Chapter 1

Some Simple Examples

1.1. Introduction

In this chapter, we give three simple examples of mechanical models in the three domains that are concerned in this book: friction problems (section 1.2), impact problems (section 1.3) and friction problems in the stochastic context (section 1.4). Each of them treats a simple mechanical system, a mass with one degree of freedom resting on a support plane. This chapter constitutes an informal introduction to the rest of the chapters where the theory will be developed with different application examples of the work.

1.2. Frictions

1.2.1. *Coulomb's law*

We recall Coulomb's friction law: we consider a solid resting on a rough ground plane, which exerts a reaction $\vec{R}$ on that solid. This reaction decomposes itself on a normal component $\vec{N}$, perpendicular to the ground, and a tangent component $\vec{T}$ (see Figure 1.1). As long as the ratio T/N does not exceed a certain limit f_0, there is adherence and the solid remains at rest. Once the value is reached, there is a slip, we have $T/N = f_0$ and the tangential force is opposed to the relative velocity between the solid and the ground. We note the adherence condition being that the vector $\vec{R}$ remains in a cone. The value f_0 depends on the nature of the ground and the solid. We assume here that the dynamic friction coefficient (in slip phase or dynamic phase) is equal to the static friction coefficient (in adherence phase or static phase). Occasionally, this static coefficient of friction is supposed to be larger than the dynamic, which can be numerically dangerous (see section 7.8). For more details, we can consult section 8.5 of [GIE 85].

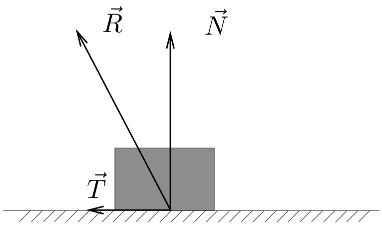


Figure 1.1. *Coulomb's friction*

We now assume that the solid moves along an axis and we note x the abscissa of the solid. We can then assume the normal of the force N to be constant. In this case, the cone transforms into an interval: the tangential force is named g and the velocity of the solid is $\dot{x}$. We introduce the function sign defined by:

$$\operatorname{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases} \tag{1.1}$$

Then there is a number $\alpha > 0$ such that:

$$\text{if } \dot{x} = 0, g \in [-\alpha, \alpha], \tag{1.2a}$$

$$\text{if } \dot{x} \neq 0, g = -\alpha \operatorname{sign}(\dot{x}). \tag{1.2b}$$

We remark that [1.2] is not equivalent to $g = -\alpha \operatorname{sign}(\dot{x})$, which implies that in the static phase, we have $g = 0$! (see section 1.2.2). Nevertheless, we can rewrite [1.2] under a more condensed form, by introducing a multivalued operator, meaning an application of $\mathbb{R}$ in the set of $\mathbb{R}$ parts. Let us then define the operator σ by:

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0. \end{cases} \tag{1.3}$$

We can also see it as part of $\mathbb{R}^2$, represented in Figure 1.2. We then have:

$$g \in -\alpha\sigma(\dot{x}). \tag{1.4}$$

We next identify the graph of $\mathbb{R}^2$ notions and of multivalued operators on $\mathbb{R}$. We will then discuss the σ graph defined by [1.3]. We note that in the dynamic phase (when $\dot{x}$ is non-zero), the force g is equal to $-\alpha \operatorname{sign}(\dot{x})$ and that in the static phase (when $\dot{x}$ is null), g belongs to $[-\alpha, \alpha]$. We will return to the graphs in more detail in Chapter 2.

Let us assume now that the studied solid has a mass m, an external force F acting on it and that we are given two initial conditions:

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0. \tag{1.5a}$$

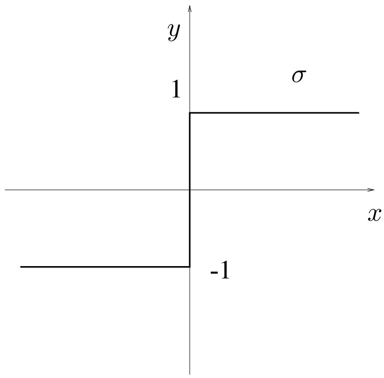


Figure 1.2. *The operator σ*

The fundamental principle of dynamics gives:

$$m\ddot{x} = F + g. \tag{1.5b}$$

1.2.2. *Differential equation with univalued operator and usual sign*

Before studying a differential inclusion with Coulomb's law friction force in section 1.2.3, let us now look at what happens if we choose to define the function g, thanks to the "usual" sign defined by [1.1], meaning:

$$g(t) = -\alpha \operatorname{sign}(\dot{x}(t)). \tag{1.6}$$

We will see that the problem formed by equations [1.5] and [1.6] equivalent to:

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{1.7a}$$

$$m\ddot{x}(t) + \alpha \operatorname{sign}(\dot{x}(t)) = F(t), \tag{1.7b}$$

is ill-posed and that the analytical resolution of this simple problem does not provide the solution! We note here that by setting $u_0 = \dot{x}(0)$ and $u = \dot{x}$, [1.7] is equivalent to:

$$u(0) = u_0, \tag{1.8a}$$

$$\forall t \in [0, T], \quad \dot{u}(t) + \frac{\alpha}{m} \operatorname{sign}(u(t)) = \frac{1}{m} F(t). \tag{1.8b}$$

To simplify, we assume that

$$m = \alpha = 1. \tag{1.9}$$

We then have the differential equation:

$$u(0) = u_0, \tag{1.10a}$$

$$\forall t \in [0, T], \quad \dot{u}(t) + \operatorname{sign}(u(t)) = F(t). \tag{1.10b}$$

Let us now consider a special case:

$$u_0 = 0, \quad F(t) = t. \tag{1.11}$$

1) We will first assume – which is physically well conceived – that the mass is at rest for a certain amount of time, which is necessary for obtaining an external force large enough to move it. Let us assume then:

$$\exists T > 0\colon \quad \forall t \in]0, T], \quad u(t) = 0, \tag{1.12}$$

Hence, $\dot{u} = t$ in $[0, T]$ and then $u(t) = t^2/2 > 0$, as soon as $t > 0$. This is contradictory to our starting hypothesis [1.12] (and against intuition).

2) Let us then try the other hypothesis:

$$\forall T > 0, \quad \exists t_0 \in]0, T]\colon \quad u(t_0) \neq 0. \tag{1.13}$$

We note:

$$\varepsilon = \operatorname{sign}(u(t_0)) \in \{-1, 1\}. \tag{1.14}$$

We can set T such that:

$$T < \frac{1}{2}. \tag{1.15}$$

We define t_1, the smallest real number of $[0, t_0[$ such that u is the sign of $u(t_0)$ on $]t_1, t_0]$. By continuity,

$$u(t_1) = 0. \tag{1.16}$$

We therefore have by integration of [1.10b] on $[t_1, t_0]$:

$$u(t_0) - u(t_1) + \varepsilon(t_0 - t_1) = \frac{1}{2}\left(t_0^2 - t_1^2\right),$$

Then,

$$u(t_0) = \frac{1}{2}\left(t_0^2 - t_1^2\right) - \varepsilon(t_0 - t_1) = (t_0 - t_1)\left(\frac{1}{2}(t_0 + t_1) - \varepsilon\right),$$

and therefore,

$$\operatorname{sign}(u(t_0)) = \frac{1}{2}(t_0 + t_1) - \varepsilon.$$

We still have two cases: either $\varepsilon = 1$ and from [1.15]

$$\operatorname{sign}(u(t_0)) = \frac{1}{2}(t_0 + t_1) - 1 < t_0 - 1 < \frac{1}{2} - 1 = -\frac{1}{2},$$

which contradicts [1.14]. Thus, $\varepsilon = -1$ and

$$\mathrm{sign}(u(t_0)) = \frac{1}{2}(t_0 + t_1) + 1 > 1,$$

which contradicts [1.14].

