READING

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The Term Structure and Interest Rate Dynamics

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LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**a.** describe relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve;
☐	**b.** describe the forward pricing and forward rate models and calculate forward and spot prices and rates using those models;
☐	**c.** describe how zero-coupon rates (spot rates) may be obtained from the par curve by bootstrapping;
☐	**d.** describe the assumptions concerning the evolution of spot rates in relation to forward rates implicit in active bond portfolio management;
☐	**e.** describe the strategy of riding the yield curve;
☐	**f.** explain the swap rate curve and why and how market participants use it in valuation;
☐	**g.** calculate and interpret the swap spread for a given maturity;
☐	**h.** describe the Z-spread;
☐	**i.** describe the TED and Libor–OIS spreads;
☐	**j.** explain traditional theories of the term structure of interest rates and describe the implications of each theory for forward rates and the shape of the yield curve;

(continued)

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	**k.** describe modern term structure models and how they are used;
☐	**l.** explain how a bond's exposure to each of the factors driving the yield curve can be measured and how these exposures can be used to manage yield curve risks;
☐	**m.** explain the maturity structure of yield volatilities and their effect on price volatility.

1 INTRODUCTION

Interest rates are both a barometer of the economy and an instrument for its control. The term structure of interest rates—market interest rates at various maturities—is a vital input into the valuation of many financial products. The goal of this reading is to explain the term structure and interest rate dynamics—that is, the process by which the yields and prices of bonds evolve over time.

A spot interest rate (in this reading, "spot rate") is a rate of interest on a security that makes a single payment at a future point in time. The forward rate is the rate of interest set today for a single-payment security to be issued at a future date. Section 2 explains the relationship between these two types of interest rates and why forward rates matter to active bond portfolio managers. Section 2 also briefly covers other important return concepts.

The swap rate curve is the name given to the swap market's equivalent of the yield curve. Section 3 describes in more detail the swap rate curve and a related concept, the swap spread, and describes their use in valuation.

Sections 4 and 5 describe traditional and modern theories of the term structure of interest rates, respectively. Traditional theories present various largely qualitative perspectives on economic forces that may affect the shape of the term structure. Modern theories model the term structure with greater rigor.

Section 6 describes yield curve factor models. The focus is a popular three-factor term structure model in which the yield curve changes are described in terms of three independent movements: level, steepness, and curvature. These factors can be extracted from the variance–covariance matrix of historical interest rate movements.

A summary of key points concludes the reading.

2 SPOT RATES AND FORWARD RATES

In this section, we will first explain the relationships among spot rates, forward rates, yield to maturity, expected and realized returns on bonds, and the shape of the yield curve. We will then discuss the assumptions made about forward rates in active bond portfolio management.

At any point in time, the price of a risk-free single-unit payment (e.g., \$1, €1, or £1) at time T is called the **discount factor** with maturity T, denoted by $P(T)$. The yield to maturity of the payment is called a **spot rate**, denoted by $r(T)$. That is,

$$P(T) = \frac{1}{\left[1 + r(T)\right]^T} \tag{1}$$

The discount factor, $P(T)$, and the spot rate, $r(T)$, for a range of maturities in years $T > 0$ are called the **discount function** and the **spot yield curve** (or, more simply, **spot curve**), respectively. The spot curve represents the term structure of interest rates at any point in time. Note that the discount function completely identifies the spot curve and vice versa. The discount function and the spot curve contain the same set of information about the time value of money.

The spot curve shows, for various maturities, the annualized return on an option-free and default-risk-free **zero-coupon bond** (**zero** for short) with a single payment of principal at maturity. The spot rate as a yield concept avoids the complications associated with the need for a reinvestment rate assumption for coupon-paying securities. Because the spot curve depends on the market pricing of these option-free zero-coupon bonds at any point in time, the shape and level of the spot yield curve are dynamic—that is, continually changing over time.

As Equation 1 suggests, the default-risk-free spot curve is a benchmark for the time value of money received at any future point in time as determined by the market supply and demand for funds. It is viewed as the most basic term structure of interest rates because there is no reinvestment risk involved; the stated yield equals the actual realized return if the zero is held to maturity. Thus, the yield on a zero-coupon bond maturing in year T is regarded as the most accurate representation of the T-year interest rate.

A **forward rate** is an interest rate that is determined today for a loan that will be initiated in a future time period. The term structure of forward rates for a loan made on a specific initiation date is called the **forward curve**. Forward rates and forward curves can be mathematically derived from the current spot curve.

Denote the forward rate of a loan initiated T^* years from today with tenor (further maturity) of T years by $f(T^*,T)$. Consider a forward contract in which one party to the contract, the buyer, commits to pay the other party to the contract, the seller, a forward contract price, denoted by $F(T^*,T)$, at time T^* years from today for a zero-coupon bond with maturity T years and unit principal. This is only an agreement to do something in the future at the time the contract is entered into; thus, no money is exchanged between the two parties at contract initiation. At T^*, the buyer will pay the seller the contracted forward price value and will receive from the seller at time $T^* + T$ the principal payment of the bond, defined here as a single currency unit.

The **forward pricing model** describes the valuation of forward contracts. The no-arbitrage argument that is used to derive the model is frequently used in modern financial theory; the model can be adopted to value interest rate futures contracts and related instruments, such as options on interest rate futures.

The no-arbitrage principle is quite simple. It says that tradable securities with identical cash flow payments must have the same price. Otherwise, traders would be able to generate risk-free arbitrage profits. Applying this argument to value a forward contract, we consider the discount factors—in particular, the values $P(T^*)$ and $P(T^* + T)$ needed to price a forward contract, $F(T^*,T)$. This forward contract price has to follow Equation 2, which is known as the forward pricing model.

$$P(T^* + T) = P(T^*)F(T^*,T) \tag{2}$$

To understand the reasoning behind Equation 2, consider two alternative investments: (1) buying a zero-coupon bond that matures in $T^* + T$ years at a cost of $P(T^*+ T)$, and (2) entering into a forward contract valued at $F(T^*,T)$ to buy at T^* a zero-coupon bond with maturity T at a cost today of $P(T^*)F(T^*,T)$. The payoffs for the two investments at time $T^* + T$ are the same. For this reason, the initial costs of the investments have to be the same, and therefore, Equation 2 must hold. Otherwise, any trader could sell the overvalued investment and buy the undervalued investment with the proceeds to generate risk-free profits with zero net investment.

Working the problems in Example 1 should help confirm your understanding of discount factors and forward prices. Please note that the solutions in the examples that follow may be rounded to two or four decimal places.

EXAMPLE 1

Spot and Forward Prices and Rates (1)

Consider a two-year loan (T = 2) beginning in one year (T^* = 1). The one-year spot rate is $r(T^*) = r(1) = 7\% = 0.07$. The three-year spot rate is $r(T^* + T) = r(1 + 2) = r(3) = 9\% = 0.09$.

1 Calculate the one-year discount factor: $P(T^*) = P(1)$.
2 Calculate the three-year discount factor: $P(T^* + T) = P(1 + 2) = P(3)$.
3 Calculate the forward price of a two-year bond to be issued in one year: $F(T^*,T) = F(1,2)$.
4 Interpret your answer to Problem 3.

Solution to 1:

Using Equation 1,

$$P(1) = \frac{1}{(1 + 0.07)^1} = 0.9346$$

Solution to 2:

$$P(3) = \frac{1}{(1 + 0.09)^3} = 0.7722$$

Solution to 3:

Using Equation 2,

$0.7722 = 0.9346 \times F(1,2)$.

$F(1,2) = 0.7722 \div 0.9346 = 0.8262$.

Solution to 4:

The forward contract price of $F(1,2) = 0.8262$ is the price, agreed on today, that would be paid one year from today for a bond with a two-year maturity and a risk-free unit-principal payment (e.g., \$1, €1, or £1) at maturity. As shown in the solution to 3, it is calculated as the three-year discount factor, $P(3) = 0.7722$, divided by the one-year discount factor, $P(1) = 0.9346$.

2.1 The Forward Rate Model

This section uses the forward rate model to establish that when the spot curve is upward sloping, the forward curve will lie above the spot curve, and that when the spot curve is downward sloping, the forward curve will lie below the spot curve.

The forward rate $f(T^*,T)$ is the discount rate for a risk-free unit-principal payment $T^* + T$ years from today, valued at time T^*, such that the present value equals the forward contract price, $F(T^*,T)$. Then, by definition,

$$F(T^*,T) = \frac{1}{\left[1 + f(T^*,T)\right]^T} \tag{3}$$

By substituting Equations 1 and 3 into Equation 2, the forward pricing model can be expressed in terms of rates as noted by Equation 4, which is the **forward rate model**:

$$\left[1 + r(T^* + T)\right]^{(T^*+T)} = \left[1 + r(T^*)\right]^{T^*}\left[1 + f(T^*,T)\right]^T \tag{4}$$

Thus, the spot rate for $T^* + T$, which is $r(T^* + T)$, and the spot rate for T^*, which is $r(T^*)$, imply a value for the T-year forward rate at T^*, $f(T^*,T)$. Equation 4 is important because it shows how forward rates can be extrapolated from spot rates; that is, they are implicit in the spot rates at any given point in time.[1]

Equation 4 suggests two interpretations or ways to look at forward rates. For example, suppose $f(7,1)$, the rate agreed on today for a one-year loan to be made seven years from today, is 3%. Then 3% is the

- reinvestment rate that would make an investor indifferent between buying an eight-year zero-coupon bond or investing in a seven-year zero-coupon bond and at maturity reinvesting the proceeds for one year. In this sense, the forward rate can be viewed as a type of breakeven interest rate.
- one-year rate that can be locked in today by buying an eight-year zero-coupon bond rather than investing in a seven-year zero-coupon bond and, when it matures, reinvesting the proceeds in a zero-coupon instrument that matures in one year. In this sense, the forward rate can be viewed as a rate that can be locked in by extending maturity by one year.

Example 2 addresses forward rates and the relationship between spot and forward rates.

EXAMPLE 2

Spot and Forward Prices and Rates (2)

The spot rates for three hypothetical zero-coupon bonds (zeros) with maturities of one, two, and three years are given in the following table.

Maturity (*T*)	**1**	**2**	**3**
Spot rates	$r(1)$ = 9%	$r(2)$ = 10%	$r(3)$ = 11%

1. Calculate the forward rate for a one-year zero issued one year from today, $f(1,1)$.
2. Calculate the forward rate for a one-year zero issued two years from today, $f(2,1)$.
3. Calculate the forward rate for a two-year zero issued one year from today, $f(1,2)$.
4. Based on your answers to 1 and 2, describe the relationship between the spot rates and the implied one-year forward rates.

1 An approximation formula that is based on taking logs of both sides of Equation 4 and using the approximation $\ln(1 + x) \approx x$ for small x is $f(T^*,T) \approx [(T^* + T)r(T^* + T) - T^*r(T^*)]/T$. For example, $f(1,2)$ in Example 2 could be approximated as $(3 \times 11\% - 1 \times 9\%)/2 = 12\%$, which is very close to 12.01%.

Solution to 1:

$f(1,1)$ is calculated as follows (using Equation 4):

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$
$$(1 + 0.10)^2 = (1 + 0.09)^1[1 + f(1,1)]^1$$
$$f(1,1) = \frac{(1.10)^2}{1.09} - 1 = 11.01\%$$

Solution to 2:

$f(2,1)$ is calculated as follows:

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$
$$(1 + 0.11)^3 = (1 + 0.10)^2[1 + f(2,1)]^1$$
$$f(2,1) = \frac{(1.11)^3}{(1.10)^2} - 1 = 13.03\%$$

Solution to 3:

$f(1,2)$ is calculated as follows:

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$
$$(1 + 0.11)^3 = (1 + 0.09)^1[1 + f(1,2)]^2$$
$$f(1,2) = \sqrt[2]{\frac{(1.11)^3}{1.09}} - 1 = 12.01\%$$

Solution to 4:

The upward-sloping zero-coupon yield curve is associated with an upward-sloping forward curve (a series of increasing one-year forward rates because 13.03% is greater than 11.01%). This point is explained further in the following paragraphs.

The analysis of the relationship between spot rates and one-period forward rates can be established by using the forward rate model and successive substitution, resulting in Equations 5a and 5b:

$$[1 + r(T)]^T = [1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)] \tag{5a}$$

$$r(T) = \{[1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)]\}^{(1/T)} - 1 \tag{5b}$$

Equation 5b shows that the spot rate for a security with a maturity of $T > 1$ can be expressed as a geometric mean of the spot rate for a security with a maturity of $T = 1$ and a series of $T - 1$ forward rates.

