
State of the Art

The construction of models in the multi-mechanism (MM) family uses a series of generic tools (“building bricks”) that are also introduced in the classical plastic or viscoplastic formulations. They are recalled here, which allows us to define the main notations in the book. Since various MM models can be either seen as purely phenomenological or present a physical background dealing with multiphase models, scale transition rules are also recalled. The models have been applied to a large set of materials, some of them being submitted to large deformation. A simple framework has been used in this case, which is also presented here. The chapter concludes with a brief description of the main steps in the development of MM models. All these elements relate to the mechanical aspects of the models. Nevertheless, some of them are developed with a view to providing a specific material microstructure. This is the reason why the first section of this chapter is devoted to the description of deformation mechanisms at the microscale.

1.1. Motivation from the microstructure

Shear deformation and volume change observed at the macroscale have various origins at the microscale, depending on material microstructure. Metallic alloys usually deform by dislocation slip, climbing and grain boundary sliding, for instance. Each of these deformation sources leads to a type of nonlinearity and a specific shape for the constitutive equations. In the following, the basic mechanisms of plastic deformation are enumerated for (1) single crystals, (2) polycrystals, (3) amorphous polymers and (4) semi-crystalline polymers:

1) *Single crystals*: the plastic deformation of metallic single crystals results from the evolution of the crystal network.

- Deformation by slip: this mechanism of deformation occurs when dislocations move on certain crystallographic (dense) planes and directions (slip system). According to Schmid's law, a slip system becomes active provided the resolved shear stress reaches a threshold value called the critical resolved shear stress. Slip produces slip lines, so that the classical models taking into account crystal plasticity represent the average of the deformation produced in a small material element. Dislocation may interact with a number of defects that are present in the material.

- Mechanical twinning: twinning is a deformation mechanism mainly observed at low temperature. Crystal twinning occurs when a crystal can jump from its initial configuration to a new one, involving the original grain and a twinned part, with an interface compatible with the two crystal networks. The deformation process is time independent, and produces locally both instantaneous shear deformation and eventually volumetric change. Twinning is one of the most active deformation mechanisms in crystals involving a reduced number of slip systems. It is pronounced in HCP crystals, and also observed at low temperature in higher symmetry BCC and FCC crystals.

2) *Polycrystals*: the grains of polycrystalline aggregates do not deform in the same way as single crystals, due to the local plastic flow that generates multiaxial stress states. The behavior is affected by grain boundaries, and by the presence of subgrain boundaries within the grains. Particles may be placed on purpose in the material, in order to produce hardening. The related mechanisms are solid solution strengthening or dislocation–precipitate interaction (Figure 1.1). Hard second-phase particles and ductile phases will change the behavior according to their size, shape, number and spatial distribution. In the classical literature, local fields in the phases are estimated by means of two opposite assumptions, uniform stress or uniform strain. This is a key issue of the MM models and it will be discussed in detail in the next chapters. In addition to the previous effects, grain boundary sliding may become a significant mechanism in specific cases, like the presence of nanograins, at high temperature that promotes diffusion at grain boundaries.

3) *Amorphous polymers*: they are polymers whose molecular structure lacks a definite repeating form, shape or structure. In some glassy polymers, the tensile stress–strain curve can be separated into three regimes: (i) initially,

the deformation is linear until the yield stress is reached, (ii) the stress decreases due to softening associated with the formation of a neck and (iii) molecular orientation provides a mechanism for hardening that predominates at large strains (Figure 1.2).

4) *Semi-crystalline polymers*: semi-crystalline polymers (SCP) belong to a family of materials that combine a crystalline molecular structure and amorphous regions. Their microstructure can then be seen as a two-phase composite material consisting of crystalline and amorphous phases. Spherulites are spherical regions inside non-branched linear polymers. They are composed of highly ordered lamellae. The macroscopically homogeneous deformation results from various deformation mechanisms in the heterogeneous microstructure (Figure 1.3). The main deformation mechanisms responsible for the plastic deformation of SCP are crystallographic in nature. For small levels of deformation, the crystals are distorted but the crystalline lamellae are not damaged, whereas, under large plastic deformation, the distortion in the spherulitic region creates a strong preferential orientation. The elastic and viscoplastic behavior of SCP is mainly affected by the degree of crystallinity, the initial microstructure and the evolution in this microstructure during the deformation process.

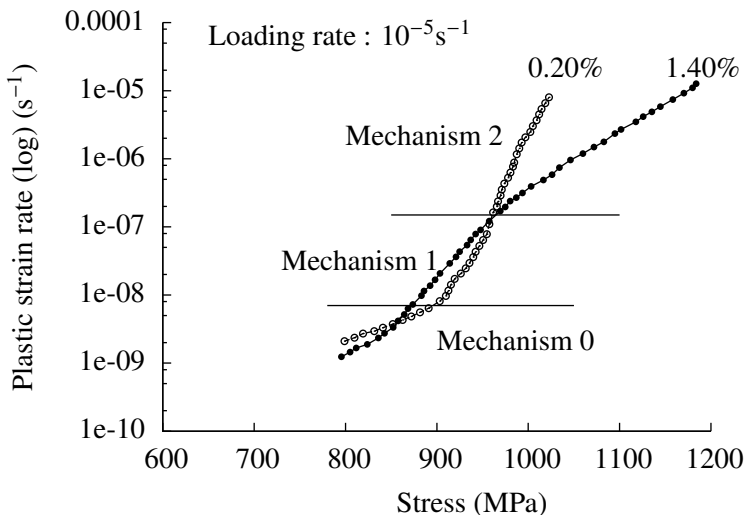


Figure 1.1. N18 alloy: stress relaxation at 650°C for two levels of initial plastic strain [SAÏ 04]

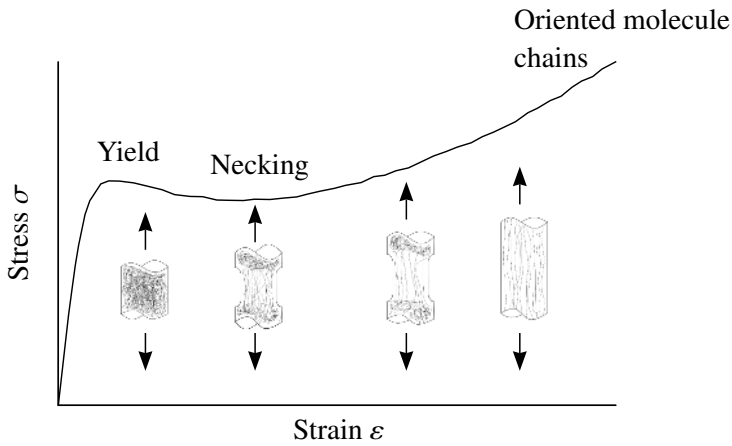


Figure 1.2. Typical stress–strain curve for glassy polymers

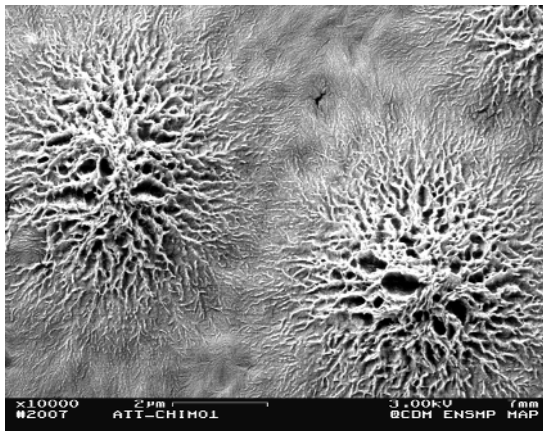


Figure 1.3. SEM observation of the spherulitic microstructure of a semi-crystalline polymer, polyamide 6 (PA6) [REG 09b]

Depending on the application, the MM models presented later in this book can be seen as *multi-mechanism*, *multi-phase* or *multi-potential*. The corresponding terminology can be characterized as follows:

- “Multi-phase” is used when different phases can be distinguished in a material such as the SCP.

– “Multi-mechanism” is used when different regimes can be observed in a material. These behaviors may be linked to different strain or stress ranges or different temperatures. The term “multi-mechanism” can also be used if nonlinearities on the macroscale come from different sources of deformation at the crystal level.

– “Multi-potential” refers to potentials in the thermodynamic formalism.

Beside the mechanisms related to metallic materials and polymers enumerated above, phase transformation is a source of deformation that deserves to be included in the MM modeling. “Phase transformation” refers to solid–solid metallurgical transformations that change the volume fraction of crystallographic phases. According to experimental observations, the transformation can be classified into (i) diffusional transformations, (ii) displacive transformations and (iii) hybrid transformations:

– In *diffusional transformations*, the diffusion of one or several species allows the creation of a new balance between several phases (like austenite–ferrite in steels) or to change the morphology of precipitates (like gamma prime in Ni-base alloys). The resulting microstructure will minimize the energy of the full system.

– The *martensitic transformation* is a time-independent phase transformation that occurs in some metals, resulting in the formation of martensite. The martensite is generally formed as thin plates with a predefined orientation with respect to the parent phase.

– *Hybrid transformations* incorporate characteristics related to both diffusional and displacive transformations. The ferrous bainitic transformation is a typical example because of its morphology, kinetics and mechanism of growth.

All these transformations are regulated by thermodynamic considerations. The driving forces for the transformation depend on temperature and deformation. Twinning-induced plasticity (TWIP) is an example where the phase change is produced by deformation (Figure 1.4). The well-known TWIP steels are increasingly used due to their high strength and good ductility.