In conclusion, none of these hypotheses are compatible with the model, yet they correspond to what we expect as possible answers to the problem. The explanation can be found in the expression of an incorrect sign. Besides, the calculation of the friction force with this "usual" expression of the sign is not correct from a physical point of view: for a mass at rest, the friction force would be null!

Another question arises: is it possible to define for the usual sign simple numerical schemes that would provide reasonable solutions, at least from a physical point of view? Let us use the same example, the same expression of the sign and test the construction of numerical schemes.

1) Let us start with an explicit Euler numerical scheme (see, e.g. [BAS 03, CRO 84, SCH 01]) for problems [1.10]–[1.11]. By discretizing, we then introduce the sequences $u_n \simeq u(t_n)$, and $t_n = nh$, for $n \in \{0, ..., N\}$, with $h = T/N$ as the time step (chosen constant) and $u_0 = u(0) = 0$. We get $u_0 = 0$ and

$$\forall n \in \{0, ..., N-1\}, \quad \frac{u_{n+1} - u_n}{h} + \mathrm{sign}(u_n) = nh, \tag{1.17}$$

or finally

$$\forall n \in \{0, ..., N-1\}, \quad u_{n+1} = u_n + h\,(hn - \mathrm{sign}(u_n)). \tag{1.18}$$

Figure 1.3 shows the results obtained for two values of h. Some oscillations emerge; at each time value, the function approached changes sign and it can no longer remain at 0. We also note that the amplitude of the oscillations seem to decrease with h. This issue can be demonstrated in the following way: we generally assume that:

$$u_0 \in [-2h, 2h] \text{ and } \forall t \in [0, T], \quad |F(t)| \leq 1. \tag{1.19}$$

Then, let $(v_n)_{0 \leq n \leq N}$ be defined by:

$$\forall n \in \{0, ..., N-1\}, \quad v_n = \frac{u_n}{h}. \tag{1.20}$$

The scheme [1.17] is then written as:

$$\forall n \in \{0, ..., N-1\}, \quad v_{n+1} = -\,\mathrm{sign}(v_n) + v_n + F(t_n). \tag{1.21}$$

Showing by induction on $n \in \{0, ..., N\}$ that:

$$\forall n \in \{0, ..., N\}, \quad v_n \in [-2, 2]. \tag{1.22}$$

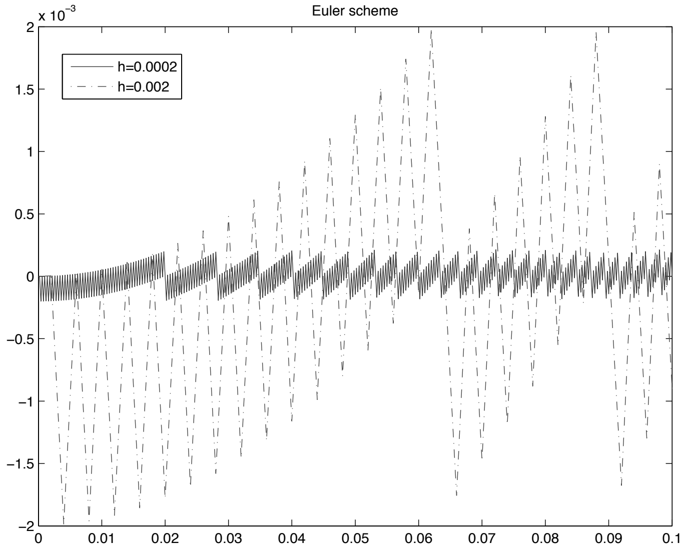


Figure 1.3. *Two approximations of [1.10] and [1.11] given by the explicit Euler scheme [1.18]*

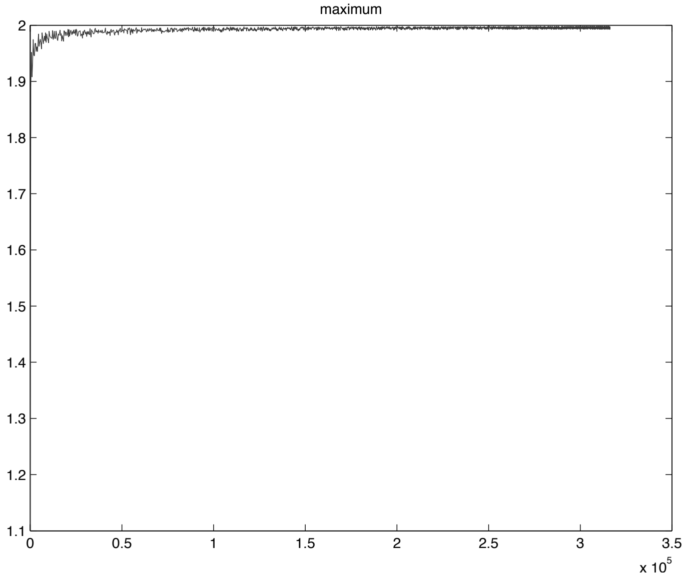
For $n = 0$, this comes from the initial condition [1.19]. Let us assume that [1.20] is true for n. We can observe that the function $x \mapsto -\operatorname{sign}(x) + x$ maps the interval $[-2, 2]$ in $[-1, 1]$. Thus, $-\operatorname{sign}(v_n) + v_n$ belongs to $[-1, 1]$; since $F(t_n) \in [-1, 1]$, we deduce from this that by summing, $v_{n+1} \in [-2, 2]$. We then deduce from [1.22] that:

$$\forall n \in \{0, ..., N\}, \quad |u_n| \leq 2h. \qquad [1.23]$$

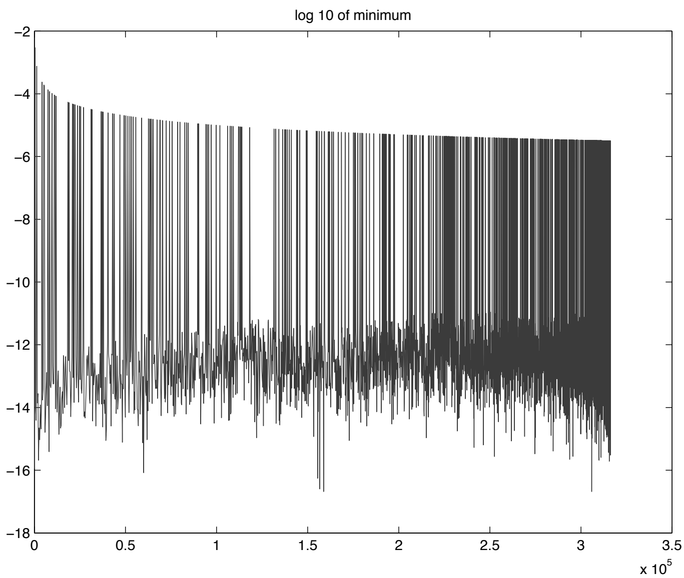
Numerically, we have drawn for N describing a logarithmic interval $[10^{n_{\min}}, 10^{n_{\max}}]$ curve $(N, \max_{0\leq n\leq N}(|u_n/h|))$ (see Figure 1.4(a)), which corroborates [1.23]. The oscillations are more difficult to demonstrate formally. We remark that $u_1 = 0$; we have then drawn in Figure 1.4(b)) the curve $(N, \min_{2\leq n\leq N}(|u_n/h|))$, which shows that the minimum seems to be comprised between two curves, one above corresponding to non-zero values (of the order of 10^{-5}).