Whether the relationship in Equation 5b holds in practice is an important consideration for active portfolio management. If an active trader can identify a series of short-term bonds whose actual returns will exceed today's quoted forward rates, then the total return over his or her investment horizon would exceed the return on a maturity-matching, buy-and-hold strategy. Later, we will use this same concept to discuss dynamic hedging strategies and the local expectations theory.

Examples 3 and 4 explore the relationship between spot and forward rates.

EXAMPLE 3

Spot and Forward Prices and Rates (3)

Given the data and conclusions for $r(1)$, $f(1,1)$, and $f(2,1)$ from Example 2:

$r(1) = 9\%$

$f(1,1) = 11.01\%$

$f(2,1) = 13.03\%$

Show that the two-year spot rate of $r(2) = 10\%$ and the three-year spot rate of $r(3) = 11\%$ are geometric averages of the one-year spot rate and the forward rates.

Solution:

Using Equation 5a,

$$[1 + r(2)]^2 = [1 + r(1)][1 + f(1,1)]$$

$$r(2) = \sqrt[2]{(1 + 0.09)(1 + 0.1101)} - 1 \approx 10\%$$

$$[1 + r(3)]^3 = [1 + r(1)][1 + f(1,1)][1 + f(2,1)]$$

$$r(3) = \sqrt[3]{(1 + 0.09)(1 + 0.1101)(1 + 0.1303)} - 1 \approx 11\%$$

We can now consolidate our knowledge of spot and forward rates to explain important relationships between the spot and forward rate curves. The forward rate model (Equation 4) can also be expressed as Equation 6.

$$\left\{\frac{[1 + r(T^* + T)]}{[1 + r(T^*)]}\right\}^{\frac{T^*}{T}}[1 + r(T^* + T)] = [1 + f(T^*,T)] \quad (6)$$

To illustrate, suppose $T^* = 1$, $T = 4$, $r(1) = 2\%$, and $r(5) = 3\%$; the left-hand side of Equation 6 is

$$\left(\frac{1.03}{1.02}\right)^{\frac{1}{4}}(1.03) = (1.0024)(1.03) = 1.0325$$

so $f(1,4) = 3.25\%$. Given that the yield curve is upward sloping—so, $r(T^* + T) > r(T^*)$—Equation 6 implies that the forward rate from T^* to T is greater than the long-term $(T^* + T)$ spot rate: $f(T^*,T) > r(T^* + T)$. In the example given, 3.25% > 3%. Conversely, when the yield curve is downward sloping, then $r(T^* + T) < r(T^*)$ and the forward rate from T^* to T is lower than the long-term spot rate: $f(T^*,T) < r(T^* + T)$. Equation 6 also shows that if the spot curve is flat, all one-period forward rates are equal to the spot rate. For an upward-sloping yield curve—$r(T^* + T) > r(T^*)$—the forward rate rises as T^* increases. For a downward-sloping yield curve—$r(T^* + T) < r(T^*)$—the forward rate declines as T^* increases.

EXAMPLE 4

Spot and Forward Prices and Rates (4)

Given the spot rates $r(1) = 9\%$, $r(2) = 10\%$, and $r(3) = 11\%$, as in Examples 2 and 3:

1 Determine whether the forward rate $f(1,2)$ is greater than or less than the long-term rate, $r(3)$.

2 Determine whether forward rates rise or fall as the initiation date, T^*, for the forward rate is increased.

Solution to 1:

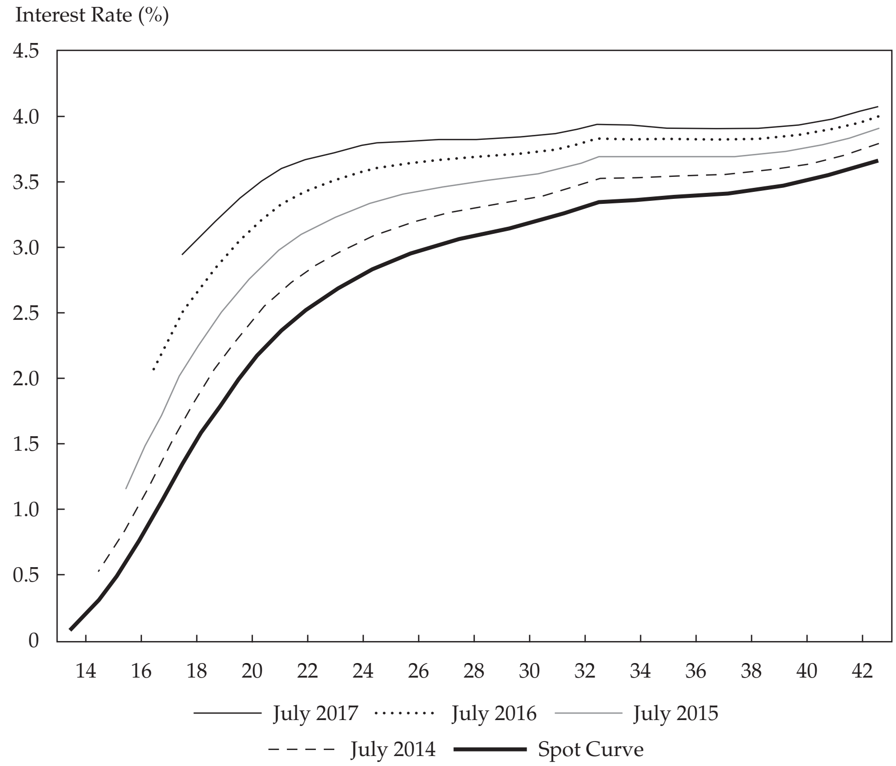
The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$, or in general, $r(T^* + T) > r(T^*)$. Thus, the forward rate will be greater than the long-term rate, or $f(T^*,T) > r(T^* + T)$. Note from Example 2 that $f(1,2) = 12.01\% > r(1 + 2) = r(3) = 11\%$.

Solution to 2:

The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$. Thus, the forward rates will rise with increasing T^*. This relationship was shown in Example 2, in which $f(1,1) = 11.01\%$ and $f(2,1) = 13.03\%$.

These relationships are illustrated in Exhibit 1, using actual data. The spot rates for US Treasuries as of 31 July 2013 are represented by the lowest curve in the exhibit, which was constructed using interpolation between the data points, shown in the table following the exhibit. Note that the spot curve is upward sloping. The spot curve and the forward curves for the end of July 2014, July 2015, July 2016, and July 2017 are also presented in Exhibit 1. Because the yield curve is upward sloping, the forward curves lie above the spot curve and increasing the initiation date results in progressively higher forward curves. The highest forward curve is that for July 2017. Note that the forward curves in Exhibit 1 are progressively flatter at later start dates because the spot curve flattens at the longer maturities.

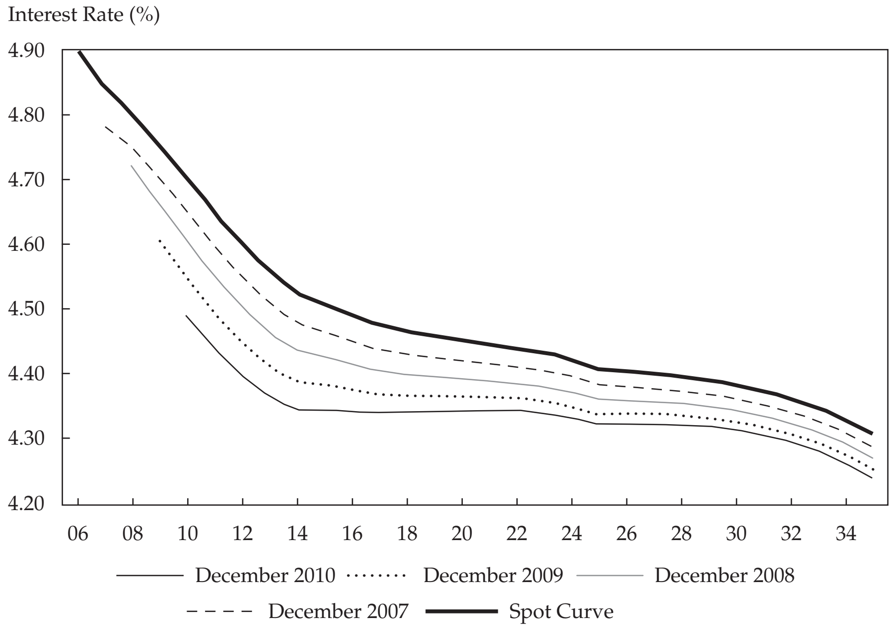
Exhibit 1 Spot Curve vs. Forward Curves, 31 July 2013



Maturity (years)	1	2	3	5	7	10	20	30
Spot rate (%)	0.11	0.33	0.61	1.37	2.00	2.61	3.35	3.66

When the spot yield curve is downward sloping, the forward yield curve will be below the spot yield curve. Spot rates for US Treasuries as of 31 December 2006 are presented in the table following Exhibit 2. We used linear interpolation to construct the spot curve based on these data points. The yield curve data were also somewhat modified to make the yield curve more downward sloping for illustrative purposes. The spot curve and the forward curves for the end of December 2007, 2008, 2009, and 2010 are presented in Exhibit 2.

Exhibit 2 Spot Curve vs. Forward Curves, 31 December 2006 (Modified for Illustrative Purposes)



Maturity (years)	1	2	3	5	7	10	20	30
Spot rate (%)	4.90	4.82	4.74	4.70	4.60	4.51	4.41	4.31

The highest curve is the spot yield curve, and it is downward sloping. The results show that the forward curves are lower than the spot curve. Postponing the initiation date results in progressively lower forward curves. The lowest forward curve is that dated December 2010.

An important point that can be inferred from Exhibit 1 and Exhibit 2 is that forward rates do not extend any further than the furthest maturity on today's yield curve. For example, if yields extend to 30 years on today's yield curve, then three years hence, the most we can model prospectively is a bond with 27 years to final maturity. Similarly, four years hence, the longest maturity forward rate would be $f(4,26)$.

In summary, when the spot curve is upward sloping, the forward curve will lie above the spot curve. Conversely, when the spot curve is downward sloping, the forward curve will lie below the spot curve. This relationship is a reflection of the basic mathematical truth that when the average is rising (falling), the marginal data point must be above (below) the average. In this case, the spot curve represents an average over a whole time period and the forward rates represent the marginal changes between future time periods.[2]

We have thus far discussed the spot curve and the forward curve. Another curve important in practice is the government par curve. The **par curve** represents the yields to maturity on coupon-paying government bonds, priced at par, over a range of maturities. In practice, recently issued ("on the run") bonds are typically used to create the par curve because new issues are typically priced at or close to par.

2 Extending this discussion, one can also conclude that when a spot curve rises and then falls, the forward curves will also rise and then fall.

The par curve is important for valuation in that it can be used to construct a zero-coupon yield curve. The process makes use of the fact that a coupon-paying bond can be viewed as a portfolio of zero-coupon bonds. The zero-coupon rates are determined by using the par yields and solving for the zero-coupon rates one by one, in order from earliest to latest maturities, via a process of forward substitution known as **bootstrapping**.

WHAT IS BOOTSTRAPPING?

The practical details of deriving the zero-coupon yield are outside the scope of this reading. But the meaning of bootstrapping cannot be grasped without a numerical illustration. Suppose the following yields are observed for annual coupon sovereign debt:

Par Rates:

One-year par rate = 5%, Two-year par rate = 5.97%, Three-year par rate = 6.91%, Four-year par rate = 7.81%. From these we can bootstrap zero-coupon rates.

Zero-Coupon Rates:

The one-year zero-coupon rate is the same as the one-year par rate because, under the assumption of annual coupons, it is effectively a one-year pure discount instrument. However, the two-year bond and later-maturity bonds have coupon payments before maturity and are distinct from zero-coupon instruments.

The process of deriving zero-coupon rates begins with the two-year maturity. The two-year zero-coupon rate is determined by solving the following equation in terms of one monetary unit of current market value, using the information that $r(1)$ = 5%:

$$1 = \frac{0.0597}{(1.05)} + \frac{1 + 0.0597}{\left[1 + r(2)\right]^2}$$

In the equation, 0.0597 and 1.0597 represent payments from interest and principal and interest, respectively, per one unit of principal value. The equation implies that $r(2)$ = 6%. We have bootstrapped the two-year spot rate. Continuing with forward substitution, the three-year zero-coupon rate can be bootstrapped by solving the following equation, using the known values of the one-year and two-year spot rates of 5% and 6%:

$$1 = \frac{0.0691}{(1.05)} + \frac{0.0691}{(1.06)^2} + \frac{1 + 0.0691}{\left[1 + r(3)\right]^3}$$

Thus, $r(3)$ = 7%. Finally the four-year zero-coupon rate is determined to be 8% by using

$$1 = \frac{0.0781}{(1.05)} + \frac{0.0781}{(1.06)^2} + \frac{0.0781}{(1.07)^3} + \frac{1 + 0.0781}{\left[1 + r(4)\right]^4}$$

In summary, $r(1)$ = 5%, $r(2)$ = 6%, $r(3)$ = 7%, and $r(4)$ = 8%.