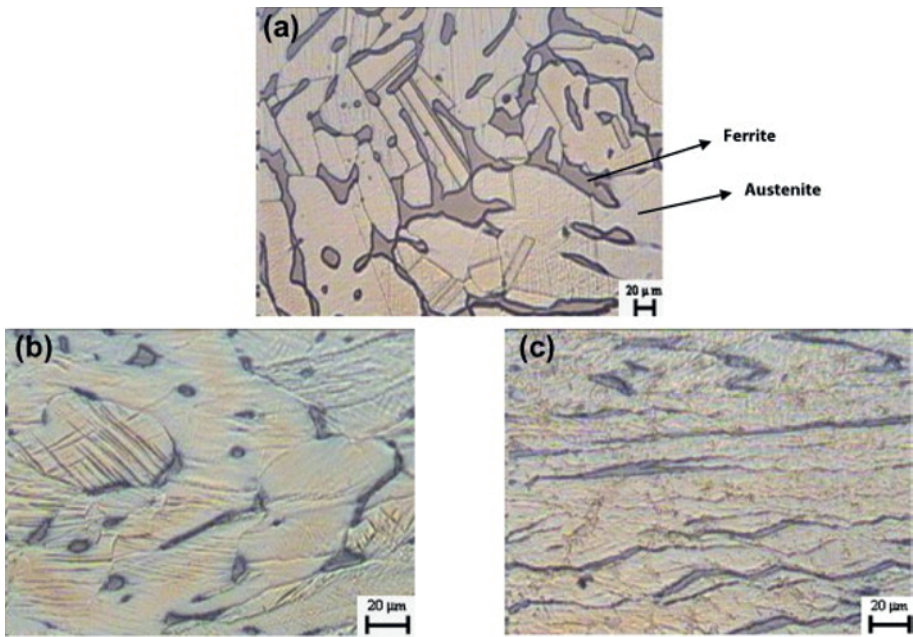


Figure 1.4. *The optical microstructures of an experimental alloy: (a) as-received dual-phase TWIP steel, (b) 40% cold-rolled steel and (c) 70% cold-rolled steel [TOR 11]. For a color version of the figure, see www.iste.co.uk/cailletaud/multi-mechanism.zip*

1.2. Building bricks

This section provides an analytical presentation of the various numerical tools that will be used in any model development. Having these elements in hand, the plastic and viscoplastic models can be constructed by assembling various blocks. What is needed is the definition of:

- the initial yield surface, defined by a plasticity criterion;
- the way in which this surface will change during inelastic flow, defined by hardening rules. Classical models introduce isotropic and/or kinematic hardening. Specific models are able to reproduce surface distortions;
- the flow direction and intensity. They come directly from the two previous items in the case of time-independent plasticity. The intensity is defined by a viscosity rule for viscoplastic models.

For the sake of brevity, most of the blocks are presented without long comments, which allows us to mention rather a large number of possibilities.

1.2.1. *Criteria*

A yield criterion is the function f characterizing the surface that delimits the elasticity domain under multiaxial stress states. For the case of materials with no hardening, it is defined in the space formed by the six components of the stress tensor, or by the eigen stresses or stress invariants. Hardening will deform the surface. It means that, when hardening is present, the function f depends not only on stress, but also on hardening variables. Only initial yield is addressed for the moment. After defining the notations, two tables will first introduce classical choices for isotropic materials and then for anisotropic materials.

- The most general form of the criterion is as follows:

$$f(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}) = 0$$

- The principal stresses σ_1, σ_2 and σ_3 can also be used in some cases:

$$f = (\sigma_1, \sigma_2, \sigma_3) = 0.$$

- The three invariants of the stress tensor or the two invariants of its deviator can also be of interest, specifically for isotropic materials:

$$f(I_1, I_2, I_3) = 0 \quad \text{or} \quad f(J_2, J_3) = 0$$

where I_1 is the first principal invariant of the stress tensor and I_2, I_3 (resp. J_2, J_3) are the second and third invariants of the stress tensor (resp. its deviatoric part).

Depending on whether the plastic flow depends on the hydrostatic pressure or not, different classes of models can be considered:

- For a large class of materials, specifically those that deform by crystallographic slip, the hydrostatic pressure does not influence plastic behavior. The models can then be expressed in terms of deviatoric stress $\mathbf{s}$

and its invariants instead of the stress tensor $\underline{\sigma}$:

$$\underline{\mathbf{s}} = \underline{\sigma} - (I_1/3)\underline{\mathbf{I}} \quad [1.1]$$

$$J_1 = Tr(\underline{\mathbf{s}}) = 0 \quad [1.2]$$

$$J_2 = (1/2)Tr(\underline{\mathbf{s}}^2) \quad [1.3]$$

$$J_3 = (1/3)Tr(\underline{\mathbf{s}}^3) \quad [1.4]$$

– On the other hand, polymeric materials, metallic alloys experiencing damage, or powders, show a significant sensitivity to hydrostatic pressure. f is then function of the stress invariants:

$$I_1 = Tr(\underline{\sigma}) \quad [1.5]$$

$$I_2 = (1/2)Tr(\underline{\sigma}^2) \quad [1.6]$$

$$I_3 = (1/3)Tr(\underline{\sigma}^3) \quad [1.7]$$

The most widely used criterion is the von Mises criterion [MIS 13], even though it was formulated after Maxwell (1865) and Huber (1904). It can be applied to isotropic materials that are insensitive to hydrostatic pressure. For this case, the simplest formulation based on invariants introduces the second invariant of the deviatoric stress tensor. In the following, the critical variable retained to express the criterion will be J instead of J_2 :

$$f(\underline{\sigma}) = J - \sigma_y \quad \text{with} \quad J = \left((3/2) \underline{\mathbf{s}} : \underline{\mathbf{s}} \right)^{1/2} \quad [1.8]$$

where σ_y is the initial yield limit. The interest of using J is that the 3D expression (used for structural computations) is the same as the 1D expression (used for the material parameter identification), provided the absolute value of the stress (1D) is replaced by J (3D).

Another very simple criterion is due to Tresca [TRE 64]. It can be expressed as a function of the eigenstresses as follows:

$$f(\underline{\sigma}) = \text{Max}_{i,j} |\sigma_i - \sigma_j| - \sigma_y \quad [1.9]$$

or as a combination of the second and third invariants:

$$f(\sigma) = J \sin\left(\theta + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} \sigma_y \quad [1.10]$$

where θ is the so-called Lode angle:

$$\cos(3\theta) = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} = \left(\frac{K}{J}\right)^3 \quad \text{with} \quad K^3 = \frac{27}{2} J_3 \quad [1.11]$$

Table 1.1 shows a list of classical criteria for isotropic materials. A non-exhaustive list of anisotropic yield criteria is given in Table 1.2.

Isotropic criteria without hydrostatic pressure von Mises [MIS 13] $f(\sigma) = \left(\frac{1}{2}(\sigma_1 - \sigma_2 ^2 + \sigma_2 - \sigma_3 ^2 + \sigma_3 - \sigma_1 ^2)\right)^{1/2} - \sigma_y$ Hosford [HOS 72] $f(\sigma) = \left(\frac{1}{2}(\sigma_1 - \sigma_2 ^m + \sigma_2 - \sigma_3 ^m + \sigma_3 - \sigma_1 ^m)\right)^{1/m} - \sigma_y$ Cazacu04 [CAZ 04] $f(\sigma) = J_2^{3/2} - c J_3 - (\tau_y)^3$ τ_y is the yield stress in pure shear. c is a material parameter related to the uniaxial yield stresses in tension and compression. Cazacu06 [CAZ 06] $f(\sigma) = (s_1 - k s_1)^a + (s_2 - k s_2)^a + (s_3 - k s_3)^a - F$ a and k are material parameters. F gives the size of the elastic domain. Drucker $f(\sigma) = t(\sigma) - \sigma_y$ $t(\sigma) = \frac{J}{2} \left((1 + K) + (1 - K) \left(\frac{J_3}{J} \right)^3 \right)$ K is a material parameter $0.778 < K \leq 1$ Criteria taking into account the hydrostatic pressure Mohr–Coulomb [MOH 00] $f(\sigma) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3) \sin \Phi - 2C \cos \Phi$ The material parameter C is called cohesion. The angle Φ denotes the internal friction in the material. Drucker–Prager $f(\sigma) = (1 - \alpha) t(\sigma) + \alpha I_1(\sigma) - \sigma_y$ t is already defined in the table above. α is a material parameter.
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Table 1.1. Examples of plastic isotropic criteria

Nouailhas [NOU 92]

$$f(\sigma) = \left(\left(\left[\frac{3}{2}(I_1 + 2a_4 I_4) \right]^2 + 3a_8 I_8 \right)^3 - (a_6 I_6)^4 \right)^{1/12} - \sigma_y$$

$$I_1 = s_{11}^2 + s_{22}^2 + s_{33}^2, I_6 = s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11}$$

$$I_4 = s_{12}^2 + s_{23}^2 + s_{31}^2, I_8 = s_{12}^4 + s_{23}^4 + s_{31}^4$$

Hill [HIL 76]

$$f(\sigma) = [F(\sigma_{11} - \sigma_{22})^2 + G(\sigma_{22} - \sigma_{33})^2 + H(\sigma_{33} - \sigma_{11})^2 + 2L\sigma_{12}^2 + 2M\sigma_{23}^2 + 2N\sigma_{13}^2]^{1/2} - \sigma_y$$

