and only if system [1.24] has no output-decoupling zero. In particular, if [1.24] is the state-space system [1.10], the state x is observable from $\begin{bmatrix} u \\ y \end{bmatrix}$ if and only if the system [1.10] is completely observable in the sense of Kalman (definition 11(ii)).

DEMONSTRATION.— This is a consequence of theorem 40 and definition 25(i). ■

REMARK 42.— The statement of corollary 41 becomes untrue with, for example $W = \mathbb{C}[t]$, since $(s - a)w = 0 \Rightarrow w = 0$ if $w \in \mathbb{C}[t]$ and $a \neq 0$, while $s - a$ is not invertible in $\mathbb{R}[s]$ (see infra, theorem 51).

1.6. Module of a system

1.6.1. Using modules in control theory

In 1969, Kalman was the first to introduce module theory in control theory ([KAL 69], Chapter 10), and various authors have followed his approach. A summary of this can be found in [SCR 89], section 6.

By writing:

$$W = \mathcal{E}(\mathbb{R}) \oplus \left\{ \sum_{\substack{i \in \mathbb{N} \\ \text{finite}}} a_i \delta_{t_i}^{(k_i)} \right\} \quad (a_i \in \mathbb{R})$$

and $\mathbf{A} = \mathbb{R}[s]$, in 1980, Hinrichsen and Prätzel [HIS 80] used the fact that $\ker_W(R\bullet)$ is an $\mathbf{A}$ -module if s acts on any element w of W as follows:

$$s.w = \partial w \tag{1.41}$$

(this formalizes the substitutions that have already been indicated several times). They further demonstrated that this module is a torsion module and is finitely generated if and only if R is right-regular. Similarly, Blomberg and Ylinen [BLO 83] noted in 1983 that a space of signals W such that $\mathcal{E}(\mathbb{R})$ or $\mathcal{D}'(\mathbb{R})$ is an $\mathbf{A}$ -module. They demonstrated the importance of assuming this module W to be “regular”, that is, faithful and divisible ([BLS 17], sections 2.3.2(III) and 3.3.1(II)), and “rich”, that is, containing exponential polynomials²⁰. Since $\mathbf{A}$ is a principal ideal domain, divisibility of W is equivalent to its injectivity, and the space of exponential

²⁰ This is not exactly how a *rich* space of signals is defined in [BLO 83], section 5.1, but this is indeed the property that is used in the rest of the cited work.

polynomials is the canonical $\mathbf{A}$ -cogenerator W_0 up to isomorphism (ibid., section 3.3.3(III)); the hypotheses of [BLO 83] are thus the same as assuming that W is an injective cogenerator.

In the rest of this book, unless otherwise stated, $\mathbf{A} = \mathbb{R}[s]$; the $\mathbf{A}$ -modules are left modules for the action [1.41] regarding continuous-time systems, and for the action:

$$s.w = \sigma w \quad [1.42]$$

regarding discrete-time systems. As for discrete-time systems, it is more common to use z to designate the indeterminate. It is then preferable to write $\mathbf{A} = \mathbb{R}[z]$, the action of z on any element w of W being:

$$z.w = \sigma w \quad [1.43]$$

1.6.2. The Fliessian approach

Fliess [FLI 90] reintroduced module theory in control theory in 1990, but in a very different way from what was just discussed, inherited from specialists of the “ $\mathbf{D}$ -module theory” in mathematics [KAS 95], that is, modules over a ring $\mathbf{D}$ of differential operators (possibly partial differential operators) such as $\mathbb{R}[\partial]$. Given the system of equations [1.34], consider the $\mathbf{A}$ -module:

$$M = \text{coker}_{\mathbf{A}}(\bullet R) := \mathbf{A}^{1 \times k} / (\mathbf{A}^{1 \times q} R) \quad [1.44]$$

where $\bullet R$ denotes the right-multiplication by the operator R and $\mathbf{A} = \mathbb{R}[s]$ with action [1.41] or [1.42].