In the preceding discussion, we considered an upward-sloping (spot) yield curve (Exhibit 1) and an inverted or downward-sloping (spot) yield curve (Exhibit 2). In developed markets, yield curves are most commonly upward sloping with diminishing marginal increases in yield for identical changes in maturity; that is, the yield curve "flattens" at longer maturities. Because nominal yields incorporate a premium for expected inflation, an upward-sloping yield curve is generally interpreted as reflecting a market expectation of increasing or at least level future inflation (associated with relatively strong economic growth). The existence of risk premiums (e.g., for the greater interest rate risk of longer-maturity bonds) also contributes to a positive slope.

An inverted yield curve (Exhibit 2) is somewhat uncommon. Such a term structure may reflect a market expectation of declining future inflation rates (because a nominal yield incorporates a premium for expected inflation) from a relatively high current level. Expectations of declining economic activity may be one reason that inflation might be anticipated to decline, and a downward-sloping yield curve has frequently been observed before recessions.[3] A flat yield curve typically occurs briefly in the transition from an upward-sloping to a downward-sloping yield curve, or vice versa. A humped yield curve, which is relatively rare, occurs when intermediate-term interest rates are higher than short- and long-term rates.

2.2 Yield to Maturity in Relation to Spot Rates and Expected and Realized Returns on Bonds

Yield to maturity (YTM) is perhaps the most familiar pricing concept in bond markets. In this section, our goal is to clarify how it is related to spot rates and a bond's expected and realized returns.

How is the yield to maturity related to spot rates? In bond markets, most bonds outstanding have coupon payments and many have various options, such as a call provision. The YTM of these bonds with maturity T would not be the same as the spot rate at T. But, the YTM should be mathematically related to the spot curve. Because the principle of no arbitrage shows that a bond's value is the sum of the present values of payments discounted by their corresponding spot rates, the YTM of the bond should be some weighted average of spot rates used in the valuation of the bond.

Example 5 addresses the relationship between spot rates and yield to maturity.

EXAMPLE 5

Spot Rate and Yield to Maturity

Recall from earlier examples the spot rates $r(1) = 9\%$, $r(2) = 10\%$, and $r(3) = 11\%$. Let $y(T)$ be the yield to maturity.

1 Calculate the price of a two-year annual coupon bond using the spot rates. Assume the coupon rate is 6% and the face value is \$1,000. Next, state the formula for determining the price of the bond in terms of its yield to maturity. Is $r(2)$ greater than or less than $y(2)$? Why?

2 Calculate the price of a three-year annual coupon-paying bond using the spot rates. Assume the coupon rate is 5% and the face value is £100. Next, write a formula for determining the price of the bond using the yield to maturity. Is $r(3)$ greater or less than $y(3)$? Why?

Solution to 1:

Using the spot rates,

$$\text{Price} = \frac{\$60}{(1 + 0.09)^1} + \frac{\$1{,}060}{(1 + 0.10)^2} = \$931.08$$

Using the yield to maturity,

$$\text{Price} = \frac{\$60}{\left[1 + y(2)\right]^1} + \frac{\$1{,}060}{\left[1 + y(2)\right]^2} = \$931.08$$

3 The US Treasury yield curve inverted in August 2006, more than a year before the recession that began in December 2007. See Haubrich (2006).

Note that $y(2)$ is used to discount both the first- and second-year cash flows. Because the bond can have only one price, it follows that $r(1) < y(2) < r(2)$ because $y(2)$ is a weighted average of $r(1)$ and $r(2)$ and the yield curve is upward sloping. Using a calculator, one can calculate the yield to maturity $y(2) = 9.97\%$, which is less than $r(2) = 10\%$ and greater than $r(1) = 9\%$, just as we would expect. Note that $y(2)$ is much closer to $r(2)$ than to $r(1)$ because the bond's largest cash flow occurs in Year 2, thereby giving $r(2)$ a greater weight than $r(1)$ in the determination of $y(2)$.

Solution to 2:

Using the spot rates,

$$\text{Price} = \frac{£5}{(1+0.09)^1} + \frac{£5}{(1+0.10)^2} + \frac{£105}{(1+0.11)^3} = £85.49$$

Using the yield to maturity,

$$\text{Price} = \frac{£5}{[1+y(3)]^1} + \frac{£5}{[1+y(3)]^2} + \frac{£105}{[1+y(3)]^3} = £85.49$$

Note that $y(3)$ is used to discount all three cash flows. Because the bond can have only one price, $y(3)$ must be a weighted average of $r(1)$, $r(2)$, and $r(3)$. Given that the yield curve is upward sloping in this example, $y(3) < r(3)$. Using a calculator to compute yield to maturity, $y(3) = 10.93\%$, which is less than $r(3) = 11\%$ and greater than $r(1) = 9\%$, just as we would expect because the weighted yield to maturity must lie between the highest and lowest spot rates. Note that $y(3)$ is much closer to $r(3)$ than it is to $r(2)$ or $r(1)$ because the bond's largest cash flow occurs in Year 3, thereby giving $r(3)$ a greater weight than $r(2)$ and $r(1)$ in the determination of $y(3)$.

Is the yield to maturity the expected return on a bond? In general, it is not, except under extremely restrictive assumptions. The expected rate of return is the return one anticipates earning on an investment. The YTM is the expected rate of return for a bond that is held until its maturity, assuming that all coupon and principal payments are made in full when due and that coupons are reinvested at the original YTM. However, the assumption regarding reinvestment of coupons at the original yield to maturity typically does not hold. The YTM can provide a poor estimate of expected return if (1) interest rates are volatile; (2) the yield curve is steeply sloped, either upward or downward; (3) there is significant risk of default; or (4) the bond has one or more embedded options (e.g., put, call, or conversion). If either (1) or (2) is the case, reinvestment of coupons would not be expected to be at the assumed rate (YTM). Case (3) implies that actual cash flows may differ from those assumed in the YTM calculation, and in case (4), the exercise of an embedded option would, in general, result in a holding period that is shorter than the bond's original maturity.

The realized return is the actual return on the bond during the time an investor holds the bond. It is based on actual reinvestment rates and the yield curve at the end of the holding period. With perfect foresight, the expected bond return would equal the realized bond return.

To illustrate these concepts, assume that $r(1) = 5\%$, $r(2) = 6\%$, $r(3) = 7\%$, $r(4) = 8\%$, and $r(5) = 9\%$. Consider a five-year annual coupon bond with a coupon rate of 10%. The forward rates extrapolated from the spot rates are $f(1,1) = 7.0\%$, $f(2,1) = 9.0\%$, $f(3,1) = 11.1\%$, and $f(4,1) = 13.1\%$. The price, determined as a percentage of par, is 105.43.

The yield to maturity of 8.62% can be determined using a calculator or by solving

$$105.43 = \frac{10}{\left[1 + y(5)\right]} + \frac{10}{\left[1 + y(5)\right]^2} + \cdots + \frac{110}{\left[1 + y(5)\right]^5}$$

The yield to maturity of 8.62% is the bond's expected return assuming no default, a holding period of five years, and a reinvestment rate of 8.62%. But what if the forward rates are assumed to be the future spot rates?

Using the forward rates as the expected reinvestment rates results in the following expected cash flow at the end of Year 5:

$$10(1 + 0.07)(1 + 0.09)(1 + 0.111)(1 + 0.131) + 10(1 + 0.09)(1 + 0.011)(1 + 0.131) + 10(1 + 0.111)(1 + 0.131) + 10(1 + 0.131) + 110 \approx 162.22$$

Therefore, the expected bond return is (162.22 – 105.43)/105.43 = 53.87% and the expected annualized rate of return is 9.00% [solve $(1 + x)^5 = 1 + 0.5387$].

From this example, we can see that the expected rate of return is not equal to the YTM even if we make the generally unrealistic assumption that the forward rates are the future spot rates. Implicit in the determination of the yield to maturity as a potentially realistic estimate of expected return is a flat yield curve; note that in the formula just used, every cash flow was discounted at 8.62% regardless of its maturity.

Example 6 will reinforce your understanding of various yield and return concepts.

EXAMPLE 6

Yield and Return Concepts

1 When the spot curve is upward sloping, the forward curve:

- **A** lies above the spot curve.
- **B** lies below the spot curve.
- **C** is coincident with the spot curve.

2 Which of the following statements concerning the yield to maturity of a default-risk-free bond is *most* accurate? The yield to maturity of such a bond:

- **A** equals the expected return on the bond if the bond is held to maturity.
- **B** can be viewed as a weighted average of the spot rates applying to its cash flows.
- **C** will be closer to the realized return if the spot curve is upward sloping rather than flat through the life of the bond.

3 When the spot curve is downward sloping, an increase in the initiation date results in a forward curve that is:

- **A** closer to the spot curve.
- **B** a greater distance above the spot curve.
- **C** a greater distance below the spot curve.

Solution to 1:

A is correct. Points on a spot curve can be viewed as an average of single-period rates over given maturities whereas forward rates reflect the marginal changes between future time periods.

Solution to 2:

B is correct. The YTM is the discount rate that, when applied to a bond's promised cash flows, equates those cash flows to the bond's market price and the fact that the market price should reflect discounting promised cash flows at appropriate spot rates.

Solution to 3:

C is correct. This answer follows from the forward rate model as expressed in Equation 6. If the spot curve is downward sloping (upward sloping), increasing the initiation date (T^*) will result in a forward curve that is a greater distance below (above) the spot curve. See Exhibit 1 and Exhibit 2.

2.3 Yield Curve Movement and the Forward Curve

This section establishes several important results concerning forward prices and the spot yield curve in anticipation of discussing the relevance of the forward curve to active bond investors.

The first observation is that the forward contract price remains unchanged as long as future spot rates evolve as predicted by today's forward curve. Therefore, a change in the forward price reflects a deviation of the spot curve from that predicted by today's forward curve. Thus, if a trader expects that the future spot rate will be lower than what is predicted by the prevailing forward rate, the forward contract value is expected to increase. To capitalize on this expectation, the trader would buy the forward contract. Conversely, if the trader expects the future spot rate to be higher than what is predicted by the existing forward rate, then the forward contract value is expected to decrease. In this case, the trader would sell the forward contract.

Using the forward pricing model defined by Equation 2, we can determine the forward contract price that delivers a T-year-maturity bond at time T^*, $F(T^*,T)$ using Equation 7 (which is Equation 2 solved for the forward price):

$$F(T^*,T) = \frac{P(T^* + T)}{P(T^*)} \tag{7}$$

Now suppose that after time t, the new discount function is the same as the forward discount function implied by today's discount function, as shown by Equation 8.

$$P^*(T) = \frac{P(t + T)}{P(t)} \tag{8}$$

Next, after a lapse of time t, the time to expiration of the contract is $T^* - t$, and the forward contract price at time t is $F^*(t,T^*,T)$. Equation 7 can be rewritten as Equation 9:

$$F^*(t,T^*,T) = \frac{P^*(T^* + T - t)}{P^*(T^* - t)} \tag{9}$$

Substituting Equation 8 into Equation 9 and adjusting for the lapse of time t results in Equation 10:

$$F^*(t,T^*,T) = \frac{\dfrac{P(t + T^* + T - t)}{P(t)}}{\dfrac{P(t + T^* - t)}{P(t)}} = \frac{P(T^* + T)}{P(T^*)} = F(T^*,T) \tag{10}$$

Equation 10 shows that the forward contract price remains unchanged as long as future spot rates are equal to what is predicted by today's forward curve. Therefore, a change in the forward price is the result of a deviation of the spot curve from what is predicted by today's forward curve.