Hu [HU 05]

$$f(\sigma) = \left(J_2^{(1)} \right)^2 + \left(J_2^{(2)} \right)^2 + \left(J_2^{(3)} \right)^2 - \frac{1}{C^2}$$

$$\begin{pmatrix} \left(J_2^{(1)} \right)^2 \\ \left(J_2^{(2)} \right)^2 \\ \left(J_2^{(3)} \right)^2 \end{pmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_{11}^2 & a_{22}^2 & a_{33}^2 & a_{12}^2 & a_{23}^2 & a_{31}^2 \\ b_{11}^2 & b_{22}^2 & b_{33}^2 & b_{12}^2 & b_{23}^2 & b_{31}^2 \end{bmatrix} \begin{pmatrix} (\sigma_{11} - \sigma_{22})^2 \\ (\sigma_{22} - \sigma_{33})^2 \\ (\sigma_{33} - \sigma_{11})^2 \\ 6\sigma_{12}^2 \\ 6\sigma_{23}^2 \\ 6\sigma_{31}^2 \end{pmatrix}$$

Gotoh [GOT 77]

$$f(\sigma) = A_0(\sigma_{11} + \sigma_{22})^2 + \Sigma$$

$$\Sigma = [A_1\sigma_{11}^4 + A_2\sigma_{11}^3\sigma_{22}^2 + A_3\sigma_{11}^2\sigma_{22}^3 + A_4\sigma_{11}\sigma_{22}^4 + A_5\sigma_{22}^4 + (A_6\sigma_{11}^2 + A_7\sigma_{11}\sigma_{22} + A_8\sigma_{22}^2)\sigma_{12}^2 + A_9\sigma_{12}^4]$$

K-B [KAR 93]

$$\Phi = (1 - c)\Phi_1 + c\Phi_2 = 2Y^m$$

$$\Phi_1 = (|S_1 - S_2|^m + |S_2 - S_3|^m + |S_3 - S_1|^m)$$

$$\Phi_2 = \frac{3^m}{2^m - 1 + 1} (|S_1|^m + |S_2|^m + |S_3|^m)$$

Y is the average yield stress in uniaxial tension

$\mathbf{L}$ is a fourth-order tensorial operator that introduces material anisotropy

$$\mathbf{\tilde{S}} = \mathbf{L}(\sigma - \mathbf{B})$$

Banabic [BAN 04]

$$f(\sigma) = \left[a(b\Gamma + c\Psi)^{2k} + a(b\Gamma - c\Psi)^{2k} + (1 - a)(2c\Psi)^{2k} \right]^{1/2k}$$

$$\Gamma = M\sigma_{11} + N\sigma_{22}, \Psi = \sqrt{(P\sigma_{11} + Q\sigma_{22})^2 + R\sigma_{12}^2}$$

$$\text{Tsaï } f(\sigma) = f_H(\sigma) + Q(\sigma_{22} - \sigma_{33}) + P(\sigma_{11} - \sigma_{33})$$

Bron [BRO 04]

$$\bar{\sigma} = \left(\alpha (\bar{\sigma}^1)^a + (1 - \alpha) (\bar{\sigma}^2)^a \right)^{(1/a)}$$

$$\bar{\sigma}^k = (\psi^k)^{1/b^k}$$

$$\psi^1 = \frac{1}{2} (|S_2^1 - S_3^1|^{b^1} + |S_3^1 - S_1^1|^{b^1} + |S_1^1 - S_2^1|^{b^1})$$

$$\psi^2 = \frac{3b^2}{2b^2+2} (|S_1^2|^{b^2} + |S_2^2|^{b^2} + |S_3^2|^{b^2})$$

$S_{i=1,2,3}^k$ are the principal values of a modified stress deviator $\underline{\tilde{s}}^k$ defined by $\underline{\tilde{s}}^k = \underline{\mathbf{L}}^k : \underline{\sigma}$

$\underline{\mathbf{L}}^k$ are fourth-order tensorial operators that introduce material anisotropy

Yld2004-18p [BAR 05]

$$\Phi(\underline{\tilde{\mathbf{S}}}', \underline{\tilde{\mathbf{S}}}'') = |\underline{\tilde{\mathbf{S}}}'_1 - \underline{\tilde{\mathbf{S}}}''_1|^a + |\underline{\tilde{\mathbf{S}}}'_1 - \underline{\tilde{\mathbf{S}}}''_2|^a + |\underline{\tilde{\mathbf{S}}}'_1 - \underline{\tilde{\mathbf{S}}}''_3|^a + |\underline{\tilde{\mathbf{S}}}'_2 - \underline{\tilde{\mathbf{S}}}''_1|^a + |\underline{\tilde{\mathbf{S}}}'_2 - \underline{\tilde{\mathbf{S}}}''_2|^a + |\underline{\tilde{\mathbf{S}}}'_2 - \underline{\tilde{\mathbf{S}}}''_3|^a \\ + |\underline{\tilde{\mathbf{S}}}'_3 - \underline{\tilde{\mathbf{S}}}''_1|^a + |\underline{\tilde{\mathbf{S}}}'_3 - \underline{\tilde{\mathbf{S}}}''_2|^a + |\underline{\tilde{\mathbf{S}}}'_3 - \underline{\tilde{\mathbf{S}}}''_3|^a = 4\bar{\sigma}^a$$

$$\underline{\tilde{\mathbf{S}}}' = \underline{\mathbf{C}}' \underline{\tilde{\mathbf{S}}} = \underline{\mathbf{C}}' \underline{\mathbf{T}} \underline{\sigma} = \underline{\mathbf{L}}' \underline{\sigma}$$

$$\underline{\tilde{\mathbf{S}}}'' = \underline{\mathbf{C}}'' \underline{\tilde{\mathbf{S}}} = \underline{\mathbf{C}}'' \underline{\mathbf{T}} \underline{\sigma} = \underline{\mathbf{L}}'' \underline{\sigma}$$

$$\underline{\mathbf{C}}' = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & 0 & 0 & 0 \\ -\alpha_3 & 0 & -\alpha_4 & 0 & 0 & 0 \\ -\alpha_5 & -\alpha_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_8 \end{pmatrix}$$

$$\underline{\mathbf{C}}'' = \begin{pmatrix} 0 & -\alpha_{10} & -\alpha_{11} & 0 & 0 & 0 \\ -\alpha_{12} & 0 & -\alpha_{13} & 0 & 0 & 0 \\ -\alpha_{14} & -\alpha_{15} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_{18} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha_{17} \end{pmatrix}$$

$$\underline{\mathbf{T}} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 2 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Table 1.2. Examples of plastic anisotropic criteria

Figures 1.5, 1.6 and 1.7 show classical yield surfaces plotted according to some common criteria. In these examples, the yield stress is 100 MPa. The Drucker criterion is first compared with the von Mises criterion for two values of the parameter K . There is a limit value for K , since, for the smaller values, the surface is no longer convex, as shown by Figure 1.5. The shape of the yield

surface predicted by the Hosford criterion is illustrated in Figure 1.6. It can be seen that, when $m = 1$ (or m goes to infinity), the Hosford criterion reduces to the Tresca yield criterion. When $m = 2$, the Hosford criterion reduces to the von Mises yield criterion. The so-called “Yld2004-18p” anisotropic model, known to capture the anisotropic behavior of a material sheet to a reasonable degree of accuracy is compared with Hill’s anisotropic criterion in Figure 1.7. In this example, the r -values are $r_0 = r_{90} = 0.6$. The Hill parameters $F = r_0/(r_0(1 + r_{90}))$, $G = 1/(1 + r_0)$ and $H = r_0/(1 + r_0)$ can then be deduced. The “Yld2004-18p” constants correspond to a fictitious material investigated by [YOO 06]. ($a = 8, \alpha_1 = 0.7297, \alpha_2 = 0.8777, \alpha_3 = 0.4252, \alpha_4 = 0.7268, \alpha_5 = 1.1386, \alpha_6 = 1, \alpha_{10} = 1.0513, \alpha_{11} = 1.0389, \alpha_{12} = 1.3289, \alpha_{13} = 1.1775, \alpha_{14} = 0.7651, \alpha_{15} = 0.9169$).

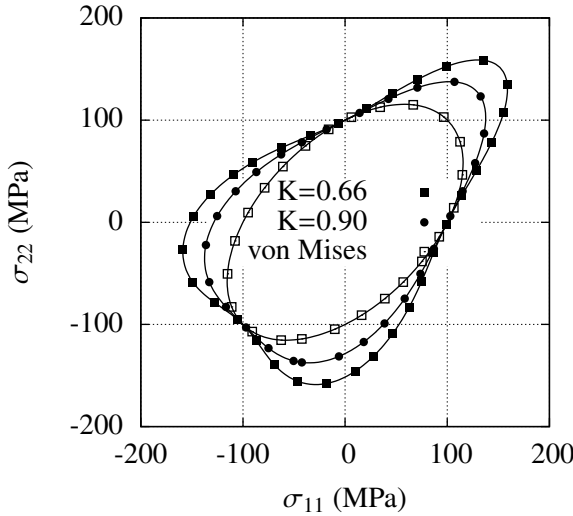


Figure 1.5. Yield surface: comparison of von Mises and Drucker criteria

1.2.2. Isotropic hardening rules

In this section, the word *isotropic* goes with the scalar variable R that defines the size of the elastic domain. Consequently, the rules providing isotropic hardening evolution also exist for anisotropic materials. First, we restrict ourselves to the case of isotropic materials. The relevant state variable that is associated with R is also a scalar. According to the way the theory is built, this variable may be the plastic work, or it may depend on the

accumulated plastic strain, or be the accumulated plastic strain itself. For the sake of simplicity, we adopt the latter assumption here. The variable will be denoted by p , with $\dot{p} = \left((2/3) \dot{\varepsilon}^p : \dot{\varepsilon}^p \right)^{1/2}$. Note that p is not exactly the accumulated inelastic strain for the MM models as it will be shown later.