We have an exact sequence ([BLS 17], section 3.1.4(I)):

$$\mathbf{A}^{1 \times q} \xrightarrow{\bullet R} \mathbf{A}^{1 \times k} \xrightarrow{\varphi} M \longrightarrow 0 \quad [1.45]$$

where φ is the canonical surjection; if $(\varepsilon_j)_{1 \leq j \leq k}$ is the canonical basis of $\mathbf{A}^{1 \times k}$, $\mathbf{w}_j = \varphi(\varepsilon_j)$ and $\mathbf{w}$ is the column with entries $\mathbf{w}_j$ ($j = 1, \dots, k$), we have relation [1.34]. In the rest of the text, if $\mathbf{w}_j$ ($j = 1, \dots, k$) are elements of an $\mathbf{A}$ -module, $[\mathbf{w}]_{\mathbf{A}}$ (or $[\mathbf{w}]$ when there is no ambiguity) denotes the $\mathbf{A}$ -module generated by the $\mathbf{w}_j$. Then, given the module $M = \text{coker}_{\mathbf{A}}(\bullet R)$, we have $M = [\mathbf{w}]_{\mathbf{A}}$ if and only if $\mathbf{w}$ satisfies the unique condition [1.34] (ibid., section 3.1.6(II)). The column $\mathbf{w}$ is a “variable” with a purely algebraic significance, contrary to the function $w : t \mapsto w(t)$ in section 1.5²¹.

²¹ It must be reiterated that $\mathbf{w}$ has no significance apart from being a column of generators of the module M . This “variable” $\mathbf{w}$ satisfies the (only one) relation $R\mathbf{w} = 0$ but the $\mathbf{w}_j$ belong

Given an $\mathbf{A}$ -module W , the shift from $M = \text{coker}_{\mathbf{A}}(\bullet R)$ to $\mathfrak{B} = \ker_W(R\bullet)$ is classic. It is carried out through Malgrange isomorphism [MAL 63] which, in the present context, can be reformulated as follows ([BLS 17], section 3.3.2): let $\mathbf{h}_W$ be the functor $\text{Hom}_{\mathbf{A}}(-, W) : \mathbf{A}\text{Mod} \rightarrow \mathbf{Ab}$ where $\mathbf{A}\text{Mod}$ and $\mathbf{Ab}$ are, respectively, the category of the left $\mathbf{A}$ -modules and the category of the abelian groups. Given the free module $\mathbf{A}^{1 \times k}$, $\mathbf{h}_W(\mathbf{A}^{1 \times k})$ is identified with W^k ; if $R \in \mathbf{A}^{q \times k}$, then $\bullet R : \mathbf{A}^{1 \times q} \rightarrow \mathbf{A}^{1 \times k}$ is a morphism of $\mathbf{A}$ -modules, and $\mathbf{h}_W(\bullet R)$ is identified with the left multiplication $R\bullet : W^k \rightarrow W^q$. These identifications are canonical (ibid.). Upon composing by $\mathbf{h}_W$, the exact sequence [1.45] yields the exact sequence:

$$W^q \xleftarrow{R\bullet} W^k \xleftarrow{\iota} \mathbf{h}_W(M) \longleftarrow 0$$

where $\iota = \mathbf{h}_W(\varphi)$ is the canonical injection. The Malgrange isomorphism is then canonical and is written $\text{Hom}_{\mathbf{A}}(M, W) \xrightarrow{\sim} \mathfrak{B} : \phi \mapsto w, w_j = \phi(\mathbf{w}_j) (j = 1, \dots, k)$, so that $\mathbf{h}_W(M)$ is canonically identified with $\mathfrak{B}$ ²².

1.6.3. Characterization in terms of modules of controllability and observability

By characterizing a system by its module $M = \text{coker}_{\mathbf{A}}(\bullet R)$, rather than by its behavior, $\mathfrak{B} = \ker_W(R\bullet)$ (with $W = \mathcal{E}(\mathbb{R})$), Fliess [FLI 92a] was led to define the system's controllability and autonomy directly from the properties of its module M , as a consequence of the following result:

THEOREM 43.— *Let $R \in \mathbf{A}^{q \times k}$ where $\mathbf{A} = \mathbb{R}[s]$, $W = \mathcal{E}(\mathbb{R})$, and the action [1.41] [FLI 92a].*