To make these observations concrete, consider a flat yield curve for which the interest rate is 4%. Using Equation 1, the discount factors for the one-year, two-year, and three-year terms are, to four decimal places,

$$P(1) = \frac{1}{(1+0.04)^1} = 0.9615$$

$$P(2) = \frac{1}{(1+0.04)^2} = 0.9246$$

$$P(3) = \frac{1}{(1+0.04)^3} = 0.8890$$

Therefore, using Equation 7, the forward contract price that delivers a one-year bond at Year 2 is

$$F(2,1) = \frac{P(2+1)}{P(2)} = \frac{P(3)}{P(2)} = \frac{0.8890}{0.9246} = 0.9615$$

Suppose the future discount function at Year 1 is the same as the forward discount function implied by the Year 0 spot curve. The lapse of time is $t = 1$. Using Equation 8, the discount factors for the one-year and two-year terms one year from today are

$$P^*(1) = \frac{P(1+1)}{P(1)} = \frac{P(2)}{P(1)} = \frac{0.9246}{0.9615} = 0.9616$$

$$P^*(2) = \frac{P(1+2)}{P(1)} = \frac{P(3)}{P(1)} = \frac{0.8890}{0.9615} = 0.9246$$

Using Equation 9, the price of the forward contract one year from today is

$$F^*(1,2,1) = \frac{P^*(2+1-1)}{P^*(2-1)} = \frac{P^*(2)}{P^*(1)} = \frac{0.9246}{0.9616} = 0.9615$$

The price of the forward contract has not changed. This will be the case as long as future discount functions are the same as those based on today's forward curve.

From this numerical example, we can see that if the spot rate curve is unchanged, then each bond "rolls down" the curve and earns the forward rate. Specifically, when one year passes, a three-year bond will return (0.9246 − 0.8890)/0.8890 = 4%, which is equal to the spot rate. Furthermore, if another year passes, the bond will return (0.9615 − 0.9246)/0.9246 = 4%, which is equal to the implied forward rate for a one-year security one year from today.

2.4 Active Bond Portfolio Management

One way active bond portfolio managers attempt to outperform the bond market's return is by anticipating changes in interest rates relative to the projected evolution of spot rates reflected in today's forward curves.

Some insight into these issues is provided by the forward rate model (Equation 4). By re-arranging terms in Equation 4 and letting the time horizon be one period, T* = 1, we get

$$\frac{[1+r(T+1)]^{T+1}}{[1+f(1,T)]^T} = [1+r(1)] \qquad \textbf{(11)}$$

The numerator of the left hand side of Equation 11 is for a bond with an initial maturity of $T + 1$ and a remaining maturity of T after one period passes. Suppose the prevailing spot yield curve after one period is the current forward curve; then, Equation 11 shows that the total return on the bond is the one-period risk-free rate. The following sidebar shows that the return of bonds of varying tenor over a one-year period is always the one-year rate (the risk-free rate over the one-year period) if the spot rates evolve as implied by the current forward curve at the end of the first year.

WHEN SPOT RATES EVOLVE AS IMPLIED BY THE CURRENT FORWARD CURVE

As in earlier examples, assume the following:

$r(1) = 9\%$

$r(2) = 10\%$

$r(3) = 11\%$

$f(1,1) = 11.01\%$

$f(1,2) = 12.01\%$

If the spot curve one year from today reflects the current forward curve, the return on a zero-coupon bond for the one-year holding period is 9%, regardless of the maturity of the bond. The computations below assume a par amount of 100 and represent the percentage change in price. Given the rounding of price and the forward rates to the nearest hundredth, the returns all approximate 9%. However, with no rounding, all answers would be precisely 9%.

The return of the one-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 91.74 and is worth the par amount of 100 at maturity.

$$\left(100 \div \frac{100}{1+r(1)}\right) - 1 = \left(100 \div \frac{100}{1+0.09}\right) - 1 = \frac{100}{91.74} - 1 = 9\%$$

The return of the two-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 82.64. One year from today, the two-year bond has a remaining maturity of one year. Its price one year from today is 90.08, determined as the par amount divided by 1 plus the forward rate for a one-year bond issued one year from today.

$$\left(\frac{100}{1+f(1,1)} \div \frac{100}{\left[1+r(2)\right]^2}\right) - 1 = \left(\frac{100}{1+0.1101} \div \frac{100}{(1+0.10)^2}\right) - 1$$

$$= \frac{90.08}{82.64} - 1 = 9\%$$

The return of the three-year zero-coupon bond over the one-year holding period is 9%. The bond is purchased at a price of 73.12. One year from today, the three-year bond has a remaining maturity of two years. Its price one year from today of 79.71 reflects the forward rate for a two-year bond issued one year from today.

$$\left(\frac{100}{\left[1+f(1,2)\right]^2} \div \frac{100}{\left[1+r(3)\right]^3}\right) - 1 =$$

$$\left(\frac{100}{(1+0.1201)^2} \div \frac{100}{(1+0.11)^3}\right) - 1 = \frac{79.71}{73.12} - 1 \cong 9\%$$

This numerical example shows that the return of a bond over a one-year period is always the one-year rate (the risk-free rate over the one period) if the spot rates evolve as implied by the current forward curve.

But if the spot curve one year from today differs from today's forward curve, the returns on each bond for the one-year holding period will not all be 9%. To show that the returns on the two-year and three-year bonds over the one-year holding period are not 9%, we assume that the spot rate curve at Year 1 is flat with yields of 10% for all maturities.

The return on a one-year zero-coupon bond over the one-year holding period is

$$\left(100 \div \frac{100}{1+0.09}\right) - 1 = 9\%$$

The return on a two-year zero-coupon bond over the one-year holding period is

$$\left(\frac{100}{1+0.10} \div \frac{100}{(1+0.10)^2}\right) - 1 = 10\%$$

The return on a three-year zero-coupon bond over the one-year holding period is

$$\left(\frac{100}{(1+0.10)^2} \div \frac{100}{(1+0.11)^3}\right) - 1 = 13.03\%$$

The bond returns are 9%, 10%, and 13.03%. The returns on the two-year and three-year bonds differ from the one-year risk-free interest rate of 9%.

Equation 11 provides a total return investor with a means to evaluate the cheapness or expensiveness of a bond of a certain maturity. If any one of the investor's expected future spot rates is lower than a quoted forward rate for the same maturity, then (all else being equal) the investor would perceive the bond to be undervalued in the sense that the market is effectively discounting the bond's payments at a higher rate than the investor is and the bond's market price is below the intrinsic value perceived by the investor.

Another example will reinforce the point that if a portfolio manager's projected spot curve is above (below) the forward curve and his or her expectation turns out to be true, the return will be less (more) than the one-period risk-free interest rate.

For the sake of simplicity, assume a flat yield curve of 8% and that a trader holds a three-year bond paying annual coupons based on a 8% coupon rate. Assuming a par value of 100, the current market price is also 100. If today's forward curve turns out to be the spot curve one year from today, the trader will earn an 8% return.

If the trader projects that the spot curve one year from today is above today's forward curve—for example, a flat yield curve of 9%—the trader's expected rate of return is 6.24%, which is less than 8%:

$$\frac{8 + \frac{8}{1+0.09} + \frac{108}{(1+0.09)^2}}{100} - 1 = 6.24\%$$

If the trader predicts a flat yield curve of 7%, the trader's expected return is 9.81%, which is greater than 8%:

$$\frac{8 + \frac{8}{1+0.07} + \frac{108}{(1+0.07)^2}}{100} - 1 = 9.81\%$$

As the gap between the projected future spot rate and the forward rate widens, so too will the difference between the trader's expected return and the original yield to maturity of 8%.

This logic is the basis for a popular yield curve trade called **riding the yield curve** or **rolling down the yield curve**. As we have noted, when a yield curve is upward sloping, the forward curve is always above the current spot curve. If the trader does not believe that the yield curve will change its level and shape over an investment horizon, then buying bonds with a maturity longer than the investment horizon would provide a total return greater than the return on a maturity-matching strategy. The total return of the bond will depend on the spread between the forward rate and the spot rate as well as the maturity of the bond. The longer the bond's maturity, the more sensitive its total return is to the spread.

In the years following the 2008 financial crisis, many central banks around the world acted to keep short-term interest rates very low. As a result, yield curves subsequently had a steep upward slope (see Exhibit 1). For active management, this provided a big incentive for traders to access short-term funding and invest in long-term bonds. Of course, this trade is subject to significant interest rate risk, especially the risk of an unexpected increase in future spot rates (e.g., as a result of a spike in inflation). Yet, such a carry trade is often made by traders in an upward-sloping yield curve environment.[4]

In summary, when the yield curve slopes upward, as a bond approaches maturity or "rolls down the yield curve," it is valued at successively lower yields and higher prices. Using this strategy, a bond can be held for a period of time as it appreciates in price and then sold before maturity to realize a higher return. As long as interest rates remain stable and the yield curve retains an upward slope, this strategy can continuously add to the total return of a bond portfolio.

Example 7 address how the preceding analysis relates to active bond portfolio management.

EXAMPLE 7

Active Bond Portfolio Management

1 The "riding the yield curve" strategy is executed by buying bonds whose maturities are:

  A equal to the investor's investment horizon.

  B longer than the investor's investment horizon.

  C shorter than the investor's investment horizon.

2 A bond will be overvalued if the expected spot rate is:

  A equal to the current forward rate.

  B lower than the current forward rate.

  C higher than the current forward rate.

3 Assume a flat yield curve of 6%. A three-year £100 bond is issued at par paying an annual coupon of 6%. What is the portfolio manager's expected return if she predicts that the yield curve one year from today will be a flat 7%?

  A 4.19%

  B 6.00%

  C 8.83%

4 A forward contract price will increase if:

  A future spot rates evolve as predicted by current forward rates.

4 Carry trades can take many forms. Here, we refer to a maturity spread carry trade in which the trader borrows short and lends long in the same currency. The maturity spread carry trade is used frequently by hedge funds. There are also cross-currency and credit spread carry trades. Essentially, a carry trade involves simultaneously borrowing and lending to take advantage of what a trader views as being a favorable interest rate differential.

B future spot rates are lower than what is predicted by current forward rates.

C future spot rates are higher than what is predicted by current forward rates.

Solution to 1:

B is correct. A bond with a longer maturity than the investor's investment horizon is purchased but then sold prior to maturity at the end of the investment horizon. If the yield curve is upward sloping and yields do not change, the bond will be valued at successively lower yields and higher prices over time. The bond's total return will exceed that of a bond whose maturity is equal to the investment horizon.

Solution to 2:

C is correct. If the expected discount rate is higher than the forward rate, then the bond will be overvalued. The expected price of the bond is lower than the price obtained from discounting using the forward rate.

Solution to 3:

A is correct. Expected return will be less than the current yield to maturity of 6% if yields increase to 7%. The expected return of 4.19% is computed as follows:

$$\frac{6 + \dfrac{6}{1 + 0.07} + \dfrac{106}{(1 + 0.07)^2}}{100} - 1 \approx 4.19\%$$

Solution to 4:

B is correct. The forward rate model can be used to show that a change in the forward contract price requires a deviation of the spot curve from that predicted by today's forward curve. If the future spot rate is lower than what is predicted by the prevailing forward rate, the forward contract price will increase because it is discounted at an interest rate that is lower than the originally anticipated rate.

3 THE SWAP RATE CURVE

Section 2 described the spot rate curve of default-risk-free bonds as a measure of the time value of money. The swap rate curve, or swap curve for short, is another important representation of the time value of money used in the international fixed-income markets. In this section, we will discuss how the swap curve is used in valuation.

3.1 The Swap Rate Curve

Interest rate swaps are an integral part of the fixed-income market. These derivative contracts, which typically exchange, or swap, fixed-rate interest payments for floating-rate interest payments, are an essential tool for investors who use them to speculate or modify risk. The size of the payments reflects the floating and fixed rates, the amount of principal—called the notional amount, or notional—and the maturity of the swap. The interest rate for the fixed-rate leg of an interest rate swap is known as the **swap rate**. The level of the swap rate is such that the swap has zero value at the initiation of the swap agreement. Floating rates are based on some short-term reference interest rate, such as three-month or six-month dollar Libor (London Interbank Offered

Rate); other reference rates include euro-denominated Euribor (European Interbank Offered Rate) and yen-denominated Tibor (Tokyo Interbank Offered Rate). Note that the risk inherent in various floating reference rates varies according to the risk of the banks surveyed; for example, the spread between Tibor and yen Libor was positive as of October 2013, reflecting the greater risk of the banks surveyed for Tibor. The yield curve of swap rates is called the **swap rate curve**, or, more simply, the **swap curve**. Because it is based on so-called **par swaps**, in which the fixed rates are set so that no money is exchanged at contract initiation—the present values of the fixed-rate and benchmark floating-rate legs being equal— the swap curve is a type of par curve. When we refer to the "par curve' in this reading, the reference is to the government par yield curve, however.

The swap market is a highly liquid market for two reasons. First, unlike bonds, a swap does not have multiple borrowers or lenders, only counterparties who exchange cash flows. Such arrangements offer significant flexibility and customization in the swap contract's design. Second, swaps provide one of the most efficient ways to hedge interest rate risk. The Bank for International Settlements (BIS) estimated that the notional amount outstanding on interest rate swaps was about US$370 trillion in December 2012.[5]

Many countries do not have a liquid government bond market with maturities longer than one year. The swap curve is a necessary market benchmark for interest rates in these countries. In countries in which the private sector is much bigger than the public sector, the swap curve is a far more relevant measure of the time value of money than is the government's cost of borrowing.