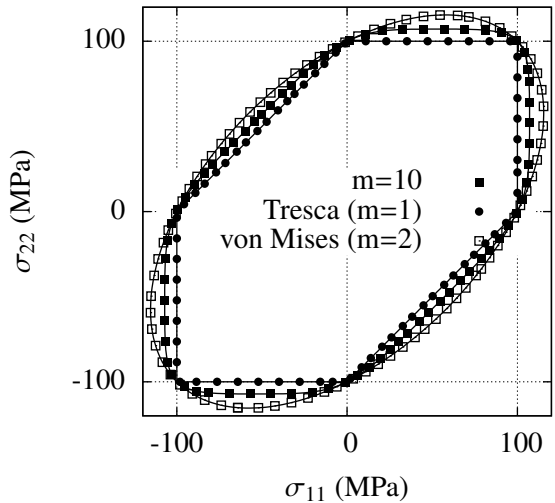


Figure 1.6. *Yield surface: comparison of von Mises, Tresca and Hosford criteria*

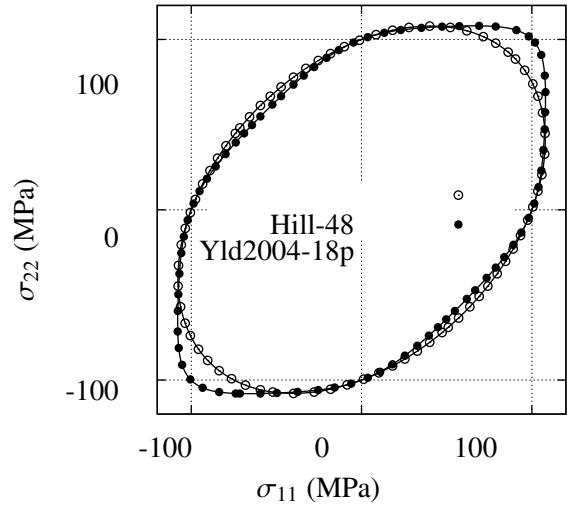


Figure 1.7. *Yield surface: comparison of the anisotropic criteria Hill-48 and Yld2004-18p*

In the framework of time-independent plasticity, the simplest model uses the following von Mises yield function:

$$f(\underline{\sigma}, R) = J(\underline{\sigma}) - R - R_0 \quad [1.12]$$

where R_0 is a function of the initial size of the yield surface. Besides the simplest rules (constant size, $R = 0$, or linear evolution of the yield surface, $R = Hp$), classical rules take into account softening or hardening of materials:

$$R = Q(1 - e^{-bp}) \quad [1.13]$$

where the material parameter Q is positive for the hardening effect and negative for the softening effect, and b controls the rate of saturation of the hardening with respect to plastic flow. To allow fine tuning of isotropic hardening, the following two-term evolution rule introduces short- and long-term mechanisms, with possible positive or negative values for Q_1 and Q_2 :

$$R = Q_1(1 - e^{-b_1p}) + Q_2(1 - e^{-b_2p}) \quad [1.14]$$

Isotropic hardening has been used in the literature to account for specific effects due to loading path. We have retained the most popular effects, namely non-proportional effect and strain memory effect. The description of the mechanical hardening has also been associated with microstructural changes, so that isotropic hardening is classically linked to dislocation densities, as shown at the end of this section.

1.2.2.1. *Non-proportional effect*

Non-proportional loading paths have been found to produce additional hardening, if compared with reference one-dimensional cyclic loadings. The landmark paper in this field [LAM 78] used combined axial–torsional fatigue tests, and specifically “out-of-phase” tests, where the sinusoidal signals for the axial strain and the shear strain present a 90° phase lag. The hardening was compared with “in-phase” loadings, with a zero phase lag. “Additional hardening” is present if the equivalent stress range obtained in “out-of-phase” tests is larger than that for “in-phase” tests with the same equivalent strain range. After initial studies on copper and 304 stainless steel, dozens of studies have shown that the effect is present each time the stacking fault energy (SFE) is large: in these conditions, the dislocation pair must

remain close to each other, so that they can easily change their slip planes. This mechanism is reinforced by non-proportionality, and leads to additional hardening. Consequently, the largest effects are observed on FCC materials with high SFE.

A specific variable is created to generate a ranking of the various loading path histories. It must be determined by an ordinary differential equation (ODE), and take a fixed reference value for in-phase loadings. A measurement of the non-proportionality comes from the relative direction of the actual state and the actual rate, in terms of stress, or plastic strain rate, for instance. Benallal and Marquis [BEN 87] have tested many different angles, before proposing a model based on the angle α between the actual plastic strain rate and the deviatoric stress rate. The model has a good quality/price ratio, even if it is not able to describe cross-effect for certain specific loading histories. This angle is equal to zero for any proportional loading.

The isotropic strain hardening variable R then takes the following form:

$$R = Q_1(1 - e^{-b_1 p}) + Q_2(1 - e^{-b_2 p}) \quad [1.15]$$

Q_1 , b_1 and b_2 are material parameters. Q_2 is used to express the non-proportionality through the angle α :

$$\dot{Q}_2 = [(d - f)A + f](Q_s - Q_2)\dot{p} \quad [1.16]$$

$$Q_s = \frac{gAQ_m + (1 - A)Q_0}{gA + (1 - A)} \quad [1.17]$$

$$\cos \alpha = \frac{\dot{\underline{s}} : \dot{\underline{\varepsilon}}^p}{\sqrt{\dot{\underline{s}} : \dot{\underline{s}}} \sqrt{\dot{\underline{\varepsilon}}^p : \dot{\underline{\varepsilon}}^p}} \quad [1.18]$$

$$A = 1 - \cos^2 \alpha \quad [1.19]$$

d , f , g , Q_0 and Q_m are material parameters.

It is worth noting that the evaluation of the non-proportionality by means of the angle between plastic strain rate and the deviatoric stress rate has two main drawbacks related to certain specific situations. This angle will be left undefined each time the deviatoric stress rate is zero; this case corresponds, for instance, to classical creep tests. The second drawback can be seen in situations

where non-proportional effects may exist. Such cases may be encountered in multistep tests such as cyclic tension compression (sequence 1) followed by cyclic torsion (sequence 2). In such a configuration, if, after the first sequence, an adequate path is applied in order to come back to zero axial stress and strain [JIA 08, MUR 82], the application of the second sequence may be accompanied by extra hardening despite the fact that $\alpha=0$. When the MM model is used for simulating creep tests, we let $A=0$, which corresponds to proportional loading conditions.

1.2.2.2. Strain memory effect

After non-proportionality of strain path, we consider here the effect of the strain range history. For the previous case, it has been demonstrated that the variation in the strain state is able to activate mechanisms that are not present for in-phase loadings. In the present case, the application of a large prestrain, either monotonic or cyclic, produces an additional hardening in subsequent loading cases of smaller strain ranges.

The experimental demonstration of this effect on 316 stainless steel is shown in [CHA 79]. It is shown that the value of the stress range for a given cyclic strain range is augmented if a larger cyclic strain range has been applied first. It means that the value of the parameter Q that controls the maximum reached by the isotropic hardening variable R of equation [1.13] should in fact depend on strain history. An additional variable, q , is then introduced [CHA 79, OHN 82] in order to keep the memory of the maximum inelastic strain range. A plasticity-type formalism is used in the plastic strain space, with a non-hardening surface F defined by

$$F(\boldsymbol{\varepsilon}^p, \boldsymbol{\xi}, q) = \frac{2}{3} J(\boldsymbol{\varepsilon}^p - \boldsymbol{\xi}) - q \quad [1.20]$$

The memory surface is activated each time plastic strain overpasses its previous maximum. Its transformation is controlled by two variables, the tensor $\boldsymbol{\xi}$ that defines its center, and the scalar q , figuring its radius. The respective evolution rules can be written as follows:

$$\dot{q} = \eta H(F) \langle \mathbf{n} : \mathbf{n}^* \rangle \dot{p} \quad \dot{\boldsymbol{\xi}} = \frac{3}{2} (1 - \eta) H(F) \langle \mathbf{n} : \mathbf{n}^* \rangle \mathbf{n}^* \dot{p} \quad [1.21]$$

$$\mathbf{n}^* = \left(\partial F / \partial \boldsymbol{\varepsilon}^p \right) / \left\| \partial F / \partial \boldsymbol{\varepsilon}^p \right\| \quad \mathbf{n} = \left(\partial f / \partial \boldsymbol{\sigma} \right) / \left\| \partial f / \partial \boldsymbol{\sigma} \right\| \quad [1.22]$$

where η is a material parameter ($0 < \eta < 1$) and where $\underline{\mathbf{n}}$ and $\underline{\mathbf{n}}^*$ are, respectively, the unit normal to the classical yield surface and to the non-hardening region. $H(\cdot)$ stands for the Heaviside function. A value of 1 for η would mean that the material has a “total memory” of the strain path, so that the value of the strain memory reaches the maximum plastic strain value. Lower values for η mean that the increase in the plastic range is partially accommodated by a translation of the memory surface, so that only a fraction of it is memorized in the variable. This “partial memory” case corresponds to the experimental observations.

The variable q is finally used to update the value of Q , which has an impact on the size of the elastic domain:

$$\dot{R} = b(Q - R)\dot{p} \quad \text{and} \quad \dot{Q} = 2\mu(Q_M - Q)\dot{q} \quad [1.23]$$

1.2.2.3. Dislocation density-based hardening rules

When plastic flow is governed by dislocation motion, a series of mechanistic models are available in the literature. Almost all of them are based on the assumption that flow stress is proportional to the square root of the dislocation density, at a temperature higher than $0.2T_m$. As a result, an approximate form of the hardening curve is

$$\sigma = \sigma_y + M\alpha\mu b \sqrt{\rho} \quad [1.24]$$

where M is the Taylor factor, μ the shear modulus, b the modulus of the Burgers vector, α a material constant and ρ the dislocation density [MEC 81].