1) *The following conditions are equivalent:*

i) *the system with behavior $\mathfrak{B} = \ker_W(R\bullet)$ is controllable à la Willems (definition 30);*

ii) *the module M is free.*

2) *The following conditions are equivalent:*

i') *the system is autonomous à la Willems (definition 31);*

ii') *the module M is torsion (i.e. $M = \mathcal{T}(M)$), where $\mathcal{T}(M)$ denotes the torsion submodule of M : see [BLS 17], section 3.1.10(II)).*

to the *cokernel* of $(\bullet R)$. On the contrary, w , which satisfies $Rw = 0$, and thus *formally* has the same relation as $\mathbf{w}$, belongs to the *kernel* of $(R\bullet) : W^k \rightarrow W^q$. In [OBE 20, POM 01], the “variables” like $\mathbf{w}$ are called “observables” (like in quantum mechanics where the observables are operators – position and momentum operators, etc.), but we will avoid the use of this terminology, which can lead to confusion in control theory, when discussing the questions of observability.

²² The “variable” or “observable” $\mathbf{w}_j$ is the operator $\text{Hom}_{\mathbf{A}}(M, W) \longrightarrow W : \phi \mapsto w_j$.

DEMONSTRATION.— This results from lemmas 35 and 37. ■

According to Fliess [FLI 90], consider a system characterized by its module $M = \text{coker}_{\mathbf{A}}(\bullet R)$, and assume that among the generators of M , specific variables, called “input” and “output” variables, have been distinguished, respectively, $\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{y}_1, \dots, \mathbf{y}_p$ (in section 2.1.1, it will be seen that the submodule $[\mathbf{u}]_{\mathbf{A}}$ generated by $\mathbf{u}_1, \dots, \mathbf{u}_m$ cannot be just any submodule). Let $[\mathbf{u}, \mathbf{y}]_{\mathbf{A}}$ be the submodule generated by all these variables. The “control system” $(M, \mathbf{u}, \mathbf{y})$ (see Preface) is said to be *observable* if all variables of the system are $\mathbf{A}$ -linear combinations of $\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{y}_1, \dots, \mathbf{y}_p$; in other words:

DEFINITION 44.— The “control system” $(M, \mathbf{u}, \mathbf{y})$ is observable if $M = [\mathbf{y}, \mathbf{u}]_{\mathbf{A}}$ [FLI 90].

1.7. The formalism of algebraic analysis

1.7.1. Nature of algebraic analysis

In general, algebraic analysis uses algebra to resolve analytic problems. The central idea is almost as old as the creation of infinitesimal calculus by Newton and Leibniz in the 17th century. And all the preceding formalisms, in some way, fall under algebraic analysis. However, this term only appeared in mathematics toward the late 1950s, introduced by M. Sato, when he created the theory of hyperfunctions ([BLS 18], sections 4.4.2, 5.4.5 and 5.4.6); since then, algebraic analysis covers algebraic methods to study systems of linear differential operators (often with partial derivatives).

1.7.2. Oberst’s contribution and its consequences

(I) The use of algebraic analysis

It came about in a very important article by Oberst [OBE 90], published in 1990 (thus, contemporaneous with [FLI 90]), which discussed multidimensional systems such as [1.34] with $\partial = (\partial_1, \dots, \partial_n)$, $\partial_i = \partial/\partial x_i$ and $W = \mathcal{E}(\mathbb{R}^n)$ or $W = \mathcal{D}'(\mathbb{R}^n)$.

Multidimensional systems, as such, are not the subject of this book²³, although they have been studied in many works, too many to cite here (see for example [POM 01, ZER 00a] and the references given in these). Nevertheless, the general process in [OBE 90] is applicable to many other types of systems and it is therefore fundamental.

²³ See [BLS 17], Preface, note 2.

Oberst essentially highlighted the fact that the information about the behavior [1.37] and the information on the $\mathbf{A}$ -module [1.44] of a system are equivalent when W is an injective cogenerator (compare with section 1.6.1, and see sections A.3.1 and A.3.2 for further details).

It was then seen that injectivity of W is a condition that can often be overcome [BLS 09].