Finally for each value of N, we can calculate the number $P(N)$ of sign changes of u_n defined by the number of values $n \in \{2, \ldots, N-1\}$ such as $u_n u_{n+1} < 0$. We have $P(N) \leq N - 2$. We have drawn the curve $(N, P(N))$ in Figure 1.5. On this curve, the points seem perfectly aligned: the correlation is equal to $r = 0.9999999999074$, for a slope $a = 0.5000000232$ and an ordinate at the origin $b = -1.3748$. We then have:

$$P(N) \approx 0.5000000232N - 1.3748.$$



a) $\max_{0\le n\le N}(|u_n/h|)$ versus N



b) $\log_{10}(\min_{2\le n\le N}(|u_n/h|))$ versus N

Figure 1.4. *Extreme values of* $|u_n/h|$ *versus* N *for* $N \in [10^{n_{\min}}, 10^{n_{\max}}]$ *with* $n_{\min} = 1$ *and* $n_{\max} = 5.5$

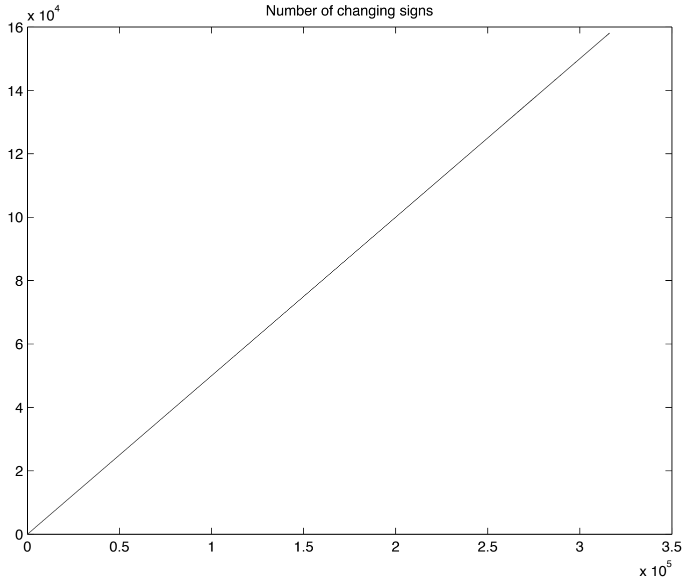


Figure 1.5. *Number $P(N)$ of changing signs of u_n versus N*

2) Let us continue with a numerical implicit Euler scheme for the same problem [1.10]–[1.11]. In this case, with the same notations, [1.17] must be replaced by:

$$\forall n \in \{0, ..., N-1\}, \quad \frac{u_{n+1} - u_n}{h} + \operatorname{sign}(u_{n+1}) = nh, \tag{1.24}$$

with $u_0 = 0$. Let us examine the possible states and demonstrate that u_n is not defined for $n \geq 2$. Let us assume that N is large enough so that:

$$h < 1. \tag{1.25}$$

For $n = 0$, [1.24] gives:

$$\frac{u_1}{h} + \operatorname{sign}(u_1) = 0. \tag{1.26}$$

We can check that $u_1 = 0$ is a solution of [1.26]. On the other hand, $u_1 > 0$ is impossible since [1.26] would give $u_1/h = -1$. Likewise, $u_1 < 0$ is impossible since [1.26] would give $u_1/h = 1$. Thus, for $n = 1$, [1.24] gives:

$$\frac{u_2}{h} + \operatorname{sign}(u_2) = h. \tag{1.27}$$

The $u_2 = 0$ case is impossible since [1.27] would give $0 = h$. Likewise, the $u_2 > 0$ case is impossible since [1.27] would give:

$$u_2 = (h-1)h < 0,$$

according to [1.25]. Finally, the $u_2 < 0$ case is impossible because [1.27] would give:

$$u_2 = (h+1)h > 0.$$

3) We have also tested the resolution of [1.10] and [1.11] with two Matlab© solvers (Figure 1.6).

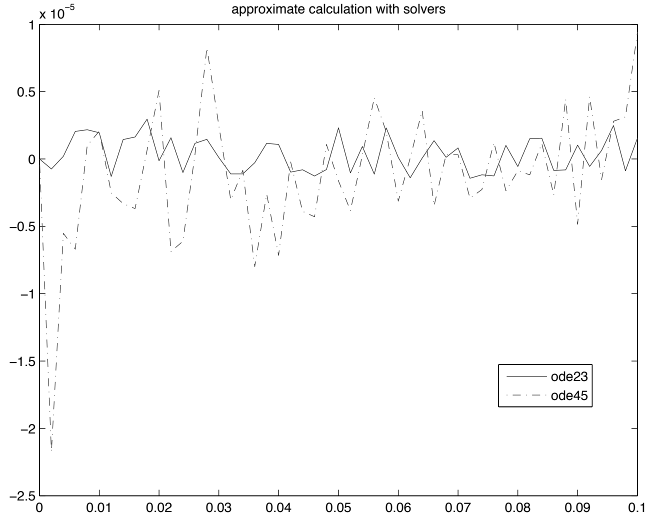


Figure 1.6. *Two approximations of [1.10] and [1.11] given by two Matlab © solvers*

Oscillations still appear and the velocity is non-zero. For the more precise second solver (ode23), the amplitude of the oscillations is weaker.

We have drawn in Figure 1.7 the results given by explicit Euler and two Matlab© solvers, this time on $[0, 2]$. Oscillations still appear and the velocity is non-zero. For the more precise second solver (ode23), the amplitude of the oscillations is smaller. Beyond 1 (see Figure 1.7(b)), the oscillations disappear: we are in the case where the point material slips with a friction force equal to -1 and there are no more problems.

COMMENT 1.1.– The astute reader will understand that even if the numerical solution given by [1.10] and [1.11] approaches zero, it cannot tend toward a solution of [1.11], since the second member $F(t) - \text{sign}(u)$ is not continuous in u and so the usual theorems of convergence of numerical schemes [BAS 03, CRO 84, SCH 01] cannot be applied.

Thus, it is clear that if we look at the consistency of the results provided by these schemes with the usual sign, they are not capable of returning correct behavior from a mathematical or physics point of view.

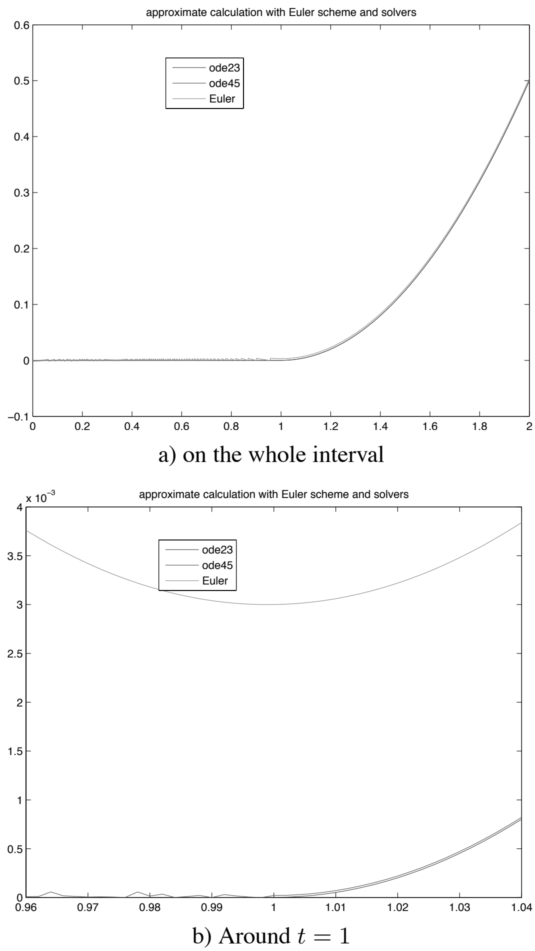


Figure 1.7. *Three approximations of [1.10] and [1.11] given by Explicit Euler numerical scheme and two Matlab© solvers on* $[0, 2]$

Let us take advantage of this simple example to demonstrate why the usual sign is not adapted to good modeling. We consider mass m initially at rest on the support subjected to an external slowly growing force $F(t)$ starting from zero. As long as the force is not large enough to overcome the friction effect (let us say at an interval of time $[0, T]$, $T > 0$), the mass will stay motionless. In [1.10] and [1.11], the term $-\alpha\,\mathrm{sign}(u(t))$ corresponds to Coulomb's friction force. Clearly on $[0, T]$, this amounts to: $0 = F(t)$, which is not coherent with the hypothesis on $f(t)$, except for $t = 0$. It appears that we must develop the possibility for the Coulomb's friction model to not be univalued: as the mass is not set in motion, $\mathrm{sign}(u(t)) = \mathrm{sign}(0)$ must take multiple values allowing $F(t)$ to balance. This leads us to consider the friction term to be modeled as a graph (the graph of a function then being a special case). The motion equation will then be described not by the previous equality but by a differential inclusion, as shown in section 1.2.3.