1.2.3. Kinematic hardening rules (KHR)

The concept of internal stress was developed independently by researchers in metallurgy and mechanics in the 1960s. The idea supporting the concept is that, due to material heterogeneities, the effective stress that is applied to deformation mechanisms differs from the macroscopic stress. On a microstructural level, several kinematic hardening types should be considered, since the kinematic effect is present on a given level each time heterogeneities are present, namely kinematic hardening is created at the intergranular level, due to residual stress redistribution between the various grains. Kinematic hardening is also present inside grains, due to dislocation–precipitate or dislocation–dislocation interactions. Adopting the mechanical point of view,

the kinematic hardening rule specifies the translation of the yield surface, then the shape of the hysteresis loop for cyclic loadings, and also regulates the ratcheting effect.

Kinematic hardening rules (KHR) have been considered as the main features introduced in the MM models, and their coupling is a critical aspect of the model behavior. It is then interesting to analyze the individual response of the various formulations with respect to uniaxial and biaxial loading (i.e. tension–torsion loading).

Following the basic nonlinear hardening rule of Armstrong–Frederick (AF) [ARM 66], a major contribution to the model evolution has been made by Chaboche [CHA 08] who proposed among other ideas the decomposition of the total backstress into a sum of several variables. A generalized form of the backstress increment was proposed by [JIA 96].

$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p$	[PRA 56]
$\dot{\underline{\mathbf{X}}} = (\underline{\sigma} - \underline{\mathbf{X}}) \dot{p}$	[ZIE 59]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C(p) \dot{\underline{\epsilon}}^p - D(p) \underline{\mathbf{X}} \dot{p}$	[MRO 67]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \underline{\mathbf{X}} \dot{p}$	[ARM 66, CHA 77]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \phi(p) \underline{\mathbf{X}} \dot{p} - D^* [J(\underline{\mathbf{X}})]^{m-1} \underline{\mathbf{X}}$	[MAL 72, CHA 75]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D(\underline{\mathbf{X}} - \underline{\mathbf{Y}}) \dot{p}$	[ROU 85a]
$\dot{\underline{\mathbf{Y}}} = \frac{1}{K} (\underline{\mathbf{X}} - \underline{\mathbf{Y}}) \left(\frac{3}{2} \dot{\underline{\mathbf{X}}} : \dot{\underline{\mathbf{X}}} \right)^{1/2}$	
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - \frac{D^2}{C} [J(\underline{\mathbf{X}})]^{m-1} \underline{\mathbf{X}} \dot{p}$	[HEN 87]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \phi(p) \underline{\mathbf{X}} \dot{p}$	[MAR 89]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \left(1 - \frac{X_I}{J(\underline{\mathbf{X}})} \right) \underline{\mathbf{X}} \dot{p}$	[CHA 91]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \left(\frac{J(\underline{\mathbf{X}})}{C/D} \right)^m \left\langle \dot{\underline{\epsilon}}^p : \frac{\underline{\mathbf{X}}}{J(\underline{\mathbf{X}})} \right\rangle \underline{\mathbf{X}}$	[OHN 93]
$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\epsilon}}^p - D \left((1 - \delta)(\underline{\mathbf{X}} : \underline{\mathbf{n}}) \underline{\mathbf{n}} + \delta \underline{\mathbf{X}} \right) \dot{p}$	[BUR 87, DEL 95]
<> denotes the MacCauley brackets $\langle u \rangle = u$ if $u > 0$ and $\langle u \rangle = 0$ if $u < 0$	
$\phi(p) = \phi_\infty + (1 - \phi_\infty) e^{-\omega p}$	$J(\underline{\mathbf{x}}) = \sqrt{\frac{3}{2} \underline{\mathbf{x}} : \underline{\mathbf{x}}}$

Table 1.3. Historical review of KHR for unified models

As far as ratcheting is concerned, the basic AF nonlinear kinematic hardening rule is known to overpredict the amount of accumulated strain. This is why many authors have proposed modifications of the NLK rule (Table 1.3). Ohno and Wang [OHN 93] assumed that the dynamic recovery term in backstress evolution vanishes for low plastic strain ranges. Chaboche [CHA 91] introduces a nonlinearity in the recall term via a threshold, that will affect the limit state producing plastic shakedown. Burlet and Cailletaud [BUR 87] and Delobelle *et al.* [DEL 95] used a combined model in which the nonlinear recall term of the AF model is superimposed to a radial fading memory term. This compromise was also used by Chen and Jiao [CHE 04] who proposed a modified kinematic hardening rule connecting the dynamic recovery term of [OHN 93] and the evanescence term of the model of [BUR 87].

1.2.4. Plastic modulus

In the following, the constitutive equations of four classical models [MRO 67, DAF 76, CHA 86, LUB 91] are briefly recalled. The resulting plastic modulus, that is, $H = d\sigma/d\varepsilon^p = \dot{\sigma}/\dot{\varepsilon}^p$, the slope of the curve $\sigma = h(\varepsilon^p)$, for one-dimensional loading, is expressed for each case.

1.2.4.1. Mroz's model [MRO 67]

Mroz's model involves multiple yield surfaces, with linear hardening for each of them. The predicted stress-strain curve is then piecewise linear. The resulting curve tends to be a smooth nonlinear curve when the number of yield surfaces increases. The model is based on several surfaces $f_0, f_1, \dots, f_a$, with f_0 being the initial yield surface:

$$f_i = J(\sigma - \mathbf{X}_i) - R \quad [1.25]$$

In the following, $f_a = 0$ denotes the active surface (loading surface) and the inner surfaces $f_0, f_1, \dots, f_{a-1}$ are tangent at the loading point:

$$f_0 = f_1 = \dots = f_a = 0 \quad [1.26]$$

The plastic strain rate is given by

$$\dot{\varepsilon}^p = \mathbf{n}^a \dot{p}^a \quad \text{with} \quad \mathbf{n}^a = \frac{\partial f_a}{\partial \sigma} \quad \text{and} \quad \dot{p}^a = \frac{\langle \mathbf{n}^a : \dot{\varepsilon} \rangle}{C^a \mathbf{n}^a : \mathbf{n}^a} \quad [1.27]$$

The chosen rule imposes the same translation for the inner surfaces:

$$\dot{\tilde{\mathbf{X}}}^1 = \dot{\tilde{\mathbf{X}}}^2 = \dots = \dot{\tilde{\mathbf{X}}}^{a-1} = \dot{\tilde{\mathbf{X}}}^a \quad [1.28]$$

$$\dot{\tilde{\mathbf{X}}}^a = \underline{\Delta}_a^{a+1} \dot{\mu} \quad [1.29]$$

$$\underline{\Delta}_a^{a+1} = \frac{(R^{a+1} - R^a) \underline{\sigma} - (R^{a+1} \underline{\mathbf{X}}^a - R^a \underline{\mathbf{X}}^{a+1})}{R^a} \quad [1.30]$$

$$\dot{\mu} = \frac{\langle \underline{\mathbf{n}}^a : \dot{\underline{\sigma}} \rangle}{\underline{\mathbf{n}}^a : \underline{\Delta}_a^{a+1}} \quad [1.31]$$

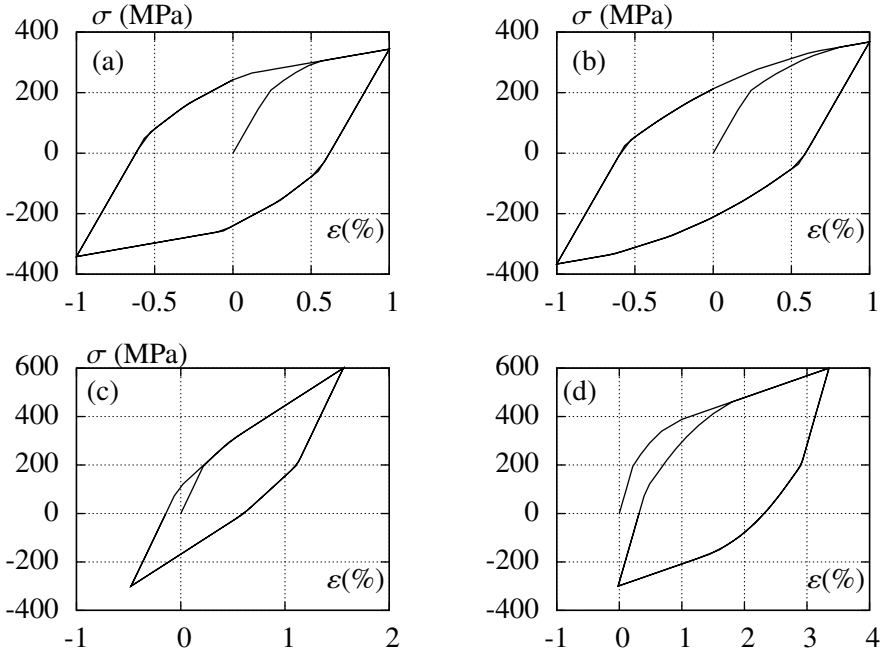


Figure 1.8. Response of Mroz's model. Top: symmetric loading under prescribed strain; bottom: unsymmetric loading under prescribed stress: (a) 4 surfaces, $R_{\varepsilon-1}$; (b) 8 surfaces, $R_{\varepsilon-1}$; (c) 4 surfaces $-300 < \sigma < 600$; (d) 8 surfaces $-300 < \sigma < 600$

Mroz's model is a generalization of the linear kinematic hardening rule. It makes it possible to correctly describe the nonlinear stress-strain behavior and the Bauschinger effect. However, under asymmetric stress loading, shakedown takes place after the first cycle. Figure 1.8 shows the response of the model

for a strain controlled test and a ratcheting test, using four surfaces and eight surfaces, respectively. The material parameters used for these simulations are given in Table 1.4.