(II) Application to controllability

With $\mathbf{A} = \mathbb{R}[s]$ and the action [1.41] or [1.43], we have the following result (see sections A.3.3 and A.3.4):

THEOREM 45.— *Let W be a cogenerator $\mathbf{A}$ -module (i.e. satisfying the equivalent conditions of theorem 325 in section A.3.4(II)) and $R_1 \in \mathbf{A}^{q_1 \times k_1}$. The following conditions are equivalent:*

- i) *the module $M = \text{coker}_{\mathbf{A}}(\bullet R_1)$ is free;*
- ii) *the module M is torsion-free;*
- iii) *there exists a matrix $R_2 \in \mathbf{A}^{k_1 \times k_2}$, which may be chosen to be left-invertible, such that $\mathfrak{B}_1 = \mathfrak{B}_2$ where $\mathfrak{B}_1 = \ker_W(R_1 \bullet)$ and:*

$$\mathfrak{B}_2 = \text{im}_W(R_2 \bullet) := \{w \in W^{k_1} : \exists v \in W^{k_2}, w = R_2 v\}$$

DEMONSTRATION.— See theorem 327 in section A.3.4(III) (like any principal ideal domain, $\mathbb{R}[s]$ is Bézout; further, it is Noetherian, therefore W is cogenerator for $\mathbf{A}\mathbf{Mod}$ if and only if W is cogenerator for the subcategory of finitely presented modules $\mathbf{A}\mathbf{Mod}^{fp}$). ■

The following definition is from Willems [WIL 89]:

DEFINITION 46.— *A behavior $\mathfrak{B}_1 = \ker_W(R_1 \bullet)$, satisfying the conditions of theorem 45 is said to have an image representation, called observable if R_2 is left-invertible²⁴.*

REMARK 47.— *A system whose behavior admits an observable image representation is sometimes said to be strongly controllable [ROC 97] (this syntagma may also have other meanings in other contexts). To the exact sequence:*

$$0 \longleftarrow \mathfrak{B} \xleftarrow{R_2 \bullet} \mathfrak{Z} \longleftarrow 0$$

where $\mathfrak{Z} \cong W^{k_2}$ there corresponds the exact sequence:

$$0 \longrightarrow M \xrightarrow{\bullet R_2} [\mathbf{z}] \longrightarrow 0$$

²⁴ This term is justified by theorem 51 given further on in the text.

with $\mathbf{h}_W([\mathbf{z}]) = \mathbf{3}$, $[\mathbf{z}] \cong \mathbf{A}^{1 \times k_2}$. Therefore, $\bullet R_2 : M \rightarrow [\mathbf{z}]$ is an $\mathbf{A}$ -isomorphism. By identification, the k_2 components of $\mathbf{z}$ constitute a basis of the free $\mathbf{A}$ -module M , called a flat output of the system [FLI 95].

COROLLARY 48.— *Let W be a Willemsian $\mathbf{A}$ -module, cogenerator for ${}_{\mathbf{A}}\mathbf{Mod}^{fp}$ (with the action [1.41]). Then, for any matrix $R \in \mathbf{A}^{q \times k}$, the system with behavior $\mathfrak{B} = \ker_W(R\bullet)$ is controllable à la Willems (definition 30) if and only if $\mathfrak{B}$ admits an observable image representation.*

DEMONSTRATION.— This is a consequence of the demonstration of lemma 35, remark 38 and theorem 45. ■

EXAMPLE 49.—

1) *The Willemsian $\mathbf{A}$ -module $\mathcal{D}(\mathbb{R})$ is not divisible. Indeed, there exists a function $v \in \mathcal{D}(\mathbb{R})$ satisfying the following conditions: $v(t) \geq 0$ ($t \in \mathbb{R}$), $v(t) = 0$ ($v \notin [0, 1]$), $\int_{-\infty}^{+\infty} v(t) dt = 1$. But then equation $\partial w = v$ (with unknown w) does not admit any solution in $\mathcal{D}(\mathbb{R})$.*

2) *Therefore, the $\mathbf{A}$ -module $\mathcal{D}(\mathbb{R})$ is not injective and, consequently, nor is $\mathcal{E}(\mathbb{R}) \oplus \mathcal{D}(\mathbb{R})$ ([BLS 17], section 3.3.1(II)).*

3) *On the other hand, the Willemsian $\mathbf{A}$ -module $\mathcal{E}(\mathbb{R})$ is cogenerator, therefore $\mathcal{E}(\mathbb{R}) \oplus \mathcal{D}(\mathbb{R})$ is a Willemsian cogenerator $\mathbf{A}$ -module and corollary 48 is applicable with $W = \mathcal{E}(\mathbb{R}) \oplus \mathcal{D}(\mathbb{R})$.*