In Asia, the swap markets and the government bond markets have developed in parallel, and both are used in valuation in credit and loan markets. In South Korea, the swap market is active out to a maturity of 10 years, whereas the Japanese swap market is active out to a maturity of 30 years. The reason for the longer maturity in the Japanese government market is that the market has been in existence for much longer than the South Korean market.

According to the *2013 CIA World Fact Book*, the size of the government bond market relative to GDP is 214.3% for Japan but only 46.9% for South Korea. For the United States and Germany, the numbers are 73.6% and 81.7%, and the world average is 64%. Even though the interest rate swap market in Japan is very active, the US interest rate swap market is almost three times larger than the Japanese interest rate swap market, based on outstanding amounts.

3.2 Why Do Market Participants Use Swap Rates When Valuing Bonds?

Government spot curves and swap rate curves are the chief reference curves in fixed-income valuation. The choice between them can depend on multiple factors, including the relative liquidity of these two markets. In the United States, where there is both an active Treasury security market and a swap market, the choice of a benchmark for the time value of money often depends on the business operations of the institution using the benchmark. On the one hand, wholesale banks frequently use the swap curve to value assets and liabilities because these organizations hedge many items on their balance sheet with swaps. On the other hand, retail banks with little exposure to the swap market are more likely to use the government spot curve as their benchmark.

5 Because the amount outstanding relates to notional values, it represents far less than $370 trillion of default exposure.

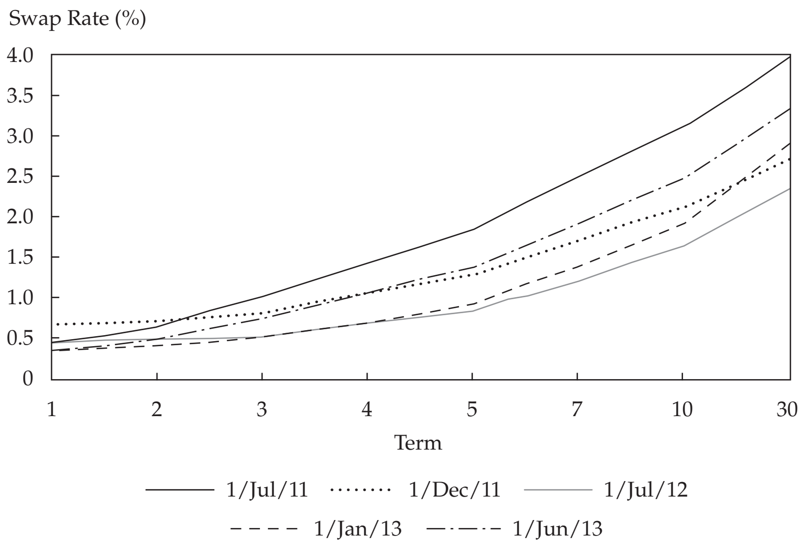
Let us illustrate how a financial institution uses the swap market for its internal operations. Consider the case of a bank raising funds using a certificate of deposit (CD). Assume the bank can borrow $10 million in the form of a CD that bears interest of 1.5% for a two-year term. Another $10 million CD offers 1.70% for a three-year term. The bank can arrange two swaps: (1) The bank receives 1.50% fixed and pays three-month Libor minus 10 bps with a two-year term and $10 million notional, and (2) the bank receives 1.70% fixed and pays three-month Libor minus 15 bps with a three-year term and a notional amount of $10 million. After issuing the two CDs and committing to the two swaps, the bank has raised $20 million with an annual funding cost for the first two years of three-month Libor minus 12.5 bps applied to the total notional amount of $20 million. The fixed interest payments received from the counterparty to the swap are paid to the CD investors; in effect, fixed-rate liabilities have been converted to floating-rate liabilities. The margins on the floating rates become the standard by which value is measured in assessing the total funding cost for the bank.

By using the swap curve as a benchmark for the time value of money, the investor can adjust the swap spread so that the swap would be fairly priced given the spread. Conversely, given a swap spread, the investor can determine a fair price for the bond. We will use the swap spread in the following section to determine the value of a bond.

3.3 How Do Market Participants Use the Swap Curve in Valuation?

Swap contracts are non-standardized and are simply customized contracts between two parties in the over-the-counter market. The fixed payment can be specified by an amortization schedule or to be coupon paying with non-standardized coupon payment dates. For this section, we will focus on zero-coupon bonds. The yields on these bonds determine the swap curve, which, in turn, can be used to determine bond values. Examples of swap par curves are given in Exhibit 3.

Exhibit 3 Historical Swap Curves



Note: Horizontal axis is not drawn to scale. (Such scales are commonly used as an industry standard because most of the distinctive shape of yield curves is typically observed before 10 years.)

Each forward date has an associated discount factor that represents the value today of a hypothetical payment that one would receive on the forward date, expressed as a fraction of the hypothetical payment. For example, if we expect to receive ₩10,000 (10,000 South Korean won) in one year and the current price of the security is ₩9,259.30, then the discount factor for one year would be 0.92593 (= ₩9,259.30/₩10,000). Note that the rate associated with this discount factor is 1/0.92593 −1 ≈ 8.00%.

To price a swap, we need to determine the present value of cash flows for each leg of the transaction. In an interest rate swap, the fixed leg is fairly straightforward because the cash flows are specified by the coupon rate set at the time of the agreement. Pricing the floating leg is more complex because, by definition, the cash flows change with future changes in interest rates. The forward rate for each floating payment date is calculated by using the forward curves.

Let $s(T)$ stand for the swap rate at time T. Because the value of a swap at origination is set to zero, the swap rates must satisfy Equation 12. Note that the swap rates can be determined from the spot rates and the spot rates can be determined from the swap rates.

$$\sum_{t=1}^{T} \frac{s(T)}{\left[1 + r(t)\right]^{t}} + \frac{1}{\left[1 + r(T)\right]^{T}} = 1 \tag{12}$$

The right side of Equation 12 is the value of the floating leg, which is always 1 at origination. The swap rate is determined by equating the value of the fixed leg, on the left-hand side, to the value of the floating leg.

Example 8 addresses the relationship between the swap rate curve and spot curve.

EXAMPLE 8

Determining the Swap Rate Curve

Suppose a government spot curve implies the following discount factors:

$P(1) = 0.9524$

$P(2) = 0.8900$

$P(3) = 0.8163$

$P(4) = 0.7350$

Given this information, determine the swap rate curve.

Solution:

Recall from Equation 1 that $P(T) = \frac{1}{\left[1 + r(T)\right]^T}$. Therefore,

$$r(T) = \left\{\frac{1}{\left[P(T)\right]}\right\}^{(1/T)} - 1$$

$$r(1) = \left(\frac{1}{0.9524}\right)^{(1/1)} - 1 = 5.00\%$$

$$r(2) = \left(\frac{1}{0.8900}\right)^{(1/2)} - 1 = 6.00\%$$

$$r(3) = \left(\frac{1}{0.8163}^{(1/3)}\right) - 1 = 7.00\%$$

$$r(4) = \left(\frac{1}{0.7350}\right)^{(1/4)} - 1 = 8.00\%$$

Using Equation 12, for $T = 1$,

$$\frac{s(1)}{\left[1 + r(1)\right]^1} + \frac{1}{\left[1 + r(1)\right]^1} = \frac{s(1)}{(1 + 0.05)^1} + \frac{1}{(1 + 0.05)^1} = 1$$

Therefore, $s(1) = 5\%$.

For $T = 2$,

$$\frac{s(2)}{\left[1 + r(1)\right]^1} + \frac{s(2)}{\left[1 + r(2)\right]^2} + \frac{1}{\left[1 + r(2)\right]^2} = \frac{s(2)}{(1 + 0.05)^1} + \frac{s(2)}{(1 + 0.06)^2} + \frac{1}{(1 + 0.06)^2} = 1$$

Therefore, $s(2) = 5.97\%$.

For $T = 3$,

$$\frac{s(3)}{\left[1 + r(1)\right]^1} + \frac{s(3)}{\left[1 + r(2)\right]^2} + \frac{s(3)}{\left[1 + r(3)\right]^3} + \frac{1}{\left[1 + r(3)\right]^3} =$$

$$\frac{s(3)}{(1 + 0.05)^1} + \frac{s(3)}{(1 + 0.06)^2} + \frac{s(3)}{(1 + 0.07)^3} + \frac{1}{(1 + 0.07)^3} = 1$$

Therefore, $s(3) = 6.91\%$.

For $T = 4$,

$$\frac{s(4)}{\left[1 + r(1)\right]^1} + \frac{s(4)}{\left[1 + r(2)\right]^2} + \frac{s(4)}{\left[1 + r(3)\right]^3} + \frac{s(4)}{\left[1 + r(4)\right]^4} + \frac{1}{\left[1 + r(4)\right]^4} =$$

$$\frac{s(4)}{(1 + 0.05)^1} + \frac{s(4)}{(1 + 0.06)^2} + \frac{s(4)}{(1 + 0.07)^3} + \frac{s(4)}{(1 + 0.08)^4} + \frac{1}{(1 + 0.08)^4} = 1$$

Therefore, $s(4) = 7.81\%$.

Note that the swap rates, spot rates, and discount factors are all mathematically linked together. Having access to data for one of the series allows you to calculate the other two.

3.4 The Swap Spread

The swap spread is a popular way to indicate credit spreads in a market. The **swap spread** is defined as the spread paid by the fixed-rate payer of an interest rate swap over the rate of the "on-the-run" (most recently issued) government security with the same maturity as the swap.[6]

Often, fixed-income prices will be quoted in SWAPS +, for which the yield is simply the yield on an equal-maturity government bond plus the swap spread. For example, if the fixed rate of a five-year fixed-for-float Libor swap is 2.00% and the five-year Treasury is yielding 1.70%, the swap spread is 2.00% − 1.70% = 0.30%, or 30 bps.

For euro-denominated swaps, the government yield used as a benchmark is most frequently bunds (German government bonds) with the same maturity. Gilts (UK government bonds) are used as a benchmark in the United Kingdom. CME Group began clearing euro-denominated interest rate swaps in 2011.

A Libor/swap curve is probably the most widely used interest rate curve because it is often viewed as reflecting the default risk of private entities at a rating of about A1/A+, roughly the equivalent of most commercial banks. (The swap curve can also be influenced by the demand and supply conditions in government debt markets, among other factors.) Another reason for the popularity of the swap market is that it is unregulated (not controlled by governments), so swap rates are more comparable across different countries. The swap market also has more maturities with which to construct a yield curve than do government bond markets. Libor is used for short-maturity yields, rates derived from eurodollar futures contracts are used for mid-maturity yields, and swap rates are used for yields with a maturity of more than one year. The swap rates used are the fixed rates that would be paid in swap agreements for which three-month Libor floating payments are received.[7]

HISTORY OF THE US SWAP SPREAD, 2008–2013

Normally, the Treasury swap spread is positive, which reflects the fact that governments generally pay less to borrow than do private entities. However, the 30-year Treasury swap spread turned negative following the collapse of Lehman Brothers Holdings Inc. in September 2008. Liquidity in many corners of the credit markets evaporated during the financial crisis, leading investors to doubt the safety and security of their counterparties in some derivatives transactions. The 30-year Treasury swap spread tumbled to a record low of –62 bps in November 2008. The 30-year Treasury swap spread again turned positive in the middle of 2013. A dramatic shift in sentiment regarding the Federal Reserve outlook since early May 2013 was a key catalyst for a selloff in most bonds. The sharp rise in Treasury yields at that time pushed up funding and hedging costs for companies, which was reflected in a rise in swap rates.

To illustrate the use of the swap spread in fixed-income pricing, consider a US$1 million investment in GE Capital (GECC) notes with a coupon rate of 1 5/8% (1.625%) that matures on 2 July 2015. Coupons are paid semiannually. The evaluation date is 12 July 2012, so the remaining maturity is 2.97 years [= 2 + (350/360)]. The Treasury rates for two-year and three-year maturities are 0.525% and 0.588%,

6 The term "swap spread" is sometimes also used as a reference to a bond's basis point spread over the interest rate swap curve and is a measure of the credit and/or liquidity risk of a bond. In its simplest form, the swap spread in this sense can be measured as the difference between the yield to maturity of the bond and the swap rate given by a straight-line interpolation of the swap curve. These spreads are frequently quoted as an I-spread, ISPRD, or interpolated spread, which is a reference to a linearly interpolated yield. In this reading, the term "swap spread" refers to an excess yield of swap rates over the yields on government bonds and I-spreads to refer to bond yields net of the swap rates of the same maturities.