E	C_1	C_2	C_3	C_4	R_1	R_2	R_3	R_4
90,000	70,000	40,000	10,000	5,000	10	20	30	∞
E	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
90,000	60,000	50,000	40,000	30,000	20,000	10,000	5,000	4,000
R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	
200	230	260	290	320	350	380	∞	

Table 1.4. Material parameters (MPa) used for the simulation of Mroz's model in Figure 1.8

1.2.4.2. Dafalias and Popov's model [DAF 76]

The model proposed by Dafalias and Popov can be seen as a Mroz model with two surfaces, but with a variable plastic modulus $K = K(\delta, \delta_{in})$ instead of the constant moduli C_i . The parameter δ is the distance (in terms of von Mises equivalent) between the actual stress and its “image”, which is the point on the bounding surface with the same normal. δ_{in} is the value of δ at the initial stage of the loading process. The plastic modulus varies between K_0 and ∞ using, for example,

$$K(\delta, \delta_{in}) = K_0 + h \frac{\delta}{\delta_{in} - \delta} \quad [1.32]$$

The plastic flow is then classically derived from the normality rule

$$\dot{\underline{\epsilon}}^p = \frac{1}{K} \langle \dot{\underline{\sigma}} : \underline{\mathbf{n}} \rangle \underline{\mathbf{n}} \quad [1.33]$$

1.2.4.3. Chaboche's model [CHA 86]

The yield criterion is

$$f = J(\underline{\sigma} - \underline{\mathbf{X}}) - R - R_0 = \left[\frac{3}{2} (\underline{\mathbf{s}} - \underline{\mathbf{X}}) : (\underline{\mathbf{s}} - \underline{\mathbf{X}}) \right]^{(1/2)} - R - R_0 \quad [1.34]$$

where $\underline{\mathbf{s}}$ is the deviatoric stress tensor and $\underline{\mathbf{X}}$ is the current yield surface center in the deviatoric space. R_0 and R are respectively the initial size and the evolution of the yield surface. The plastic flow follows the normality rule

$$\dot{\underline{\epsilon}}^p = \dot{p} \frac{\partial f}{\partial \underline{\sigma}} = \dot{p} \underline{\mathbf{n}} = \frac{3}{2} \dot{p} \frac{\underline{\mathbf{s}} - \underline{\mathbf{X}}}{J(\underline{\sigma} - \underline{\mathbf{X}})} \quad [1.35]$$

In the case of a time-independent framework, the plastic multiplier is given by the consistency condition $f = df = 0$ when plastic flow occurs ($\dot{p}=0$ when $f < 0$ or $df < 0$):

$$\frac{\partial f}{\partial \underline{\sigma}} : d\underline{\sigma} + \frac{\partial f}{\partial \underline{\mathbf{X}}} : d\underline{\mathbf{X}} + \frac{\partial f}{\partial R} : dR = 0 \quad [1.36]$$

The basic nonlinear kinematic hardening rule of Armstrong–Frederick can be written as

$$\dot{\underline{\mathbf{X}}} = \frac{2}{3} C \dot{\underline{\varepsilon}}^p - D \underline{\mathbf{X}} \dot{p} \quad [1.37]$$

The isotropic hardening variable R that characterizes the size change of the elastic domain can be written in integrated form:

$$R = Q(1 - e^{-bp}) \quad [1.38]$$

The expression of the plastic multiplier $\dot{p}$ comes from the consistency condition $\dot{f} = 0$:

$$\dot{p} = \frac{\underline{\mathbf{n}} : \dot{\underline{\sigma}}}{h} \quad \text{with} \quad h = Qbe^{-bp} - D \underline{\mathbf{n}} : \underline{\mathbf{X}} \quad [1.39]$$

The plastic strain rate is then given by

$$\dot{\underline{\varepsilon}}^p = \frac{< \underline{\mathbf{n}} : \dot{\underline{\sigma}} > : \underline{\mathbf{n}}}{h} \quad [1.40]$$

For uniaxial case, assuming that only σ_{11} is non-null, and $x = (2/3)X_{11}$:

$$\dot{\varepsilon}^p = \frac{\dot{\sigma}}{C + Qbe^{-bp} - Dx} \quad [1.41]$$

A close form solution can be obtained at each half cycle, beginning at ε_0^p , x_0 :

$$\sigma = \nu \frac{C}{D} + \left(x_0 - \nu \frac{C}{D} \right) e^{-D\nu(\varepsilon^p - \varepsilon_0^p)} + Qbe^{-bp} + R_0 \quad [1.42]$$

where $\nu = \pm 1$ is positive in the tension going branch and negative in the compression going branch.

1.2.4.4. Lubliner model [LUB 91]

This approach is motivated by the behavior of some solids which, after a tensile plastic flow and a partial elastic unloading, present a new plastic flow during reloading, for a stress value lower than the level previously reached in tension (Figure 1.9). In this so-called “generalized plasticity” model, the relation between the plastic strain ε^p and the stress σ on initial loading under uniaxial stress results in the integration of the differential equation:

$$\frac{d\varepsilon^p}{d\sigma} = \frac{1}{H\beta} \langle \sigma - \sigma_y - H\varepsilon^p \rangle \quad [1.43]$$

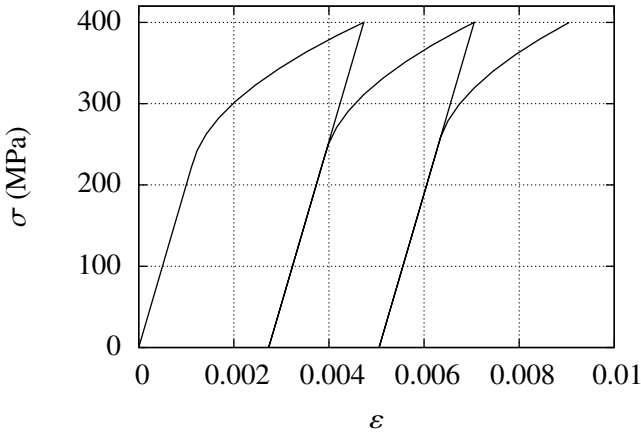


Figure 1.9. Stress–strain loading–unloading–reloading curve motivating the Lubliner model $E=200000$ MPa, $\sigma_y=230$ MPa, $H=5000$ MPa, $\beta=1000$ MPa

It can easily be demonstrated that the initial loading curve is described by

$$\varepsilon^p = \frac{\beta}{H} e^{-(\sigma - \sigma_y)/\beta} + \frac{\sigma - \sigma_y - \beta}{H} \quad [1.44]$$

For a test under prescribed stress between 0 and σ_M , the plastic flow occurs for the stress level

$$\sigma = \sigma_M + \beta \left(-1 + e^{-(\sigma - \sigma_y)/\beta} \right) \quad [1.45]$$

Lubliner *et al.* [LUB 93] have generalized the previous model to account for asymptotically perfectly plastic or strain softening by postulating the following uniaxial model:

$$\frac{d\varepsilon^p}{d\sigma} = \frac{\langle \sigma - \sigma_y - H\varepsilon^p \rangle}{H\beta + R[\beta - (\sigma - \sigma_y - H\varepsilon^p)]} \quad [1.46]$$

1.2.5. Viscosity

Viscoplastic constitutive equations are mainly characterized by a flow function to describe the relation between the viscous stress and the norm of the viscoplastic strain rate. Besides the intensively used power law (known as Norton's law), several approaches may be used to determine the flow function. Table 1.5 gives classical flow rules taken from the literature.

1.2.5.1. Portevin–Le Chatelier effect (PLC)

PLC describes a serrated stress–strain curve “jerky flow”, which some materials show as they undergo plastic deformation, specifically inhomogeneous deformation. The PLC effect is due to dynamic strain aging and can be associated with a negative strain rate sensitivity for some temperature and strain rate ranges. Two classes of models can be found in the literature:

1) According to [PEN 72], the PLC effect can be mathematically expressed by the empirical law:

$$\sigma = h\varepsilon + F(\dot{\varepsilon}) \quad [1.47]$$

h : work hardening rate considered constant and positive

$F(\dot{\varepsilon})$: function that gives rise to a strain rate dependency of σ .

$\dot{\varepsilon}$ is summed up over all cross-sections along the specimen length. For deformation with imposed strain rate $\dot{\varepsilon}_a$

$$\dot{\varepsilon}_a = \frac{\dot{\sigma}}{M} + \frac{1}{L} \int_0^L \dot{\varepsilon}(x) dx \quad [1.48]$$

L : length of the specimen and M is the combined elastic modulus of the specimen and the testing machine.

Kubin and Estrin [KUB 85] have investigated the PLC effect in the case of loading with constant stress rate and creep loading.