1.2.3. *Differential equation with multivalued term: differential inclusion*

In this section, we demonstrate that the problem is well set if the force g is defined by [1.4]:

$$g(t) \in -\alpha\sigma\left(\dot{x}(t)\right). \qquad [1.28]$$

We are therefore seeking two functions x and g checking systems [1.5] and [1.28]. By setting:

$$g(t) = m\ddot{x}(t) - F(t),$$

this system can also be written in the equivalent following form:

$$\begin{aligned} &x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \\ &m\ddot{x}(t) - F(t) \in -\alpha\sigma\left(\dot{x}\right), \end{aligned}$$

then again by returning the symbol $\in$:

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \qquad [1.29a]$$

$$\forall t \in [0,T], \quad m\ddot{x}(t) + \alpha\sigma\left(\dot{x}(t)\right) \ni F(t). \qquad [1.29b]$$

This is not a differential equation but a differential inclusion. We will then take a closer look at this differential inclusion [1.29], where the data is F and the unknown function is x. As in section 1.2.2, by setting $u_0 = \dot{x}(0)$ and $u = \dot{x}$, we replace [1.29] by:

$$\begin{aligned} &u(0) = u_0, \\ &\forall t \in [0,T], \quad \dot{u}(t) + \frac{\alpha}{m}\sigma\left(u(t)\right) \ni \frac{1}{m}F(t). \end{aligned}$$

To simplify things, we will assume that [1.9] occurs. We therefore have a differential inclusion:

$$u(0) = u_0, \qquad [1.30a]$$

$$\forall t \in [0,T], \quad \dot{u}(t) + \sigma\left(u(t)\right) \ni F(t). \qquad [1.30b]$$

We will come back to this example, according to initial conditions and second members in section 2.2.6.1.

As in section 1.2.2, we can present several simulations. By anticipating section 2.2.6.1, we note that the inclusions [1.11] and [1.30] are of the type [2.26] with the hypotheses [2.35] where $t_a = 1$ and $T = 2$ are valid. The exact expression is then given by [2.36], thus here:

$$\forall t \in [0,T], \quad u(t) = \begin{cases} 0 & \text{if } t \leq 1, \\ \\ \dfrac{1}{2}t^2 - t + \dfrac{1}{2} & \text{if } t \geq 1. \end{cases} \qquad [1.31]$$

We note that in section 2.2.6.1, the *ad hoc* numerical implicit Euler scheme used (see [2.32]) provides the solution with an error in $0(h)$.

The results are given in Figure 1.8. In this figure, we note that in the static phase, on $[0,1]$, u is null and the numerical scheme gives a solution that is rigorously null, without oscillation.

To conclude, we therefore note that all is well here with a differential inclusion, contrary to the differential equation ill-posed in section 1.2.2.

COMMENT 1.2.– Let us note two important things here. The force g is continuous; otherwise said, the σ graph does not present a "hole". Moreover, the power of the force dissipated by friction is always negative because:

$$\begin{aligned} & g(t)\dot{x}(t) = 0, \text{ if } \dot{x}(t) = 0, \\ & g(t)\dot{x}(t) = -\alpha \operatorname{sign}\left(\dot{x}(t)\right)\dot{x}(t) = -\alpha \left|\dot{x}(t)\right|, \text{ if } \dot{x}(t) \neq 0. \end{aligned}$$

We therefore have:

$$g(t)\dot{x}(t) \leq 0. \qquad [1.32]$$

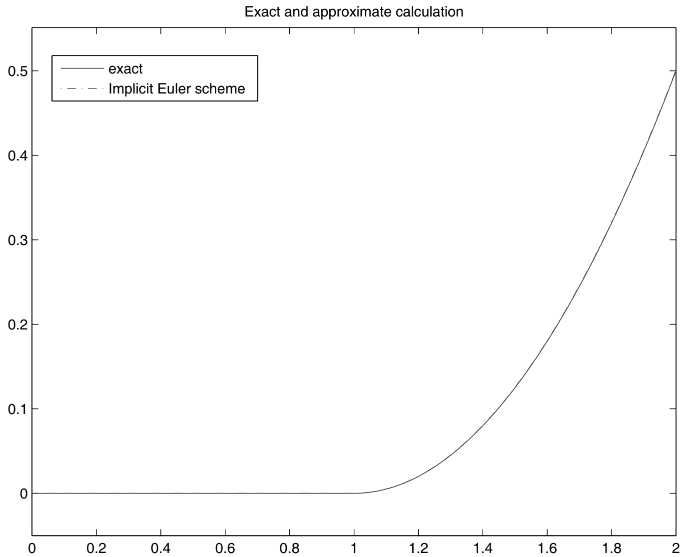
The function sign is monotone; we deduce from it the monotonicity of the σ graph that can be written as:

$$\forall x, y \in \mathbb{R}, \quad u \in \sigma(x), \quad v \in \sigma(y) \Longrightarrow (v-u)(y-x) \geq 0. \qquad [1.33]$$

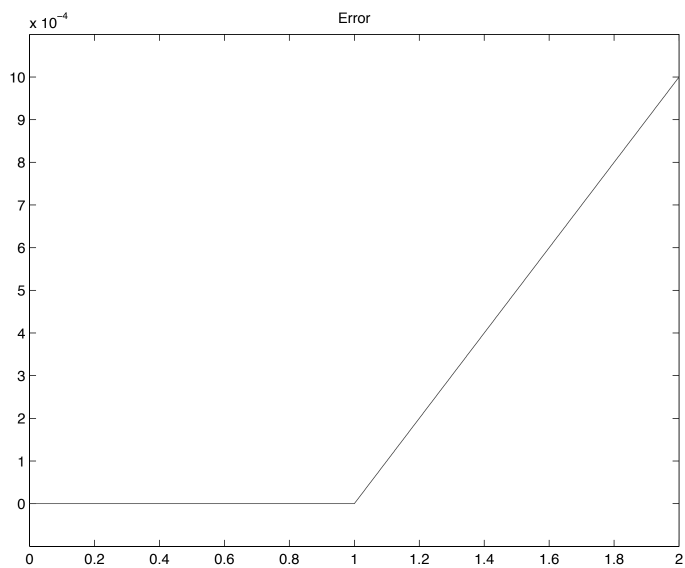
Since α is strictly positive and $0 \in \sigma(0)$, [1.32] is only a consequence of [1.28] and [1.33].

1.2.4. *Other friction laws*

Other type of frictions can be considered, which are more realistic than ideal Coulomb's friction.