7 The US dollar market uses three-month Libor, but other currencies may use one-month or six-month Libor.

respectively. By simple interpolation between these two rates, the treasury rate for 2.97 years is 0.586% [= 0.525% + (350/360)(0.588% − 0.525%)]. If the swap spread for the same maturity is 0.918%, then the yield to maturity on the bond is 1.504% (= 0.918% + 0.586%). Given the yield to maturity, the invoice price (price including accrued interest) for US$1 million face value is

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.01504}{2}\right)^{\left(1-\frac{10}{180}\right)}}+\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.01504}{2}\right)^{\left(2-\frac{10}{180}\right)}}+\cdots+$$

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.01504}{2}\right)^{\left(6-\frac{10}{180}\right)}}+\frac{1{,}000{,}000}{\left(1+\frac{0.01504}{2}\right)^{\left(6-\frac{10}{180}\right)}}=\text{US\$}1{,}003{,}954.12$$

The left side sums the present values of the semiannual coupon payments and the final principal payment of US$1,000,000. The accrued interest rate amount is US$451.39 [= 1,000,000 × (0.01625/2)(10/180)]. Therefore, the clean price (price not including accrued interest) is US$1,003,502.73 (= 1,003,954.12 – 451.39).

The swap spread helps an investor to identify the time value, credit, and liquidity components of a bond's yield to maturity. If the bond is default free, then the swap spread could provide an indication of the bond's liquidity or it could provide evidence of market mispricing. The higher the swap spread, the higher the return that investors require for credit and/or liquidity risks.

Although swap spreads provide a convenient way to measure risk, a more accurate measure of credit and liquidity is called the zero-spread (Z-spread). The **Z-spread** is the constant basis point spread that would need to be added to the implied spot yield curve so that the discounted cash flows of a bond are equal to its current market price. This spread will be more accurate than a linearly interpolated yield, particularly with steep interest rate swap curves.

USING THE Z-SPREAD IN VALUATION

Consider again the GECC semi-annual coupon note with a maturity of 2.97 years and a par value of US$1,000,000. The implied spot yield curve is

$r(0.5) = 0.16\%$

$r(1) = 0.21\%$

$r(1.5) = 0.27\%$

$r(2) = 0.33\%$

$r(2.5) = 0.37\%$

$r(3) = 0.41\%$

The Z-spread is given as 109.6 bps. Using the spot curve and the Z-spread, the invoice price is

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.0016+0.01096}{2}\right)^{\left(1-\frac{10}{180}\right)}}+\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.00021+0.01096}{2}\right)^{\left(2-\frac{10}{180}\right)}}+\cdots+$$

$$\frac{1{,}000{,}000\left(\frac{0.01625}{2}\right)}{\left(1+\frac{0.0041+0.01096}{2}\right)^{\left(6-\frac{10}{180}\right)}}+$$

$$\frac{1{,}000{,}000}{\left(1+\frac{0.0041+0.01096}{2}\right)^{\left(6-\frac{10}{180}\right)}}=\text{US\$}1{,}003{,}954.12$$

3.5 Spreads as a Price Quotation Convention

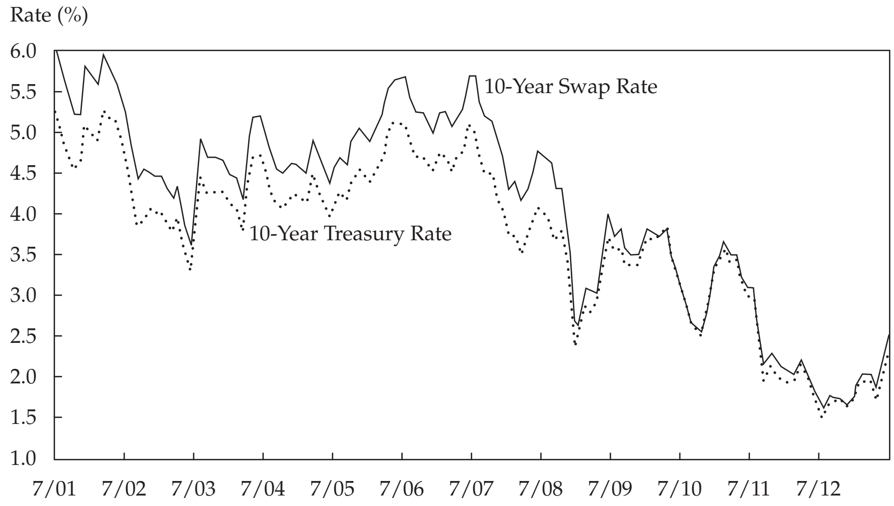
We have discussed both Treasury curves and swap curves as benchmarks for fixed-income valuation, but they usually differ. Therefore, quoting the price of a bond using the bond yield net of either a benchmark Treasury yield or swap rate becomes a price quote convention.

The Treasury rate can differ from the swap rate for the same term for several reasons. Unlike the cash flows from US Treasury bonds, the cash flows from swaps are subject to much higher default risk. Market liquidity for any specific maturity may differ. For example, some parts of the term structure of interest rates may be more actively traded with swaps than with Treasury bonds. Finally, arbitrage between these two markets cannot be perfectly executed.

Swap spreads to the Treasury rate (as opposed to the **I-spreads**, which are bond rates net of the swap rates of the same maturities) are simply the differences between swap rates and government bond yields of a particular maturity. One problem in defining swap spreads is that, for example, a 10-year swap matures in exactly 10 years whereas there typically is no government bond with exactly 10 years of remaining maturity. By convention, therefore, the 10-year swap spread is defined as the difference between the 10-year swap rate and the 10-year on-the-run government bond. Swap spreads of other maturities are defined similarly.

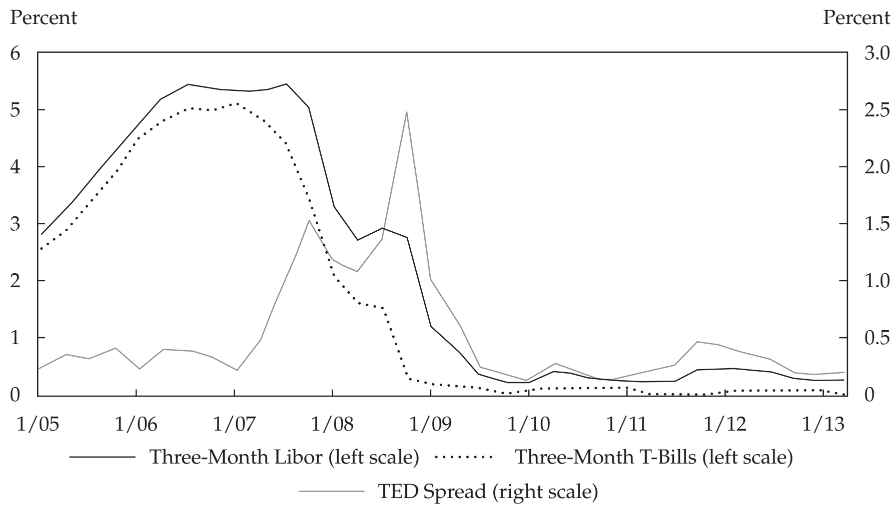
To generate the curves in Exhibit 4, we used the constant-maturity Treasury note to exactly match the corresponding swap rate. The 10-year swap spread is the 10-year swap rate less the 10-year constant-maturity Treasury note yield. Because counterparty risk is reflected in the swap rate and US government debt is considered nearly free of default risk, the swap rate is usually greater than the corresponding Treasury note rate and the 10-year swap spread is usually, but not always, positive.

Exhibit 4 10-Year Swap Rate vs. 10-Year Treasury Rate



The **TED spread** is an indicator of perceived credit risk in the general economy. TED is an acronym formed from US T-bill and ED, the ticker symbol for the eurodollar futures contract. The TED spread is calculated as the difference between Libor and the yield on a T-bill of matching maturity. An increase (decrease) in the TED spread is a sign that lenders believe the risk of default on interbank loans is increasing (decreasing). Therefore, as it relates to the swap market, the TED spread can also be thought of as a measure of counterparty risk. Compared with the 10-year swap spread, the TED spread more accurately reflects risk in the banking system, whereas the 10-year swap spread is more often a reflection of differing supply and demand conditions.

Exhibit 5 TED Spread



Another popular measure of risk is the **Libor–OIS spread**, which is the difference between Libor and the overnight indexed swap (OIS) rate. An OIS is an interest rate swap in which the periodic floating rate of the swap is equal to the geometric average of an overnight rate (or overnight index rate) over every day of the payment period. The index rate is typically the rate for overnight unsecured lending between banks—for example, the federal funds rate for US dollars, Eonia (Euro OverNight Index Average) for euros, and Sonia (Sterling OverNight Index Average) for sterling. The Libor–OIS spread is considered an indicator of the risk and liquidity of money market securities.

4 TRADITIONAL THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

This section presents four traditional theories of the underlying economic factors that affect the shape of the yield curve.

4.1 Local Expectations Theory

One branch of traditional term structure theory focuses on interpreting term structure shape in terms of investors' expectations. Historically, the first such theory is known as the **unbiased expectations theory** or **pure expectations theory**. It says that the forward rate is an unbiased predictor of the future spot rate; its broadest interpretation is that bonds of any maturity are perfect substitutes for one another. For example, buying a bond with a maturity of five years and holding it for three years has the same expected return as buying a three-year bond or buying a series of three one-year bonds.

The predictions of the unbiased expectations theory are consistent with the assumption of risk neutrality. In a risk-neutral world, investors are unaffected by uncertainty and risk premiums do not exist. Every security is risk free and yields the risk-free rate for that particular maturity. Although such an assumption leads to interesting results, it clearly is in conflict with the large body of evidence that shows that investors are risk averse.

A theory that is similar but more rigorous than the unbiased expectations theory is the **local expectations theory**. Rather than asserting that every maturity strategy has the same expected return over a given investment horizon, this theory instead contends that the expected return for every bond over short time periods is the risk-free rate. This conclusion results from an assumed no-arbitrage condition in which bond pricing does not allow for traders to earn arbitrage profits.

The primary way that the local expectations theory differs from the unbiased expectations theory is that it can be extended to a world characterized by risk. Although the theory requires that risk premiums be nonexistent for very short holding periods, no such restrictions are placed on longer-term investments. Thus, the theory is applicable to both risk-free as well as risky bonds.

Using the formula for the discount factor in Equation 1 and the variation of the forward rate model in Equation 5, we can produce Equation 13, where $P(t,T)$ is the discount factor for a T-period security at time t.

$$\frac{1}{P(t,T)} = [1 + r(1)][1 + f(1,1)][1 + f(2,1)][1 + f(3,1)]\ldots[1 + f(T-1,1)] \quad (13)$$

Using Equation 13, we can show that if the forward rates are realized, the one-period return of a long-term bond is $r(1)$, the yield on a one-period risk-free security, as shown in Equation 14.

$$\frac{P(t+1,T-1)}{P(t,T)} = 1 + r(1) \quad (14)$$

The local expectations theory extends this equation to incorporate uncertainty while still assuming risk neutrality in the short term. When we relax the certainty assumption, then Equation 14 becomes Equation 15, where the tilde (~) represents an uncertain outcome. In other words, the one-period return of a long-term risky bond is the one-period risk-free rate.

$$\frac{E\left[\tilde{P}(t+1,T-1)\right]}{P(t,T)} = 1 + r(1) \tag{15}$$

Although the local expectations theory is economically appealing, it is often observed that short-holding-period returns on long-dated bonds do exceed those on short-dated bonds. The need for liquidity and the ability to hedge risk essentially ensure that the demand for short-term securities will exceed that for long-term securities. Thus, both the yields and the actual returns for short-dated securities are typically lower than those for long-dated securities.

4.2 Liquidity Preference Theory

Whereas the unbiased expectations theory leaves no room for risk aversion, liquidity preference theory attempts to account for it. **Liquidity preference theory** asserts that **liquidity premiums** exist to compensate investors for the added interest rate risk they face when lending long term and that these premiums increase with maturity.[8] Thus, given an expectation of unchanging short-term spot rates, liquidity preference theory predicts an upward-sloping yield curve. The forward rate provides an estimate of the expected spot rate that is biased upward by the amount of the liquidity premium, which invalidates the unbiased expectations theory.

For example, the US Treasury offers bonds that mature in 30 years. However, the majority of investors have an investment horizon that is shorter than 30 years.[9] For investors to hold these bonds, they would demand a higher return for taking the risk that the yield curve changes and that they must sell the bond prior to maturity at an uncertain price. That incrementally higher return is the liquidity premium. Note that this premium is not to be confused with a yield premium for the lack of liquidity that thinly traded bonds may bear. Rather, it is a premium applying to all long-term bonds, including those with deep markets.