2) On the other hand, the PLC effect attributed to dynamic strain aging is described in [KUB 85] by including the time dependency of the solute concentration C_s at temporarily arrested dislocations:

$$\sigma = \sigma_f + M_i \left(\ln \frac{\dot{\epsilon}}{\dot{\epsilon}_0} + H'_c C_0 (K D t_a)^{2/3} \right) \quad [1.49]$$

σ_f : internal stress usually taken as a function of the forest dislocation density

M_i : instantaneous strain rate sensitivity

C_0 : alloy solute composition

D : solute diffusion coefficient

t_a : effective aging time of arrested mobile dislocations

K : constant that includes dislocation binding energy

$H'_c = \partial H / \partial C_s$

H : activation enthalpy

C_s : dislocation motion

This model was extended to the 3D case by [ZHA 01] and used by [MAZ 10]:

$$f = J(\sigma) - R(p) - P_1 C_s(p, t_a) \quad [1.50]$$

$$C_s = C_m \left(1 - e^{-P_2 p^\alpha t_a^n} \right) \quad [1.51]$$

$$R(p) = R_0 + Q(1 - e^{-bp}) \quad [1.52]$$

$$t_a = 1 - \frac{t_a}{w} \dot{p} \quad [1.53]$$

$$\dot{p} = \mu(f) = \dot{\gamma}_0 \sinh \left(\frac{< f >}{K} \right) \quad [1.54]$$

A closed-form solution giving t_a and the stress value has been reported by [MAZ 06] for a strain controlled test:

$$t_a = \frac{w}{\dot{\epsilon}} \left(1 - e^{-\frac{p}{w}} \right) + \frac{R_0}{E \dot{\epsilon}} e^{-\frac{p}{w}} \quad [1.55]$$

$$\sigma = K a \sinh \left(\frac{\dot{p}}{\dot{\gamma}_0} \right) + R_0 + Q(1 - e^{-bp}) + P_1 C_m \left(1 - e^{-P_2 p^\alpha t_a^n} \right) \quad [1.56]$$

Time-independent plasticity theory		
Plasticity	$\dot{f} = 0 \quad \dot{f} = 0$	
Norton rule		
Power law	$\dot{p} = \langle f/K \rangle^n$	[NOR 29]
Two-term law	$\dot{p} = \langle f/K_1 \rangle^{n_1} + \langle f/K_2 \rangle^{n_2}$	[CHA 89]
VBO		
Krempel	$\dot{p} = \frac{\sigma}{EK_1} \left(1 + \frac{\sigma}{K_2} \right)^{K_3}$	[KRE 86]
Strain hardening		
Strain hardening	$\dot{p} = \langle f/K \rangle^n (p + p_0)^m$	
Summation of inverse Norton rates		
Flow sum inv	$1/\dot{p} = \sum_i 1/\dot{p}_i$	
Summation of Norton terms		
Flow sum	$\dot{p} = \sum_i \dot{p}_i$	
Flow for crystal deformation		
Exponential crystal	$\dot{p} = \gamma_0 \exp \left[-\frac{F_0}{RT} [1 - \langle f/k \rangle^{n_1}]^{n_2} \right]$	
sinh function		
Miller	$\dot{p} = B \left[\sinh (f/K)^{1.5} \right]^n$	[MIL 76]
Delobelle	$\dot{p} = A \sinh (f/K)^n$	[DEL 84]
exp function		
Bodner	$\dot{p} = A \exp (-K/\sigma)^n$	[BOD 87]
Onera	$\dot{p} = \langle f/K \rangle^n \exp(\alpha \langle f/K \rangle^{n+1})$	[NOU 89]
Kocks	$\dot{p}_\infty \exp \left[-\frac{1}{C} \langle 1 - (\sigma/D)^p \rangle^q \right]$	[KOC 76]
Other flow rules		
	$\dot{p} = A \exp \left(-\frac{p + p_0}{\alpha \sigma^n} \right)$	
	$\dot{p} = \frac{1}{d^2} \frac{k_1 \sigma}{1 + k_2/(d\sigma^m)}$	

Table 1.5. Flow rules

1.3. Scale transition rules

1.3.1. General remarks on scale transition rules

Phases in the polycrystalline aggregate are defined according to crystal orientation, that is, all the grains falling in the same class of Euler angles are in the same phase. The models used to predict polycrystal behavior in plasticity differ essentially by the transition rule linking local and global stresses and strains.

Besides the simplest and most widely used models (uniform plastic strain or uniform stress), the most popular concept is the self-consistent framework proposed by [HIL 65] and revisited by many authors (see, for instance, [MOL 99]). In terms of rates, the local stress $\underline{\sigma}^g$ is expressed according to the global stress $\underline{\sigma}$, the global strain $\underline{\varepsilon}$ and the local strain $\underline{\varepsilon}^g$:

$$\underline{\dot{\sigma}}^g = \underline{\dot{\sigma}} + \underline{\mathbf{L}}^\star : (\underline{\dot{\varepsilon}} - \underline{\dot{\varepsilon}}^g) \quad [1.57]$$

The fourth-order tensor $\underline{\mathbf{L}}^\star$ takes into account the behavior of the equivalent medium of each grain. $\underline{\mathbf{L}}^\star$, also called the accommodation tensor, is defined by:

$$\underline{\mathbf{L}}^\star = \underline{\mathbf{L}}^{\text{eff}} : (\underline{\mathbf{S}}^{-1} - \underline{\mathbf{I}}) \quad [1.58]$$

where $\underline{\mathbf{S}}$ is the well-known Eshelby tensor and $\underline{\mathbf{I}}$ the fourth-order identity tensor. The tensor $\underline{\mathbf{S}}$ is written in terms of effective modulus $\underline{\mathbf{L}}^{\text{eff}}$. Thus,

$$\begin{cases} \left(\underline{\mathbf{L}} + \underline{\mathbf{L}}^\star \right) : \underline{\dot{\varepsilon}}^g = \left(\underline{\mathbf{L}}^{\text{eff}} + \underline{\mathbf{L}}^\star \right) : \underline{\dot{\varepsilon}} \\ \text{and} \\ \underline{\dot{\sigma}}^g = \underline{\mathbf{L}} : \left(\underline{\mathbf{L}} + \underline{\mathbf{L}}^\star \right)^{-1} : \left(\underline{\mathbf{L}}^{\text{eff}} + \underline{\mathbf{L}}^\star \right) : \underline{\dot{\varepsilon}} \end{cases} \quad [1.59]$$

Finally, the following expression can be obtained:

$$\underline{\mathbf{L}}^{\text{eff}} = \left\langle \underline{\mathbf{L}} : \left(\underline{\mathbf{L}} + \underline{\mathbf{L}}^\star \right)^{-1} : \left(\underline{\mathbf{L}}^{\text{eff}} + \underline{\mathbf{L}}^\star \right) \right\rangle \quad [1.60]$$

Berveiller and Zaoui [BER 79] deduced an explicit transition rule from the previous approach, for the case of a radial monotonic loading path applied to a globally isotropic behavior and isochoric plastic flow:

$$\underline{\sigma}^g = \underline{\sigma} + \mu\alpha(\underline{\sigma}, \underline{\varepsilon}^p)(\underline{\varepsilon}^p - \underline{\varepsilon}^g) \quad \text{with} \quad \frac{1}{\alpha} \simeq 1 + \frac{3\mu\varepsilon^p}{2\sigma} \quad [1.61]$$

where σ and ε^p are, respectively, the overall equivalent stress and the plastic part of overall strain in the uniaxial tension test. They also showed that equation [1.61] allows plastic accommodation in the polycrystal, meanwhile Kröner's rule (obtained with $\alpha = 1$ [KRÖ 61]) produces only elastic accommodation and too large residual stresses. The idea behind all the approaches is finally to introduce a corrective term depending on plastic strains, to compute local stresses. Nevertheless, a linear dependency of this term with respect to plastic strains gives stresses that are too large. The nonlinear accommodation can also be obtained by means of a phenomenological approach, where the difference of the local and global plastic strains present in Kröner's model is replaced by the difference between a local and a global nonlinear hardening variable. The concept of β -rule introduced a variable β^g and its average on the whole aggregate β . When applied to a material presenting uniform isotropic elasticity with a shear modulus μ , the expression of the local stress $\underline{\sigma}^g$ is:

$$\underline{\sigma}^g = \underline{\sigma} + \mu(\beta - \beta^g) \quad \text{with} \quad \beta = \langle \beta^g \rangle \quad [1.62]$$

The symbol $\langle . \rangle$ denotes the volume average. The following rule can be used:

$$\underline{\sigma}^g = \underline{\sigma} + C(\beta - \beta^g) \quad \text{with} \quad \beta = \langle \beta^g \rangle \quad [1.63]$$

where C is a material parameter. The variable β^g presents a nonlinear evolution with respect to plastic strain according to the following rule [CAI 94]:

$$\dot{\beta}^g = \dot{\varepsilon}^g - D(\beta^g - \delta\varepsilon^g)\|\dot{\varepsilon}^g\| \quad [1.64]$$

The von Mises equivalent strain rate $\|\dot{\varepsilon}^g\|$ can also be replaced by the sum of the slip rate absolute values on all slip systems $|\dot{\gamma}^s|$ [CAI 87, SAÏ 06a]

$$\dot{\beta}^g = \dot{\varepsilon}^g - D(\beta^g - \delta\varepsilon^g) \sum_{s \in \mathcal{S}} |\dot{\gamma}^s| \quad [1.65]$$

1.3.2. Scale transition rules for the MM model

According to Kröner, the physical idea behind the self-consistent framework is the assumption of perfect disorder. It means that the mixture of the two constituents is such that, in a given realization, the probability of finding phase A and phase B in a given place is totally random. As a result, phase A can be seen as an inclusion in the homogeneous equivalent medium made of A and B, and phase B can also be seen as an inclusion in the homogeneous equivalent medium made of A and B. This is why the localization rules for both phases are totally symmetric. The β _rule is generally selected for its simplicity and its versatility. It can be used for a material with N crystallographic phases, and in the case of a 2-phase material as well.

In the MM models, each mechanism I is associated with a local stress tensor σ^I calculated from a stress concentration rule. Several approaches were used to obtain local stresses and strains through different transition rules. They are recalled in the following. The thermodynamic approach of MM models has first been investigated by [CAI 95] for the case of uniform elasticity. In this framework, the model is defined from the free energy $\rho\Psi$, which defines stress and hardening variables knowing elastic strain and internal variables. The plastic strain ε^p is decomposed into the two parts $\varepsilon^1, \varepsilon^2$:

$$\varepsilon^p = A_1 \varepsilon^1 + A_2 \varepsilon^2 \quad [1.66]$$

where A_1 and A_2 can be fourth-order tensors or scalars. From a thermodynamic point of view, equation [1.66] leads *de facto* to the following form of the local stresses:

$$\begin{cases} \sigma^1 = A_1 \sigma \\ \sigma^2 = A_2 \sigma \end{cases} \quad [1.67]$$

The simplest assumption involves considering $A_1 = A_2 = 1$. This assumption is widely used by Taleb and co-workers.