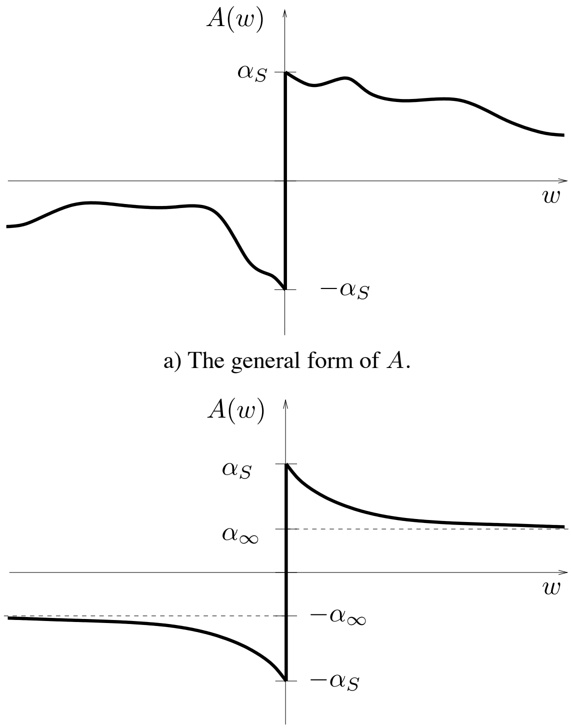


a) Approximation and exact function



b) Error

Figure 1.8. *An approximation of [1.11] and [1.30] given by the numerical scheme of Chapter 2*



a) The general form of A.

b) A usual law.

Figure 1.9. *Two types of friction laws*

In [BAS 08a], we have seen other forms of friction: we can, generalizing the inclusion [1.28], assume that:

$$g(t) \in -A\left(\dot{x}(t)\right), \qquad [1.34]$$

where the multivalued operator is of the form:

$$A(w) = \alpha_S \sigma(w) + \Psi(w), \qquad [1.35]$$

where α_S is the static coefficient of friction, the multivalued operator σ is defined by [1.3] and Ψ is a smooth function, vanishing at zero. Thus, we obtain a friction force generalizing what we have seen in equation [1.2]:

$$\text{if } \dot{x} = 0, g \in [-\alpha_S, \alpha_S], \qquad [1.36a]$$

$$\text{if } \dot{x} \neq 0, g = -\alpha_S \text{sign}\left(\dot{x}\right) - \Psi\left(\dot{x}\right). \qquad [1.36b]$$

This case corresponds to a friction force given in Figure 1.9(a). A special case often used is given in Figure 1.9(b), which emphasizes a static coefficient of friction

α_S and a limit coefficient of friction α_∞ with:

$$0 \le \alpha_\infty < \alpha_S. \qquad [1.37]$$

A particular case is given by Stibeck's formula (p. 86 of [AND 05]) where:

$$A(w) = \frac{D}{A|w|^3 + Bw^2 + C|w| + 1}\sigma(w), \qquad [1.38]$$

with A and D strictly positive and B and C positive. See, for example, Figure 1.10 that corresponds to particular values of coefficients (this figure corresponds to Figure 3.22 of [AND 05]) given by:

$$A = 0.01, \quad B = 0.2, \quad C = 0.01 \quad D = 900. \qquad [1.39]$$

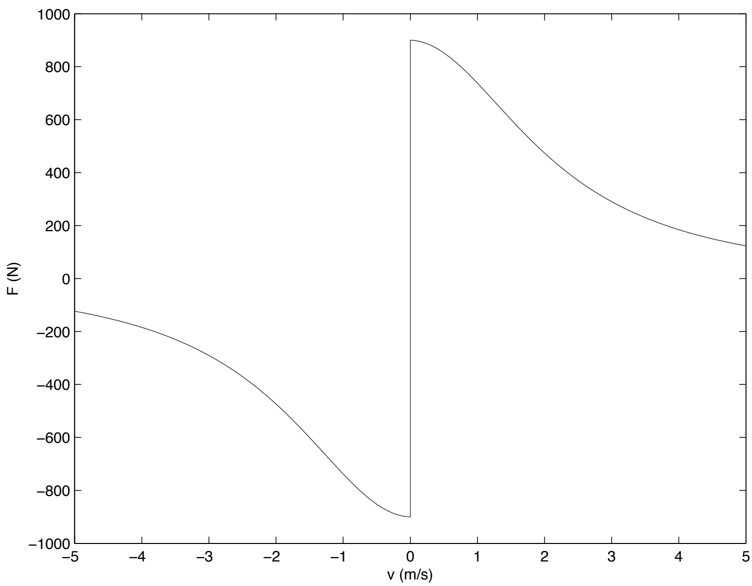


Figure 1.10. *The opposite of the force function of the velocity for Stibeck's formula with A, B, C and D defined by* [1.39]

In this book, we take a specific interest in the special case where the friction force is given by [1.4]: we will demonstrate throughout Chapter 5 that a large class of visco-elastoplastic models is covered by a similar mathematical formality, involving maximal monotone operators. These are introduced throughout Chapter 2. We will show that the problem is well set and that an *ad hoc* numerical scheme can be used, which will please numericians. Furthermore, we are in particular interested in this ideal problem as it reveals the difficulty of treating the fundamental discontinuity of the zero sign.

However, to also satisfy the engineer, we will show in section 7.6.1 that most of the ideal models studied throughout Chapter 5 can take into account the [1.35] or [1.38] type laws, provided certain precautions are taken. In section 7.6.2, we will briefly cover the case where the static coefficient can be varied and in section 7.6.3 the case where the static and dynamic coefficient can be varied.

1.3. Impact

1.3.1. *Difficulties with writing the differential equation*

This introduction is inspired by the mathematical aspect of the pioneering works by Michelle Schatzman [SCH 78]. Let us consider a simple example of a mechanical system with one degree of freedom modeled by the following problem:

$$\frac{d^2u}{dt^2}(t) + R(t) = f(t), \qquad [1.40]$$

where u denotes the displacement of the system (mass unit here), R denotes a force (Reaction) exerted on the system while it encounters an obstacle, placed to simplify $u = u_{max} = 0$ (so that the movement takes place in the half-space $u \geq u_{max} = 0$). Here, f corresponds to an external stress, which is eventually null. This model is not complete because we can intuitively see that after encountering the obstacle, a usual mechanical system undergoes a change (generally discontinuous) in velocity. This model is then completed with a shock law that operates until instants t where the system encounters the obstacle ($u(t) = u_{max} = 0$). Aside from such an instant, the "reaction" R is null.

A family of shock laws is very familiar. It consists of linking the velocity just before the shock (left-hand limit at t^-) with the velocity after the shock (right-hand limit at t^+) by a coefficient of restitution and an inversion of the sign to model the mechanical system rebound on the obstacle. By noting $\frac{du}{dt} = \dot{u}$, this law can be written as:

$$\dot{u}(t^+) = -e\dot{u}(t^-),$$

where $e \in [0, 1]$ is the coefficient of restitution. To simplify here, we consider a perfectly elastic shock $e = 1$. With initial conditions $u(0) = u_0 > 0$, and $\dot{u}(0) = v_0 < 0$, we obtain analytically:

$$\begin{aligned}
&u(t) = u_0 + v_0 t, 0 \leq t < t_1,\\
&u(t_1) = u_{max} = 0, t_1 = -\tfrac{u_0}{v_0},\\
&u(t) = -v_0(t - t_1), t > t_1.
\end{aligned}$$

The velocity verifies

$$\dot{u}(t_1^-) = v_0, \dot{u}(t_1^+) = -v_0 = -e\dot{u}(t_1^-), (e = 1). \qquad [1.41]$$