4) *For the space of compactly supported distributions $\mathcal{E}'(\mathbb{R})$, it is possible to have the same reasoning as for $\mathcal{D}(\mathbb{R})$: this Willemsian $\mathbf{A}$ -module is neither injective nor cogenerator.*

5) *Let $R_1 = s$, $W = \mathcal{D}(\mathbb{R})$. According to remark 38(3)), W is not a cogenerator $\mathbf{A}$ -module and $\mathfrak{B}_1 := \ker_W(R_1\bullet) = \{0\}$, therefore, the corresponding system is controllable à la Willems (definition 30). We have $\mathfrak{B}_1 = \mathfrak{B}_2$ where $\mathfrak{B}_2 := \text{im}_W(R_2\bullet)$, R_2 denoting the empty matrix (element of $\mathbf{A}^{0 \times 0}$: see [BKI 70], Chapter 2, section 10.1). Further, $(R_2\bullet) : W^0 = \{0\} \rightarrow W^0 = \{0\}$ is injective, therefore $\mathfrak{B}_1$ admits an injective image representation. On the other hand, $M := \text{coker}_{\mathbf{A}}(\bullet R_1) \cong \mathbf{A}/\mathbf{A}s$ is a non-zero torsion $\mathbf{A}$ -module, therefore, conditions (i) and (ii) of theorem 45 are not satisfied.*

Let $R \in \mathbf{A}^{q \times k}$ ($\mathbf{A} = \mathbb{R}[s]$), $M = \text{coker}_{\mathbf{A}}(\bullet R)$, $\mathfrak{B} = \ker_W(R\bullet)$ where W is an injective $\mathbf{A}$ -module for ${}_{\mathbf{A}}\mathbf{Mod}^{fp}$. The quotient $M/\mathcal{T}(M)$ of M by its torsion submodule is torsion-free. The exact sequence in ${}_{\mathbf{A}}\mathbf{Mod}^{fp}$:

$$0 \longrightarrow \mathcal{T}(M) \longrightarrow M \longrightarrow M/\mathcal{T}(M) \longrightarrow 0 \quad [1.46]$$

is split, namely:

$$M = \mathcal{T}(M) \oplus F \quad [1.47]$$

where $F \cong M/\mathcal{T}(M)$, and F is free because, as remarked earlier, the ring $\mathbb{R}[s]$ is a principal ideal domain ([BLS 17], section 3.4.1(III)). To this sequence there corresponds, through the exact functor $\mathbf{h}_W = \text{Hom}_{\mathbf{A}}(-, W)$, the split exact sequence ([ZER 00a], section 1.3.1):

$$0 \longleftarrow \mathfrak{B}_a \longleftarrow \mathfrak{B} \longleftarrow \mathfrak{B}_c \longleftarrow 0 \quad [1.48]$$

where (by abuse of language):

$$\mathfrak{B}_c \subset \mathfrak{B} \text{ is controllable} \quad [1.49]$$

$$\mathfrak{B}_a \cong \mathfrak{B}/\mathfrak{B}_c \text{ is autonomous} \quad [1.50]$$

and:

$$\mathfrak{B} \cong \mathfrak{B}_c \oplus \mathfrak{B}_a \quad [1.51]$$

REMARK 50.–

1) Since $\mathcal{T}(M)$ and $M/\mathcal{T}(M)$ are two finitely presented $\mathbf{A}$ -modules, there indeed exist matrices $R_c \in \mathbf{A}^{q_c \times k}$ and $R_a \in \mathbf{A}^{q_a \times k}$ such that $\mathfrak{B}_c = \ker_W(R_c \bullet)$ and $\mathfrak{B}_a = \ker_W(R_a \bullet)$.

2) If $\mathbf{A}$ is a left coherent Ore domain (section A.3.4), it will again be said that a behavior $\mathfrak{B}_c$ admitting an image representation (respectively, of the form $\mathbf{h}_W(T)$ where $T \in {}_{\mathbf{A}}\mathbf{Mod}^{fp}$ is a torsion $\mathbf{A}$ -module) is controllable (respectively, autonomous). If the $\mathbf{A}$ -module W is injective for ${}_{\mathbf{A}}\mathbf{Mod}^{fp}$, the exactness of the sequence [1.46] results in the exactness of the sequence [1.48], in the same manner as [1.49] and [1.50]. On the other hand, in general the exact sequences [1.46] and [1.48] are no longer split, in which case the isomorphism [1.51] is no longer valid.