For the specific case of two-phase alloys (Figure 1.10), A_1 and A_2 can be seen as the volume fraction of the two phases, so that $A_1 + A_2 = 1$. This kind of consideration includes the works related to the semi-crystalline polymer behavior. The contribution of each phase (amorphous and crystalline) is taken into account through the degree of crystallinity.

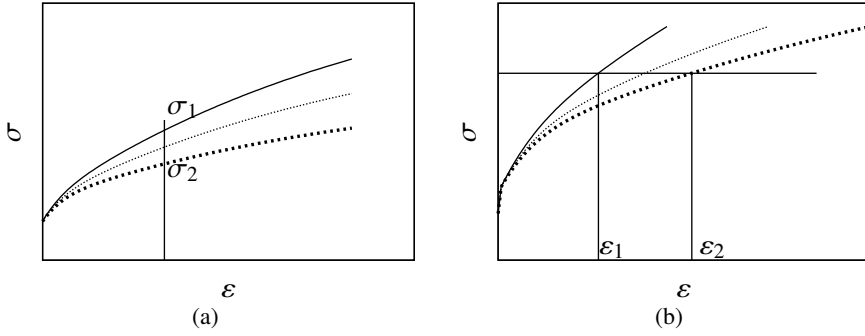


Figure 1.10. Estimation of the macroscopic behavior according to (a) uniform strain and (b) uniform stress assumption

Alternatively, the self-consistent schemes can be applied to the case of the MM model. The explicit transition rule of equation [1.61] leads to

$$\begin{cases} \tilde{\sigma}^1 = \tilde{\sigma} + A(\tilde{\varepsilon}^p - \tilde{\varepsilon}^1) \\ \tilde{\sigma}^2 = \tilde{\sigma} + A(\tilde{\varepsilon}^p - \tilde{\varepsilon}^2) \end{cases} \quad \text{with} \quad \tilde{\varepsilon}^p = A_1 \tilde{\varepsilon}^1 + A_2 \tilde{\varepsilon}^2 \quad [1.68]$$

whereas the “ β _rule”, equation [1.63], for 2-phase materials can be written as follows:

$$\begin{cases} \tilde{\sigma}^1 = \tilde{\sigma} + A(\tilde{\beta} - \tilde{\beta}_1) \\ \tilde{\sigma}^2 = \tilde{\sigma} + A(\tilde{\beta} - \tilde{\beta}_2) \end{cases} \quad \text{with} \quad \tilde{\beta} = A_1 \tilde{\beta}_1 + A_2 \tilde{\beta}_2 \quad [1.69]$$

The basic idea behind these localization rules is that phases can be hard or soft. Indeed, the softer phase shows a plastic strain $\tilde{\varepsilon}^1$ larger than the average plastic strain. Therefore, the stress level $\tilde{\sigma}^1$ supported by the soft phase will be smaller than the macroscopic stress. The hard phase will then be applied a higher stress level, $\tilde{\sigma}^2$. Accordingly, the plastic strain $\tilde{\varepsilon}^2$ will be smaller than the average plastic strain.

1.4. Large deformation

MM models are able to describe the mechanical behavior for both metallic materials and polymeric materials. For the latter, a finite strain formulation

is mandatory to correctly represent the large strains applied to the material element. Large strain formalism introduces the following standard expressions:

$$\underline{\mathbf{L}} = \dot{\underline{\mathbf{F}}} \underline{\mathbf{F}}^{-1} \quad \underline{\mathbf{D}} = \frac{1}{2} (\underline{\mathbf{L}} + \underline{\mathbf{L}}^T) \quad \underline{\mathbf{\Omega}} = \frac{1}{2} (\underline{\mathbf{L}} - \underline{\mathbf{L}}^T) \quad [1.70]$$

where $\underline{\mathbf{F}}$ is the deformation gradient, $\underline{\mathbf{L}}$ is the rate of deformation, $\underline{\mathbf{D}}$ is the stretch rate and $\underline{\mathbf{\Omega}}$ is the rotation rate.

The critical part of the algorithm is the definition of the derivatives of the stress and strain tensors. Several possibilities are known to introduce an objective derivation process. Convective transport rules of a given tensor consist of a transformation of this tensor in a rotating frame, where the derivative is applied, then applying the reverse rotation to obtain the value in the laboratory frame. Several candidates have been used in the literature:

– For the case of Green–Naghdi derivative, the rotation is defined by means of the $\underline{\mathbf{R}}$ tensor extracted from the polar decomposition $\underline{\mathbf{F}} = \underline{\mathbf{R}} \underline{\mathbf{U}}$:

$$\dot{\underline{\mathbf{T}}}^{GN} = \dot{\underline{\mathbf{T}}} + \underline{\mathbf{T}} \underline{\mathbf{R}} - \underline{\mathbf{R}} \underline{\mathbf{T}} \quad [1.71]$$

– The Jauman derivative uses the corotational frame, which is defined by a tensor $\underline{\mathbf{\Omega}}$, the spin tensor that comes from the material rotation:

$$\dot{\underline{\mathbf{T}}}^J = \dot{\underline{\mathbf{T}}} + \underline{\mathbf{T}} \underline{\mathbf{\Omega}} - \underline{\mathbf{\Omega}} \underline{\mathbf{T}} \quad [1.72]$$

The spin tensor is found by integrating the relation $\underline{\mathbf{\Omega}} = -\underline{\mathbf{Q}}^{-1} \dot{\underline{\mathbf{Q}}}$, ensuring that the rotating frame follows the material elements:

$$\dot{\underline{\mathbf{Q}}} = -\underline{\mathbf{Q}} \underline{\mathbf{\Omega}} \quad [1.73]$$

In these cases, the formulation of the material models with tensorial internal variables and even anisotropic response is the same for small perturbations or large strain assumptions [LAD 99].

Specifically, the additional partition of the strain into elastic and inelastic parts is still valid for large strains. The resulting response is an hypoelastic formulation, in agreement with a Green–Naghdi stress rate, that can be problematic, but only for very large strain values. The stress measure is the

Cauchy stress $\underline{\sigma}$ obtained using the relevant transformation, with either $\underline{\mathbf{R}}$ or $\underline{\mathbf{\Omega}}$:

$$\underline{\sigma} = \det^{-1}(\underline{\mathbf{F}}) \underline{\mathbf{R}} \underline{\mathbf{S}} \underline{\mathbf{R}}^T \quad \underline{\sigma} = \det^{-1}(\underline{\mathbf{F}}) \underline{\mathbf{Q}} \underline{\mathbf{S}} \underline{\mathbf{Q}}^T \quad [1.74]$$

1.5. Brief history of the MM models

MM models have emerged following the development of the crystallographic approaches. Taylor [TAY 38] and Bishop and Hill [BIS 51] were among the pioneers who contributed to the extension of the elasto-plastic macroscopic theory in order to study polycrystal behavior. Sanders [SAN 54], Budiansky and Wu [BUD 62] and Koiter [KOI 60] have considered that the plastic deformation is the result of slip that occurs along parallel planes or more generally involves several “independent” mechanisms. Mandel [MAN 65] has generalized this concept to the case of dependent mechanisms to investigate the uniqueness and existence of the plastic multipliers. Following his work, two kinds of approaches have been developed: (i) the polycrystalline plasticity models, see, for instance, [BUI 69, ZAO 70] and (ii) the MM models as seen by Zarka and co-workers [ZAR 72, ZAR 78, ZAR 79, ZAR 80]. These two approaches take into account, as a common root, the microstructural heterogeneity of the material that leads to internal stresses. Indeed, both approaches account for plastic accommodation induced by the difference between local (grain or mechanism) and global plastic strain. Analogy was carried out between the MM model types and the polycrystalline plasticity approach [CAI 08].

The MM approach belongs to a large class of models, which also includes “*multi-phase*” models. A non-exhaustive list of studies devoted to the “*multi-mechanism*” models type includes the works of [ZAR 79, CON 89, SAĬ 93, CAI 95, ROU 00, SAĬ 04, VEL 06, TAL 06, SAĬ 07, WOL 08, HAS 08, REG 09b, BES 09a, TAL 11, SAĬ 11a]. Investigations of the “*multi-phase*” models can be cited as: [KAR 74, VID 94, GAU 04, GAL 07, BRU 08, DEB 08, BAU 09, GAL 10, HOS 10, CLE 12, ELS 12].

Stress and strains can be expressed for the considered number of mechanisms n through m criteria. n can be equal or not to m . Nevertheless, the most common models relate to one of the following categories:

– “*Model with various mechanisms and various criteria*”: The crystal plasticity models for which the behavior of each slip system is represented individually by the resolved shear stress belong to this family. It is worth noting that the single crystal formulation is accounted for by the same set of material parameters for all the slip systems. However, if the mechanisms do not particularly refer to slip systems, the number n can reasonably be set at two or three. Otherwise, the large number of material parameters are difficult to identify. This kind of approach allows introducing several types of plastic mechanisms or phase behavior.

– “*Model with various mechanisms and one criterion*”: In this class of models, each mechanism has its own set of variables (i.e. stress, strain, hardening variables, etc.). However, a unique yield function is used to describe the different flow rules similarly to the unified approach. Even if all the mechanisms are simultaneously activated, differentiation between the level of variables can be allowed by the material parameters. The number of the isotropic material parameters is reduced compared with the first case.