The reaction $R(t)$ is null if $t \neq t_1$, and to give a direction to $R(t_1)$, we must derive the velocity (to obtain the acceleration) set as a function that can be locally integrated (piecewise constant), continuously differentiable (C^1) except at t_1 where we are situated at a point of discontinuity of the first kind. The formula for the jump distribution theory gives meaning to the equilibrium reaction opposite the acceleration:

$$R = R_1 \delta_{t_1},$$

where δ_{t_1} denotes the Dirac distribution in t_1, and R_1 is the opposite of the jump of the velocity at t_1. We are naturally seeing the context in which these models have an impact: these are differential equations in the sense of measures. But note here that it is possible to use other formulations: the context of differential inclusions (we will quickly see), or in the context of complementary problems [ACA 08, BRO 96, GLO 01], which consists of writing the previous problem in the form of equations [1.40]–[1.41] with the complementarity relations:

$$R(t).(u(t) - u_{max}) = 0, -R(t) \geq 0, u(t) - u_{max} \geq 0.$$

There are also consistent mechanical approaches that regularize the problem for physical reasons: we remove the impact, we assume that before the obstacle ($u = l$, l small) the mass encounters a very stiff spring (stiffness K, very large K), which is compressed, its velocity is cancelled and it expands to its original configuration before letting the mass rebound. In this case, we can have a (brief) passage of the mass in the obstacle. The model therefore allows us to then calculate all of the mechanical properties to consider the limiting case of an infinite stiffness. The details of the calculations for initial conditions $u(0) = u_0 > l > 0$, and $\dot{u}(0) = v_0 < 0$:

$$0 \leq t < t_{1b} = \frac{l - u_0}{v_0}, u(t) = v_0 t + u_0.$$

then with $\Omega = \sqrt{K}$:

$$\begin{aligned} &t_{1b} \leq t \leq t_{3b}, \tan(\Omega(t_{2b} - t_{1b})) = \frac{v_0}{l\Omega}, \tan(\Omega(t_{3b} - t_{2b})) = \frac{v_0}{l\Omega}, \\ &t_{1b} < t_{2b} < t_{3b}, \dot{u}(t_{2b}) = 0, u(t) = \rho \sin(\Omega(t - t_{1b}) + \theta_1), \end{aligned}$$

with θ_1 as:

$$\frac{l}{\rho} \cos(\Omega(t - t_{1b})) + \frac{v_0}{\rho\Omega} \sin(\Omega(t - t_{1b})) = \sin(\Omega(t - t_{1b}) + \theta_1),$$

and $\rho = \sqrt{l^2 + \dfrac{v_0^2}{\Omega^2}}$. Then beyond instant t_{3b}, the solution is defined by:

$$t \geq t_{3b}, u(t) = v_3(t - t_{3b}) + l,$$

with

$$\dot{u}(t_{3b}) = \rho\Omega\cos(\Omega(t_{3b} - t_{1b}) + \theta_1) = v_3.$$

By passing to the limit on K and l, for example by setting $Kl = 1$, and if K tends to $+\infty$, we have:

$$\begin{array}{l} t_{1b} \longrightarrow t_1, l \longrightarrow 0^+, t_{2b} - t_{1b} \longrightarrow 0^+, t_{3b} - t_{2b} \longrightarrow 0^+, \rho \longrightarrow 0^+, \\ \dot{u}(t_{3b}) \longrightarrow \dot{u}(t_1^+), \dot{u}(t_{1b}) \longrightarrow \dot{u}(t_1^-), \dot{u}(t_{3b}) \longrightarrow -\dot{u}(t_{1b}). \end{array}$$

We find, at the limit, all the properties of the original model.

Let us look at the mathematics. It is possible to model the reaction R by taking into account the following aspects:

- $u(t)$ must not penetrate the obstacle: $u(t) \geq u_{\max} = 0$;
- if $u(t) = u_{\max}$, the equilibrium must take the reaction into account;
- if $u(t) > u_{\max} = 0$, there is no reaction.

To model, we must add a shock law that will deliver the discontinuity of the velocity generally observed at the moment of impact (we describe shock as instantaneous). Rather than a differential equation, we write a differential inclusion. In our simple example (derived from [SCH 78]), it is written as:

$$\frac{d^2u}{dt^2} + \partial\Psi_K(u) \ni f, \qquad [1.42]$$

here with $f = 0$ to simplify and

$$\begin{array}{l} \partial\Psi_K(z) = \{0\}, z > u_{max}, \\ \partial\Psi_K(z) =]-\infty, 0], z = u_{max}, \\ \partial\Psi_K(z) = \emptyset, z < u_{max}. \end{array}$$

We show that $\partial\Psi_K$ is the sub-differential of the indicator function $K = [u_{\max}, +\infty]$:

$$\forall z, \quad \Psi_K(z) = \begin{cases} +\infty & \text{if } z \notin K, \\ 0 & \text{if } z \in K. \end{cases}$$

The mathematical treatment of the problem requires solving an approximate problem using Yosida regularization [BRE 73, SCH 78] of a multivalued operator defined on a Hilbert space, very simple here: $\mathbb{R}$. If:

$$J_\lambda = (I + \lambda\partial\phi)^{-1},$$

and if ϕ_λ is the regularization Yosida of ϕ defined by:

$$z \in \mathbb{R}, \phi_\lambda(z) = \phi(J_\lambda(z)) + \frac{[z - J_\lambda(z)]^2}{2\lambda},$$

we arrive at

$$\partial\phi_\lambda(z) = \frac{z - J_\lambda(z)}{\lambda}.$$

For $\phi = \Psi_K$, it leads to:

$$\forall z, \quad \partial\psi_\lambda(z) = \begin{cases} \dfrac{z - u_{\max}}{\lambda} & \text{if } z \leq u_{\max}, \\ 0 & \text{if } z \geq u_{\max}. \end{cases}$$

1.3.2. *Ill-posed problems*

The examples are inspired by Michelle Schatzman's works [SCH 78]. They concern ill-posed problems from a (dis)continuity or a uniqueness point of view. Let us examine and illustrate these questions on these examples more precisely.

For $u_0^h = (1-h, \frac{1}{2}+h), v_0^h = (-1, -\frac{1}{2})$ with $\mid h \mid\leq \frac{1}{2}$, we have uniqueness of the solution of the exact initially problem. The detailed solution is provided by Michelle Schatzman and shows that like the function of h, the right solution is continuous on the right-hand side, but not on the left-hand side, showing a regularity (and sensitivity) problem with regard to the data, which can also be observed with reflections of two straight edges or forming a right angle or a more open marginal angle (infinitely small).

The second problem is that of the uniqueness. We must first remember the founding work of [SCH 78], which defined a solution preserving energy. M. Schatzman is placed in a more general context than that of section 1.3.1: she considers a function ϕ, which is convex, proper and lower semi-continuous and she demonstrates a result that generalizes the existence of the solution of [1.42]. Here, we note $H = \mathbb{R}^n$, equipped with the usual scalar product $\langle ., . \rangle$ (see Appendix 1). u_0 is in the domain $D(\phi)$ of ϕ. Thus, $f \in L^2(0, T; H)$ (see Appendix 1) with T finite. The function u is a solution of the problem:

$$\frac{d^2u}{dt^2} + \partial\phi(u) \ni f, + \text{ initial conditions,} \qquad [1.43]$$

preserving energy, when u satisfies:

$$u \in W^{1,\infty}(0,T;H),$$
$$\forall t \in [0,T], u(t) \in D(\phi),$$

when μ is a measure with values in H, such that, in the sense of distributions[1]:

$$\frac{d^2u}{dt^2} + \mu = f,$$

when for all continuous v, with values in H, such as $\phi(v) \in L^1(0,T)$, we have[2]:

$$\int_0^T (\phi(v) - \phi(u))dt \geq < \mu, v - u >,$$
$$\frac{du}{dt} \text{ admits left- and right-hand limits at } t \text{ for all } t \in [0,T],$$

by correcting in 0 and in T and when energy is finally conserved:

$$\left\|\frac{du}{dt}(t^+)\right\|^2 + \phi(u(t)) = \left\|\frac{du}{dt}(t^-)\right\|^2 + \phi(u(t)) = \|u_1\|^2 + \phi(u_0) + \int_0^t \langle \dot{u}(s), f(s)\rangle ds,$$

almost everywhere on $[0,T]$ where $\dot{u}$ means du/dt, and initial conditions are satisfied in the following sense: $u(0) = u_0$, and if K_0 is the closure of $D(\phi)$ and Ψ_{K_0} the indicator function of K_0:

$$-u_1 + \frac{du}{dt}(0^+) + \partial\Psi_{K_0}(u_0) \ni 0.$$

With this definition, M. Schatzman proves the following result.