3) Nonetheless, with the definitions given in (2) above, since the $\mathbf{A}$ -module W is assumed to be injective for ${}_{\mathbf{A}}\mathbf{Mod}^{fp}$, the decomposition [1.51] remains valid if the ring $\mathbf{A}$ is left semi-hereditary ([CAR 56], Chapter I, section 6) and left- and right-Ore ([BLS 17], section 3.1.10(II)). Indeed, in this case, any finitely generated $\mathbf{A}$ -module M admits a direct sum decomposition [1.47], where F is a projective submodule (being the direct sum of a finite number of modules, each of which is projective, being isomorphic with a finitely generated left ideal ([CAR 56], Chapter I, proposition 6.1)).

(III) Application to observability

The result below is a direct consequence of the definition of a cogenerator:

THEOREM 51.— *Let $\mathbf{A} = \mathbb{R}[s]$ (or, more generally, let $\mathbf{A}$ be a left coherent ring), W be an $\mathbf{A}$ -module, the behavior be that in [1.39] and let there be the following conditions:*

- i) w' is observable from w'' ;
- ii) the matrix R' is left-invertible over $\mathbf{A}$.

Then (ii) $\Rightarrow$ (i). Reciprocally, (i) $\Rightarrow$ (ii) if W is cogenerator for $\mathbf{A}\text{Mod}^{fp}$.

DEMONSTRATION.— (ii) $\Rightarrow$ (i): If R' is left-invertible over $\mathbf{A}$, there exists a matrix S such as $SR' = I_{k'}$, therefore for any $w' \in W^{k'}$, $SR'w' = w'$, and $R'w' = 0 \Rightarrow w' = 0$.

(i) $\Rightarrow$ (ii): If $\ker_W(R'\bullet) = 0$, using the notations in section A.3.2(V), we have $(\mathbf{A}^{1 \times q} R')^\perp = 0$, therefore $(\mathbf{A}^{1 \times q} R')^{\perp\perp} = \mathbf{A}^{1 \times k'}$, and because W is cogenerator, $(\mathbf{A}^{1 \times q} R')^{\perp\perp} = \mathbf{A}^{1 \times q} R'$. Consequently, $(\bullet R')$ is surjective, and R' is left-invertible, according to lemma 318. ■

REMARK 52.— *The content of remark 42 can be explained by the fact that $W = \mathbb{C}[t]$ is not cogenerator (see example 311(iii) in section A.3.2(V)). It will be noted that it is not necessary to assume that W is an injective cogenerator for the implication (i) $\Rightarrow$ (ii) from theorem 51 to be valid.*

Observability can be reformulated using the following considerations [BLU 09]: let $P \in \mathbf{A}^{k'' \times k}$ and $(P\bullet) : W^k \rightarrow W^{k''} : \begin{bmatrix} w' \\ w'' \end{bmatrix} \mapsto w''$. The matrix P determines the restriction $(P\bullet)|_{\mathfrak{B}} : \mathfrak{B} \rightarrow W^{k''}$ (where $\mathfrak{B} = \ker_W(R\bullet)$) and the image:

$$(P\bullet)(\mathfrak{B}) = \{Pw : w \in W^k, Rw = 0\}$$

Further, $w \in \ker_W((P\bullet)|_{\mathfrak{B}})$ if and only if $w \in \mathfrak{B}$ and $Pw = 0$. Therefore:

$$\ker_W((P\bullet)|_{\mathfrak{B}}) = \left\{ w \in W^k : \begin{bmatrix} R \\ P \end{bmatrix} w = 0 \right\}$$

which leads to the following definition from Blumthaler and Oberst [BLU 09]:

DEFINITION 53.— $\mathfrak{B}$ is observable from $(P\bullet)(\mathfrak{B})$ if $\ker_W((P\bullet)|_{\mathfrak{B}}) = \{0\}$.