Liquidity preference theory fails to offer a complete explanation of the term structure. Rather, it simply argues for the existence of liquidity premiums. For example, a downward-sloping yield curve could still be consistent with the existence of liquidity premiums if one of the factors underlying the shape of the curve is an expectation of deflation (i.e., a negative rate of inflation due to monetary or fiscal policy actions). Expectations of sharply declining spot rates may also result in a downward-sloping yield curve if the expected decline in interest rates is severe enough to offset the effect of the liquidity premiums.

In summary, liquidity preference theory claims that lenders require a liquidity premium as an incentive to lend long term. Thus, forward rates derived from the current yield curve provide an upwardly biased estimate of expected future spot rates. Although downward-sloping or hump-shaped yield curves may sometimes occur, the existence of liquidity premiums implies that the yield curve will typically be upward sloping.

8 The wording of a technical treatment of this theory would be that these premiums increase monotonically with maturity. A sequence is said to be monotonically increasing if each term is greater than or equal to the one before it. Define $LP(T)$ as the liquidty premium at maturity T. If premiums increase monotonically with maturity, then $LP(T + t) \geq LP(T)$ for all $t > 0$.

9 This view can be confirmed by examining typical demand for long-term versus short-term Treasuries at auctions.

4.3 Segmented Markets Theory

Unlike expectations theory and liquidity preference theory, **segmented markets theory** allows for lender and borrower preferences to influence the shape of the yield curve. The result is that yields are not a reflection of expected spot rates or liquidity premiums. Rather, they are solely a function of the supply and demand for funds of a particular maturity. That is, each maturity sector can be thought of as a segmented market in which yield is determined independently from the yields that prevail in other maturity segments.

The theory is consistent with a world where there are asset/liability management constraints, either regulatory or self-imposed. In such a world, investors might restrict their investment activity to a maturity sector that provides the best match for the maturity of their liabilities. Doing so avoids the risks associated with an asset/liability mismatch.

For example, because life insurers sell long-term liabilities against themselves in the form of life insurance contracts, they tend to be most active as buyers in the long end of the bond market. Similarly, because the liabilities of pension plans are long term, they typically invest in long-term securities. Why would they invest short term given that those returns might decline while the cost of their liabilities stays fixed? In contrast, money market funds would be limited to investing in debt with maturity of one year or less, in general.

In summary, the segmented markets theory assumes that market participants are either unwilling or unable to invest in anything other than securities of their preferred maturity. It follows that the yield of securities of a particular maturity is determined entirely by the supply and demand for funds of that particular maturity.

4.4 Preferred Habitat Theory

The **preferred habitat theory** is similar to the segmented markets theory in proposing that many borrowers and lenders have strong preferences for particular maturities but it does not assert that yields at different maturities are determined independently of each other.

However, the theory contends that if the expected additional returns to be gained become large enough, institutions will be willing to deviate from their preferred maturities or habitats. For example, if the expected returns on longer-term securities exceed those on short-term securities by a large enough margin, money market funds will lengthen the maturities of their assets. And if the excess returns expected from buying short-term securities become large enough, life insurance companies might stop limiting themselves to long-term securities and place a larger part of their portfolios in shorter-term investments.

The preferred habitat theory is based on the realistic notion that agents and institutions will accept additional risk in return for additional expected returns. In accepting elements of both the segmented markets theory and the unbiased expectations theory, yet rejecting their extreme polar positions, the preferred habitat theory moves closer to explaining real-world phenomena. In this theory, both market expectations and the institutional factors emphasized in the segmented markets theory influence the term structure of interest rates.

PREFERRED HABITAT AND QE

The term "quantitative easing" (QE) refers to an unconventional monetary policy used by central banks to increase the supply of money in an economy when central bank and/or interbank interest rates are already close to zero. The first of three QE efforts by the US Federal Reserve began in late 2008, following the establishment of a near-zero target range for the federal funds rate. Since then, the Federal Reserve has greatly expanded its holdings of long-term securities via a series of asset purchase programs, with the goal of putting downward pressure on long-term interest rates thereby making financial conditions even more accommodative. Exhibit 6 presents information regarding the securities held by the Federal Reserve on 20 September 2007 (when all securities held by the Fed were US Treasury issuance) and 19 September 2013 (one year after the third round of QE was launched).

Exhibit 6 Securities Held by the US Federal Reserve

(US$ millions)	20 Sept. 2007	19 Sept. 2013
Securities held outright	779,636	3,448,758
US Treasury	779,636	2,047,534
Bills	267,019	0
Notes and bonds, nominal	472,142	1,947,007
Notes and bonds, inflation indexed	35,753	87,209
Inflation compensation	4,723	13,317
Federal agency	0	63,974
Mortgage-backed securities	0	1,337,520

As Exhibit 6 shows, the Federal Reserve's security holdings on 20 September 2007 consisted entirely of US Treasury securities and about 34% of those holdings were short term in the form of T-bills. On 19 September 2013, only about 59% of the Federal Reserve's security holdings were Treasury securities and none of those holdings were T-bills. Furthermore, the Federal Reserve held well over US$1.3 trillion of mortgage-backed securities (MBS), which accounted for almost 39% of all securities held.

Prior to the QE efforts, the yield on MBS was typically in the 5%–6% range. It declined to less than 2% by the end of 2012. Concepts related to preferred habitat theory could possibly help explain that drop in yield.

The purchase of MBS by the Federal Reserve essentially reduced the supply of these securities that was available for private purchase. Assuming that many MBS investors are either unwilling or unable to withdraw from the MBS market because of their investment in gaining expertise in managing interest rate and repayment risks of MBS, MBS investing institutions would have a "preferred habitat" in the MBS market. If they were unable to meet investor demand without bidding more aggressively, these buyers would drive down yields on MBS.

The case can also be made that the Federal Reserve's purchase of MBS helped reduced prepayment risk, which also resulted in a reduction in MBS yields. If a homeowner prepays on a mortgage, the payment is sent to MBS investors on a pro-rata basis. Although investors are uncertain about when such a prepayment will be received, prepayment is more likely in a declining interest rate environment.

Use Example 9 to test your understanding of traditional term structure theories.

EXAMPLE 9

Traditional Term Structure Theories

1 In 2010, the Committee of European Securities Regulators created guidelines that restricted weighted average life (WAL) to 120 days for short-term money market funds. The purpose of this restriction was to limit the ability of money market funds to invest in long-term, floating-rate securities. This action is *most* consistent with a belief in:

   A the preferred habitat theory.

   B the segmented markets theory.

   C the local expectations theory.

2 The term structure theory that asserts that investors cannot be induced to hold debt securities whose maturities do not match their investment horizon is *best* described as the:

   A preferred habitat theory.

   B segmented markets theory.

   C unbiased expectations theory.

3 The unbiased expectations theory assumes investors are:

   A risk averse.

   B risk neutral.

   C risk seeking.

4 Market evidence shows that forward rates are:

   A unbiased predictors of future spot rates.

   B upwardly biased predictors of future spot rates.

   C downwardly biased predictors of future spot rates.

5 Market evidence shows that short holding-period returns on short-maturity bonds *most* often are:

   A less than those on long-maturity bonds.

   B about equal to those on long-maturity bonds.

   C greater than those on long-maturity bonds.

Solution to 1:

A is correct. The preferred habitat theory asserts that investors are willing to move away from their preferred maturity if there is adequate incentive to do so. The proposed WAL guideline was the result of regulatory concern about the interest rate risk and credit risk of long-term, floating-rate securities. An inference of this regulatory action is that some money market funds must be willing to move away from more traditional short-term investments if they believe there is sufficient compensation to do so.

Solution to 2:

B is correct. Segmented markets theory contends that asset/liability management constraints force investors to buy securities whose maturities match the maturities of their liabilities. In contrast, preferred habitat theory asserts that investors are willing to deviate from their preferred maturities if yield differentials encourage the switch. The unbiased expectations theory makes no assumptions about maturity preferences. Rather, it contends that forward rates are unbiased predictors of future spot rates.

Solution to 3:

B is correct. The unbiased expectations theory asserts that different maturity strategies, such as rollover, maturity matching, and riding the yield curve, have the same expected return. By definition, a risk-neutral party is indifferent about choices with equal expected payoffs, even if one choice is riskier. Thus, the predictions of the theory are consistent with the existence of risk-neutral investors.

Solution to 4:

B is correct. The existence of a liquidity premium ensures that the forward rate is an upwardly biased estimate of the future spot rate. Market evidence clearly shows that liquidity premiums exist, and this evidence effectively refutes the predictions of the unbiased expectations theory.

Solution to 5:

A is correct. Although the local expectations theory predicts that the short-run return for all bonds will be equal to the risk-free rate, most of the evidence refutes that claim. Returns from long-dated bonds are generally higher than those from short-dated bonds, even over relatively short investment horizons. This market evidence is consistent with the risk–expected return trade-off that is central to finance and the uncertainty surrounding future spot rates.

5 MODERN TERM STRUCTURE MODELS

Modern term structure models provide quantitatively precise descriptions of how interest rates evolve. A model provides a sometimes simplified description of a real-world phenomenon on the basis of a set of assumptions; models are often used to solve particular problems. These assumptions cannot be completely accurate in depicting the real world, but instead, the assumptions are made to explain real-world phenomena sufficiently well to solve the problem at hand.

Interest rate models attempt to capture the statistical properties of interest rate movements. The detailed description of these models depends on mathematical and statistical knowledge well outside the scope of the investment generalist's technical preparation. Yet, these models are very important in the valuation of complex fixed-income instruments and bond derivatives. Thus, we provide a broad overview of these models in this reading. Equations for the models and worked examples are given for readers who are interested.

5.1 Equilibrium Term Structure Models

Equilibrium term structure models are models that seek to describe the dynamics of the term structure using fundamental economic variables that are assumed to affect interest rates. In the modeling process, restrictions are imposed that allow for the derivation of equilibrium prices for bonds and interest rate options. These models require the specification of a drift term (explained later) and the assumption of a functional form for interest rate volatility. The best-known equilibrium models are the **Cox–Ingersoll–Ross model**[10] and the **Vasicek model**,[11] which are discussed in the next two sections.

10 Cox, Ingersoll, and Ross (1985).
11 Vasicek (1977).

Equilibrium term structure models share several characteristics:

- *They are one-factor or multifactor models*. One-factor models assume that a single observable factor (sometimes called a state variable) drives all yield curve movements. Both the Vasicek and CIR models assume a single factor, the short-term interest rate, *r*. This approach is plausible because empirically, parallel shifts are often found to explain more than 90% of yield changes. In contrast, multifactor models may be able to model the curvature of a yield curve more accurately but at the cost of greater complexity.
- *They make assumptions about the behavior of factors*. For example, if we focus on a short-rate single-factor model, should the short rate be modeled as mean reverting? Should the short rate be modeled to exhibit jumps? How should the volatility of the short rate be modeled?
- *They are, in general, more sparing with respect to the number of parameters that must be estimated compared with arbitrage-free term structure models.* The cost of this relative economy in parameters is that arbitrage-free models can, in general, model observed yield curves more precisely.[12]

An excellent example of an equilibrium term structure model is the Cox–Ingersoll–Ross (CIR) model discussed next.

5.1.1 *The Cox–Ingersoll–Ross Model*

The CIR model assumes that every individual has to make consumption and investment decisions with their limited capital. Investing in the productive process may lead to higher consumption in the following period, but it requires sacrificing today's consumption. The individual must determine his or her optimal trade-off assuming that he or she can borrow and lend in the capital market. Ultimately, interest rates will reach a market equilibrium rate at which no one needs to borrow or lend. The CIR model can explain interest rate movements in terms of an individual's preferences for investment and consumption as well as the risks and returns of the productive processes of the economy.

As a result of this analysis, the model shows how the short-term interest rate is related to the risks facing the productive processes of the economy. Assuming that an individual requires a term premium on the long-term rate, the model shows that the short-term rate can determine the entire term structure of interest rates and the valuation of interest rate–contingent claims. The CIR model is presented in Equation 16.