1 Here, unlike the case of Appendix 1, the distribution space is smaller here than $\mathcal{D}'(\Omega; H)$: it is the dual of $C^0([0,T], H)$, named the set of Radon measurements. The reader is referred for example to [WAG 99].

2 In the two following equations, $\langle\mu, v\rangle$ signifies the bilinear form, meaning the image of the μ measurement on the continuous function v.

THEOREM 1.1.–

1) For all $f \in L^2(0,T;H)$*,* $u_0 \in D(\phi)$ *and* $u_1 \in H$*, problem* [1.43] *has a solution conserving energy in the previous sense. This solution is obtained as a strong limit in* $H^1(0,T;H)$ *and as a weak limit* $*$ *in* $W^{1,\infty}(0,T;H)$ *of a sequence extracted from a family of solutions of the problem:*

$$\frac{d^2u_\lambda}{dt^2} + \partial\phi_\lambda(u_\lambda) = f, \qquad [1.44a]$$

$$u_\lambda(0) = u_0, \qquad [1.44b]$$

$$\frac{du_\lambda}{dt}(0) = u_1. \qquad [1.44c]$$

2) Moreover, if ϕ *is Lipschitz continuous in the neighborhood of* u_0 *relative to* K_0 *and if* $-u_1 \in C$*,* C *being the tangent cone at* K_0 *in* u_0*, then:*

$$\frac{du}{dt}(t^+) = 2P_C u_1 - u_1,$$

where P_C *is the projection on* C*.*

During the proof of this result, Michelle Schatzman clarifies the sense in measure of the equation and in terms of distributions as well as the velocity properties. Following this result, she proposes the construction of function f (external solicitation) such that the problem admits two solutions. The function f is constructed analytically in a countdown time from an instant t to the initial time, and is very regular (infinitely differentiable). The two solutions are respectively:

– for the first, an infinite number of arches "bounces" with an amplitude gradually getting weaker as we go back to initial time; the "reaction" is given by atomic measures;

– for the second: the null solution, with the function f (external solicitation) as the reaction to itself.

Later, Michelle Schatzman will focus with F. Nqi on other unpleasant behaviors of such non-regular systems at another level of dynamics study [NQI 97]. These works will display throughout the simple example of a one degree of freedom oscillator with impact that the calculation schemes exposing Lyapounov are under extreme sensitivity to the parameters of the discrete system obtained via a numerical scheme, in particular the time step.

The formality of theorem 1.1 is not used in this publication, since we will often consider an analytical or numerical solution.

1.4. Probabilistic context

Let us here interest ourselves on the problem exposed in section 1.3.2 and defined in [1.30] where F is stochastic, meaning it depends on time and on a variable describing hazard, in other words it is a stochastic process, a notion that will be defined in Chapter 3. We assume for the moment that it is white noise that we will not define more rigorously in this work because it requires the introduction of a heavy mathematical framework on generalized processes (measures on the space of tempered distributions) that won't justify itself. Noting the white noise $N(t)$, the mathematical framework mentioned will allow us to give sense to the equality $N(t) = dB(t)/dt$ where B denotes Brownian motion (this process will be defined more rigorously in Chapter 3). Indeed, this equality cannot be taken in the usual sense since the trajectories of the Brownian motion are nowhere differentiable with probability 1.

Let us assume that the soliciting force is $F(t) = N(t)$. By using $N(t) = dB(t)/dt$, we interest ourselves in the following differential system:

$$u(0) = u_0, \quad [1.45a]$$

$$\forall t \in [0, T], \quad du(t) + \sigma\left(u(t)\right) dt \ni dB(t). \quad [1.45b]$$

We will then give in Chapter 3 a meaning to this multivalued stochastic differential equation. Here, we recall a result obtained in [SHR 81] with regard to the stochastic differential equation:

$$u(0) = u_0, \quad [1.46a]$$

$$\forall t \in [0, T], \quad du(t) = -\operatorname{sign}\left(u(t)\right) dt + dB(t), \quad [1.46b]$$

which is the stochastic analogous of [1.10].

The results in this stochastic context are distinguished by those presented in the deterministic context. It was indeed shown that for mathematical and physics reasons, it is necessary to consider a multivalued operator rather than the sign function. The illustration was shown in the case where the F function was initially worth 0 and increasing, such that velocity u was null for a certain time, and then took non-zero values. An equality formulation of the differential problem could not model this phenomenon and the multivalued character of the friction is necessary. In the stochastic case, Shreve provides in [SHR 81] analytical expressions for the law of solution of [1.46]. In particular, he finds a new result proved in [BEN 74] in which, with probability 1, the Lebesgue measure of the set of time t such that $u(t) = 0$ is null. In other terms, the assigned value of the sign function in 0 has no incidence for the solution [1.46]. This result is completely different for the deterministic case for which the value of 0 was crucial, it must have a range of values even. This result can be observed from a mathematical point of view by introducing the explicit numerical Euler scheme. We will see, unlike the deterministic case, that the scheme does not lead to oscillations previously observed. By discretizing, we introduce the sequences $u_n \simeq u(t_n)$, and $t_n = nh$, for $n \in \{0, ..., N\}$, with $h = T/N$ as the timestep (chosen constant) and $u_0 = u(0) = 0$ (we will not confuse the number N of timesteps with the white noise $(N(t))_{t \geq 0}$). According to $N(t)dt = dB(t)$, the explicit Euler scheme associated to [1.46] takes the following form: $u_0 = 0$ and

$$\forall n \in \{0, ..., N-1\}, \quad u_{n+1} = u_n - h\,\mathrm{sign}(u_n) + B(t_{n+1}) - B(t_n), \quad [1.47]$$

where, by the Brownian motion property (see Chapter 3), $B(t_{n+1}) - B(t_n)$ is a random variable following a centered normal law with variance h. We represent in Figure 1.11 a trajectory of the system [1.46] using the scheme [1.47] for values of $h \in \{10^{-4}, 2\,10^{-4}, 4\,10^{-4}, 5\,10^{-4}, 10^{-3}, 2\,10^{-3}\}$. It is impossible to distinguish the different obtained simulations with different timesteps. We can note the convergence, without oscillations, of the scheme in Figure 1.12 obtained by zooming in on part of the trajectory (indeed, it is difficult here again to distinguish the different simulations).

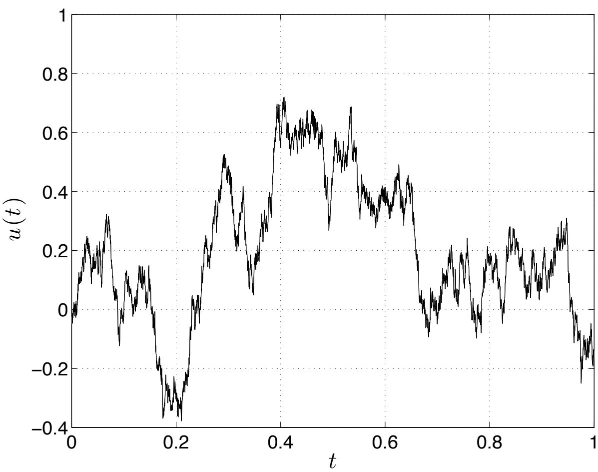


Figure 1.11. *Simulation of a trajectory of the system [1.46] using the scheme [1.47] for values of* $h \in \{10^{-4}, 2\ 10^{-4}, 4\ 10^{-4}, 5\ 10^{-4}, 10^{-3}, 2\ 10^{-3}\}$

Let us assume from now on that the soliciting force is not a white noise but a filtered white noise, meaning the output of a linear system excited by a white noise. We frequently encounter this type of noise for modeling earthquakes, swell or wind (see, e.g. [KRE 83] for a detailed presentation). A widely used seismic model is the Kanai–Tajimi model [KRE 83] that is represented in Figure 1.13.