As above, the following result is obtained:

THEOREM 54.– Let $\mathbf{A} = \mathbb{R}[s]$ (or, more generally, let $\mathbf{A}$ be a left coherent ring), W be an $\mathbf{A}$ -module, let the behavior be $\mathfrak{B} = \ker_W(R\bullet)$ and let there be the following conditions:

- i) $\mathfrak{B}$ is observable from $(P\bullet)(\mathfrak{B})$;
- ii) the matrix $\begin{bmatrix} R \\ P \end{bmatrix}$ is left-invertible over $\mathbf{A}$.

Then (ii) $\Rightarrow$ (i). Reciprocally, (i) $\Rightarrow$ (ii) if W is cogenerator for $\mathbf{A}\mathbf{Mod}^{fp}$.

(IV) $\mathbf{A}_T$ -systems

If the ring $\mathbf{A}$ is not a Bézout domain, the conditions stated in theorem 45 are no longer equivalent and we have the following generalization, motivated by the procedure adopted in [BLU 09], and most recently in [OBE 20], Chapter 9: let us assume that $\mathbf{A}$ is a coherent integral domain, let $T = \mathbf{A} - U$ where U is a union of maximal ideals of $\mathbf{A}$ and let us write $\mathbf{A}_T = T^{-1}\mathbf{A}$ (see section A.3.6). An $\mathbf{A}$ -module $M \in \mathbf{A}\mathbf{Mod}^{fp}$, $M = \text{coker}_{\mathbf{A}}(\bullet R)$, determines the $\mathbf{A}_T$ -module $M_T = \mathbf{A}_T \otimes_{\mathbf{A}} M$. Since $\mathbf{A}$ is an integral domain, the canonical map $\mathbf{A} \rightarrow \mathbf{A}_T$ is injective ([BLS 17], section 3.1.9(I)) and $\mathbf{A}$ can be embedded in $\mathbf{A}_T$; for any $a \in \mathbf{A}$, we therefore have $a/1 = a$ and with this notation, it is possible to write $M_T = \text{coker}_{\mathbf{A}_T}(\bullet R)$. Similarly, the $\mathbf{A}$ -module W determines the $\mathbf{A}_T$ -module $W_T = \mathbf{A}_T \otimes_{\mathbf{A}} W$. An element of W (i.e. a signal) is said to be T -negligible if there exists $t \in T$ such that $tw = 0$ ([BLU 09], theorem and definition 2.15; [OBE 20], definition and corollary 9.3.4). If $\varphi_T : W \rightarrow W_T$ is the canonical surjection, then the set of T -negligible signals is $\ker(\varphi_T)$ (see [A.62], section A.3.6).

If W is an injective cogenerator for $\mathbf{A}\mathbf{Mod}^{fp}$, then W_T is an injective cogenerator for $\mathbf{A}_T\mathbf{Mod}^{fp}$ (see section A.3.6, theorem 343). Let us write $\mathfrak{B}_T = \ker_{W_T}(R\bullet)$.

DEFINITION 55.– If $\mathfrak{B}$ is the behavior of the system Σ , then $\mathfrak{B}_T$ is said to be the behavior of the $\mathbf{A}_T$ -system Σ_T . The latter is interpreted as the system Σ such as it is seen when the T -negligible signals going through it are abstracted (according to [A.63], section A.3.6).

THEOREM-DEFINITION 56.– Let W be an injective cogenerator $\mathbf{A}$ -module and let $R_1 \in \mathbf{A}^{q_1 \times k_1}$.

1) The following conditions are equivalent:

- i) the $\mathbf{A}_T$ -module $M_{1T} = \text{coker}_{\mathbf{A}_T}(\bullet R_1)$ is torsion-free;
- ii) the matrix R_1 is right-invertible over $\mathbf{A}_T$;

iii) there exists a matrix $R_2 \in \mathbf{A}^{k_1 \times k_2}$ such that $(\mathfrak{B}_1)_T = (\mathfrak{B}_2)_T$ where $(\mathfrak{B}_1)_T = \ker_{W_T}(R_1 \bullet)$ and:

$$(\mathfrak{B}_2)_T = \text{im}_{W_T}(R_2 \bullet) := \left\{ w \in W_T^{k_1} : \exists v \in W_T^{k_2}, w = R_2 v \right\}$$

2) The system Σ , with behavior $\mathfrak{B}_1 = \ker_W(R_1 \bullet)$ (or with module $M_1 = \text{coker}_{\mathbf{A}}(\bullet R_1)$) is said to be T -controllable (or the $\mathbf{A}_T$ -system Σ_T is said to be controllable) if one of the above equivalent properties is satisfied.