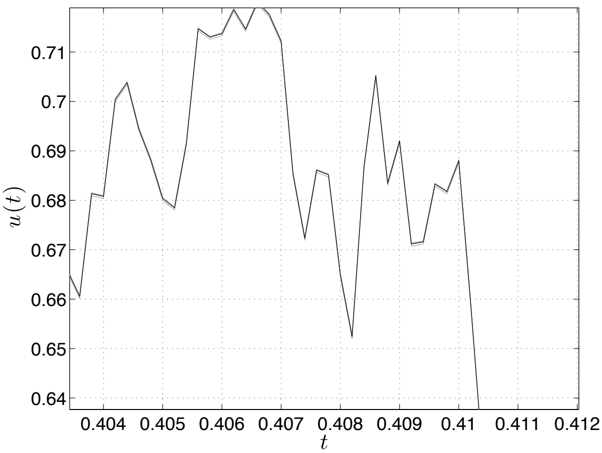


Figure 1.12. *Simulation of a trajectory of the system [1.46] using the scheme [1.47] for values in* $h \in \{10^{-4}, 2\ 10^{-4}, 4\ 10^{-4}, 5\ 10^{-4}, 10^{-3}, 2\ 10^{-3}\}$

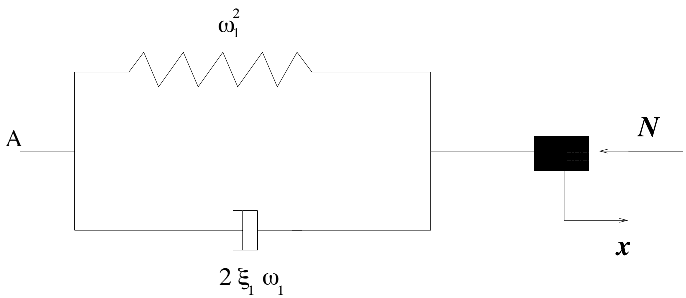


Figure 1.13. *Kanai–Tajimi model*

More precisely, the force exerted on point A, which is $-(2\xi_1\omega_1\dot{x} + \omega_1^2 x)$, with $\ddot{x}(t) + 2\xi_1\omega_1\dot{x}(t) + \omega_1^2 x(t) = N(t)$, becomes the soliciting force of the system. We therefore obtain, with $\omega_1 = 1$ and $\xi_1 = 0.5$ to simplify the study, the following system:

$$x(0) = x_0, \qquad [1.48a]$$

$$y(0) = y_0, \qquad [1.48b]$$

$$u(0) = u_0, \qquad [1.48c]$$

$$\forall t \in [0,T], \quad dx(t) = y(t)dt, \qquad [1.48d]$$

$$\forall t \in [0,T], \quad dy(t) = -y(t)dt - x(t)dt + dB(t), \qquad [1.48e]$$

$$\forall t \in [0,T], \quad du(t) + \sigma\left(u(t)\right) dt \ni -x(t)dt - y(t)dt. \qquad [1.48f]$$

To our knowledge, theoretical results analogous to those of Shreve [SHR 81] do not exist for system [1.48]. Besides, we will convince ourselves with numerical simulations that such results cannot be established. We indeed find the same difficulties encountered in the deterministic case. Let us first simulate system [1.48] using the sign function instead of the σ operator like we did in [1.46] and for the following numerical scheme of the [1.47] type: $x_0 = y_0 = u_0 = 0$ and $\forall n \in \{0, ..., N-1\}$,

$$x_{n+1} = x_n + hy_n, \tag{1.49a}$$

$$y_{n+1} = y_n - hy_n - hx_n + B(t_{n+1}) - B(t_n), \tag{1.49b}$$

$$u_{n+1} = u_n - h\,\mathrm{sign}(u_n) - hx_n - hy_n. \tag{1.49c}$$

We have represented in Figure 1.14 the simulation of a trajectory of u solution to system [1.48] using the numerical scheme [1.49] for values of $h \in \{10^{-4}, 2\,10^{-4}, 4\,10^{-4}, 5\,10^{-4}, 10^{-3}, 2\,10^{-3}\}$. We can note oscillations, like in the deterministic case.

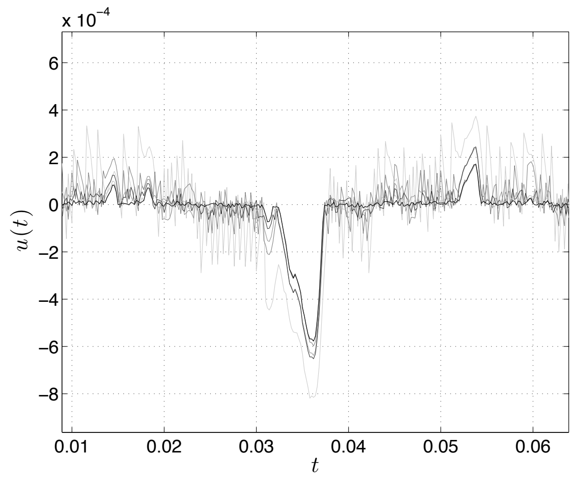


Figure 1.14. *Simulation of a trajectory of the system [1.48] using the numerical scheme [1.49] for values of* $h \in \{10^{-4}, 2\ 10^{-4}, 4\ 10^{-4}, 5\ 10^{-4}, 10^{-3}, 2\ 10^{-3}\}$

Let us now consider for system [1.48] an *ad hoc* numerical scheme, meaning specific to stochastic multivalued differential equations (see section 3.4). We will then observe that there are no oscillations like we see in Figure 1.15.

Whether in the deterministic or stochastic context, the numerical approximation of differential inclusions is to be considered with many precautions. The three following chapters are committed to defining modeling by maximal monotone multivalued operator in a rigorous mathematical context also integrating numerical aspects.

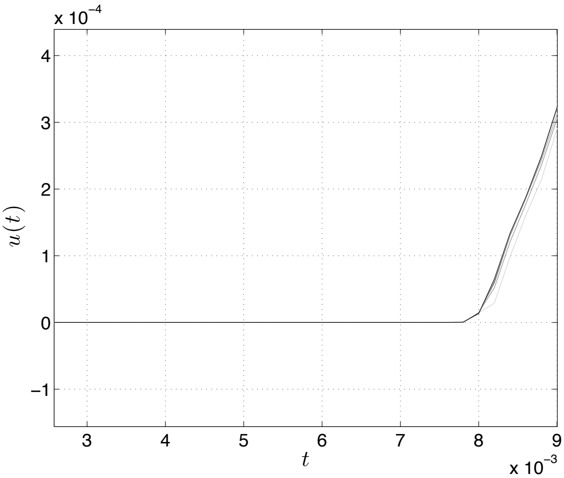


Figure 1.15. *Simulation of a trajectory of the system [1.48] using the ad hoc numerical scheme for stochastic differential inclusions for values of* $h \in \{10^{-4}, 2\ 10^{-4}, 4\ 10^{-4}, 5\ 10^{-4}, 10^{-3}, 2\ 10^{-3}\}$