DEMONSTRATION.— See theorem 327 in section A.3.4(III). ■

Defining a *strongly T -controllable* system (in the sense of remark 47) is straightforward and left to the reader.

REMARK 57.—

1) Assertion (ii) in theorem-definition 56 signifies that there exists $R_1^\dagger \in \mathfrak{M}(\mathbf{A}_T)$ ($\mathfrak{M}(\mathbf{A}_T)$ denoting the set of matrices having their entries in $\mathbf{A}_T$: see section A.4) such that $R_1 R_1^\dagger = I$. This right-inverse $R_1^\dagger$ is generally non-unique.

2) If $\mathbf{A}$ is a principal ideal domain, then so is $\mathbf{A}_T$ ([BLS 17], section 3.1.9(II)).

The same hypotheses and same notations as in theorem 56 lead to the following corollary-definition:

COROLLARY-DEFINITION 58.— Let W be an injective cogenerator $\mathbf{A}$ -module, $R \in \mathbf{A}^{q \times k}$, $P \in \mathbf{A}^{k'' \times k}$, $(\mathfrak{B})_T = \ker_{W_T}(R \bullet)$.

1) The following conditions are equivalent:

i) $\ker_W((P \bullet)|_{\mathfrak{B}_T}) = \{0\}$;

ii) the matrix $\begin{bmatrix} R \\ P \end{bmatrix}$ is left-invertible over $\mathbf{A}_T$.

2) The system Σ with behavior $\mathfrak{B}$ is said to be T -observable (or the $\mathbf{A}_T$ -system Σ_T is said to be observable) from $(P \bullet)(\mathfrak{B}_T)$ if one of the above equivalent properties is satisfied.

REMARK 59.— With $\mathbf{A} = \mathbb{R}[s]$, let $\mathbb{D}$ be a part of the complex plane, stable under complex conjugation $\alpha + i\omega \mapsto \alpha - i\omega$. If T is the set of non-zero polynomials having all their roots in $\mathbb{D}$, we have $T = \mathbf{A} - U$ where:

$$U = \bigcup_{z \in \mathbb{D}} (s - z) \quad [1.52]$$

with $\tilde{\mathbb{D}} =: \mathbb{C} - \mathbb{D}$, and every ideal $(s - z) := \mathbf{A}(s - z)$ is a maximal ideal in $\mathbf{A}$. It will sometimes be convenient to say that the system with behavior $\mathfrak{B}$ is $\mathbb{D}$ -controllable (respectively, $\mathbb{D}$ -observable from $(P\bullet)(\mathfrak{B}_T)$) when it is T -controllable (respectively, T -observable from $(P\bullet)(\mathfrak{B}_T)$). If $\mathbb{D} = \emptyset$, T is the group of units of $\mathbb{R}[s]$, then $\mathbf{A}_T = \mathbf{A}$ and the notions of T -controllability and T -observability coincide with the usual notions of controllability and observability, respectively. If T is the set of polynomials having all their roots in $\mathbb{C}_s$, a T -controllable (respectively, T -observable system from $(P\bullet)(\mathfrak{B}_T)$) is said to be stabilizable (respectively, detectable). If $\mathbb{D} = \mathbb{C}$, then $\mathbf{A}_T = \mathbb{R}(s)$; this case is possible but not of great interest. With $W = \mathcal{E}(\mathbb{R})$, the set of T -negligible signals is the complex vector space:

$$\ker(\varphi_T) = \bigoplus_{\lambda \in \mathbb{D}, \alpha \in \mathbb{Z}_+} \mathbb{C} e_{-\lambda} \mathbf{m}^\alpha$$

where $e_{-\lambda} : t \mapsto e^{\lambda t}$ and $\mathbf{m} : t \mapsto t$, hence $(e_{-\lambda} \mathbf{m}^\alpha)(t) = e^{\lambda t} t^\alpha$.