In Equation 16, the terms "*dr*" and "*dt*" mean, roughly, an infinitely small increment in the (instantaneous) short-term interest rate and time, respectively; the CIR model is an instance of a so-called continuous-time finance model. The model has two parts: (1) a deterministic part (sometimes called a "drift term"), the expression in *dt*, and (2) a stochastic (i.e., random) part, the expression in *dz*, which models risk.

$$dr = a(b - r)dt + \sigma\sqrt{r}dz \quad (16)$$

The way the deterministic part, $a(b - r)dt$, is formulated in Equation 16 ensures mean reversion of the interest rate toward a long-run value *b*, with the speed of adjustment governed by the strictly positive parameter *a*. If *a* is high (low), mean reversion to the long-run rate *b* would occur quickly (slowly). In Equation 16, for simplicity of

12 Other contrasts are more technical. They include that equilibrium models use real probabilities whereas arbitrage-free models use so-called risk-neutral probabilities.

presentation we have assumed that the **term premium** of the CIR model is equal to zero.[13] Thus, as modeled here, the CIR model assumes that the economy has a constant long-run interest rate that the short-term interest rate converges to over time.

Mean reversion is an essential characteristic of the interest rate that sets it apart from many other financial data series. Unlike stock prices, for example, interest rates cannot rise indefinitely because at very high levels, they would hamper economic activity, which would ultimately result in a decrease in interest rates. Similarly, with rare historical exceptions, nominal interest rates are non-negative. As a result, short-term interest rates tend to move in a bounded range and show a tendency to revert to a long-run value b.

Note that in Equation 16, there is only one stochastic driver, dz, of the interest rate process; very loosely, dz can be thought of as an infinitely small movement in a "random walk." The stochastic or volatility term, $\sigma\sqrt{r}dz$, follows the random normal distribution for which the mean is zero, the standard deviation is 1, and the standard deviation factor is $\sigma\sqrt{r}$. The standard deviation factor makes volatility proportional to the square root of the short-term rate, which allows for volatility to increase with the level of interest rates. It also avoids the possibility of non-positive interest rates for all positive values of a and b.[14]

Note that a, b, and σ are model parameters that have to be specified in some manner.

AN ILLUSTRATION OF THE CIR MODEL

Assume again that the current short-term rate is r = 3% and the long-run value for the short-term rate is b = 8%. As before, assume that the speed of the adjustment factor is a = 0.40 and the annual volatility is σ = 20%. Using Equation 16, the CIR model provides the following formula for the change in short-term interest rates, dr:

$$dr = 0.40(8\% - r)dt + (20\%)\sqrt{r}dz$$

Assume that a random number generator produced standard normal random error terms, dz, of 0.50, −0.10, 0.50, and −0.30. The CIR model would produce the evolution of interest rates shown in Exhibit 7. The bottom half of the exhibit shows the pricing of bonds consistent with the evolution of the short-term interest rate.

Exhibit 7 Evolution of the Short-Term Rate in the CIR Model

	Time				
Parameter	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
r	3.000%	6.732%	6.720%	9.825%	7.214%
$a(b - r) = 0.40(8\% - r)$	2.000%	0.507%	0.512%	−0.730%	
dz	0.500	−0.100	0.500	−0.300	
$\sigma\sqrt{r}dz = 20\%\sqrt{r}dz$	1.732%	−0.519%	2.592%	−1.881%	
dr	3.732%	−0.012%	3.104%	−2.611%	
$r(t + 1) = r + dr$	6.732%	6.720%	9.825%	7.214%	
YTM for Zero-Coupon Bonds Maturing in					

13 Equilibrium models, but not arbitrage-free models, assume that a term premium is required on long-term interest rates. A term premium is the additional return required by lenders to invest in a bond to maturity net of the expected return from continually reinvesting at the short-term rate over that same time horizon.

14 As long as $2ab > \sigma^2$, per Yan (2001, p. 65).

Exhibit 7 (Continued)

	Time				
Parameter	***t* = 0**	***t* = 1**	***t* = 2**	***t* = 3**	***t* = 4**
1 Year	3.862%	6.921%	6.911%	9.456%	7.316%
2 Years	4.499%	7.023%	7.015%	9.115%	7.349%
5 Years	5.612%	7.131%	7.126%	8.390%	7.327%
10 Years	6.333%	7.165%	7.162%	7.854%	7.272%
30 Years	6.903%	7.183%	7.182%	7.415%	7.219%

The simulation of interest rates starts with an interest rate of 3%, which is well below the long-run value of 8%. Interest rates generated by the model quickly move toward this long-run value. Note that the standard normal variable dz is assumed to be 0.50 in time periods $t = 0$ and $t = 2$ but the volatility term, $\sigma\sqrt{r}dz$, is much higher in $t = 2$ than in $t = 0$ because volatility increases with the level of interest rates in the CIR model.

This example is stylized and intended for illustrative purposes only. The parameters used in practice typically vary significantly from those used here.

5.1.2 *The Vasicek Model*

Although not developed in the context of a general equilibrium of individuals seeking to make optimal consumption and investment decisions, as was the case for the CIR model, the Vasicek model is viewed as an equilibrium term structure model. Similar to the CIR model, the Vasicek model captures mean reversion.

Equation 17 presents the Vasicek model:

$$dr = a(b - r)dt + \sigma dz \qquad (17)$$

The Vasicek model has the same drift term as the CIR model and thus tends toward mean reversion in the short rate, r. The stochastic or volatility term, σdz, follows the random normal distribution for which the mean is zero and the standard deviation is 1. Unlike the CIR Model, interest rates are calculated assuming that volatility remains constant over the period of analysis. As with the CIR model, there is only one stochastic driver, dz, of the interest rate process and a, b, and σ are model parameters that have to be specified in some manner. The main disadvantage of the Vasicek model is that it is theoretically possible for the interest rate to become negative.

AN ILLUSTRATION OF THE VASICEK MODEL

Assume that the current short-term rate is $r = 3\%$ and the long-run value for the short-term rate is $b = 8\%$. Also assume that the speed of the adjustment factor is $a = 0.40$ and the annual volatility is $\sigma = 2\%$. Using Equation 17, the Vasicek model provides the following formula for the change in short-term interest rates, dr:

$$dr = 0.40(8\% - r)dt + (2\%)dz$$

The stochastic term, dz, is typically drawn from a standard normal distribution with a mean of zero and a standard deviation of 1. Assume that a random number generator produced standard normal random error terms of 0.45, 0.18, −0.30, and 0.25. The Vasicek model would produce the evolution of interest rates shown in Exhibit 8.

Exhibit 8 Evolution of the Short-Term Rate in the Vasicek Model

	Time				
Parameter	***t* = 0**	***t* = 1**	***t* =2**	***t* = 3**	***t* = 4**
r	3.000%	5.900%	7.100%	6.860%	7.816%
$a(b - r)$	2.000%	0.840%	0.360%	0.456%	
dz	0.450	0.180	−0.300	0.250	
σdz	0.900%	0.360%	−0.600%	0.500%	
dr	2.900%	1.200%	−0.240%	0.956%	
$r(t + 1) = r + dr$	5.900%	7.100%	6.860%	7.816%	
YTM for Zero-Coupon Bonds Maturing in					
1 Year	3.874%	6.264%	7.253%	7.055%	7.843%
2 Years	4.543%	6.539%	7.365%	7.200%	7.858%
5 Years	5.791%	7.045%	7.563%	7.460%	7.873%
10 Years	6.694%	7.405%	7.670%	7.641%	7.876%
30 Years	7.474%	7.716%	7.816%	7.796%	7.875%

Note that the simulation of interest rates starts with an interest rate of 3%, which is well below the long-run value of 8%. Interest rates generated by the model move quickly toward this long-run value despite declining in the third time period, which reflects the mean reversion built into the model via the drift term $a(b - r)dt$.

This example is stylized and intended for illustrative purposes only. The parameters used in practice typically vary significantly from those used here.

Note that because both the Vasicek model and the CIR model require the short-term rate to follow a certain process, the estimated yield curve may not match the observed yield curve. But if the parameters of the models are believed to be correct, then investors can use these models to determine mispricings.

5.2 Arbitrage-Free Models: The Ho–Lee Model

In **arbitrage-free models**, the analysis begins with the observed market prices of a reference set of financial instruments and the underlying assumption is that the reference set is correctly priced. An assumed random process with a drift term and volatility factor is used for the generation of the yield curve. The computational process that determines the term structure is such that the valuation process generates the market prices of the reference set of financial instruments. These models are called "arbitrage-free" because the prices they generate match market prices.

The ability to calibrate models to market data is a desirable feature of any model, and this fact points to one of the main drawbacks of the Vasicek and CIR models: They have only a finite number of free parameters, and so it is not possible to specify these parameter values in such a way that model prices coincide with observed market prices. This problem is overcome in arbitrage-free models by allowing the parameters to vary deterministically with time. As a result, the market yield curve can be modeled with the accuracy needed for such applications as valuing derivatives and bonds with embedded options.

The first arbitrage-free model was introduced by Ho and Lee.[15] It uses the relative valuation concepts of the Black–Scholes–Merton option-pricing model. Thus, the valuation of interest rate contingent claims is based solely on the yield curve's shape and its movements. The model assumes that the yield curve moves in a way that is consistent with a no-arbitrage condition.

In the **Ho–Lee model**, the short rate follows a normal process, as shown in Equation 18:

$$dr_t = \theta_t dt + \sigma dz_t \tag{18}$$

The model can be calibrated to market data by inferring the form of the time-dependent drift term, θ_t, from market prices, which means the model can precisely generate the current term structure. This calibration is typically performed via a binomial lattice-based model in which at each node the yield curve can move up or down with equal probability. This probability is called the "implied risk-neutral probability." Often it is called the "risk-neutral probability," which is somewhat misleading because arbitrage-free models do not assume market professionals are risk neutral as does the local expectations theory. This is analogous to the classic Black–Scholes–Merton option model insofar as the pricing dynamics are simplified because we can price debt securities "as if" market investors were risk neutral.

To make the discussion concrete, we illustrate a two-period Ho–Lee model. Assume that the current short-term rate is 4%. The time step is monthly, and the drift terms, which are determined using market prices, are θ_1 = 1% in the first month and θ_2 = 0.80% in the second month. The annual volatility is 2%. Below, we create a two-period binomial lattice-based model for the short-term rate. In the discrete binomial model, the dz term has two possible outcomes: +1 for periods in which rates move up and –1 for periods in which rates move down. Note that the monthly volatility is

$$\sigma\sqrt{\frac{1}{t}} = 2\%\sqrt{\frac{1}{12}} = 0.5774\%$$

and the time step is

$$dt = \frac{1}{12} = 0.0833$$

$$dr_t = \theta_t dt + \sigma dz_t = \theta_t(0.0833) + (0.5774)dz_t$$

If the rate goes up in the first month,

$$r = 4\% + (1\%)(0.0833) + 0.5774\% = 4.6607\%$$

If the rate goes up in the first month and up in the second month,

$$r = 4.6607\% + (0.80\%)(0.0833) + 0.5774\% = 5.3047\%$$

If the rate goes up in the first month and down in the second month,

$$r = 4.6607\% + (0.80\%)(0.0833) - 0.5774\% = 4.1499\%$$

If the rate goes down in the first month,

$$r = 4\% + (1\%)(0.0833) - 0.5774\% = 3.5059\%$$

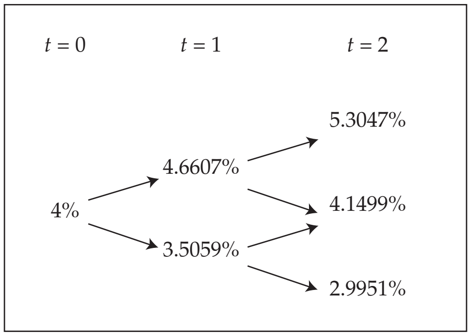
If the rate goes down in the first month and up in the second month,

$$r = 3.5059\% + (0.80\%)(0.0833) + 0.5774\% = 4.1499\%$$

If the rate goes down in the first month and down in the second month,

$$r = 3.5059\% + (0.80\%)(0.0833) - 0.5774\% = 2.9951\%$$

15 Ho and Lee (1986).



The interest rates generated by the model can be used to determine zero-coupon bond prices and the spot curve. By construction, the model output is consistent with market prices. Because of its simplicity, the Ho–Lee model is useful for illustrating most of the salient features of arbitrage-free interest rate models. Because the model generates a symmetrical ("bell-shaped" or normal) distribution of future rates, negative interest rates are possible. Note that although the volatility of the one-period rate is constant at each node point in the illustration, time-varying volatility—consistent with the historical behavior of yield curve movements—can be modeled in the Ho–Lee model because sigma (interest rate volatility) can be specified as a function of time. A more sophisticated example using a term structure of volatilities as inputs is outside the scope of this reading.

As mentioned before, models are assumptions made to describe certain phenomena and to provide solutions to problems at hand. Modern interest rate theories are proposed for the most part to value bonds with embedded options because the values of embedded options are frequently contingent on interest rates. The general equilibrium models introduced here describe yield curve movement as the movement in a single short-term rate. They are called one-factor models and, in general, seem empirically satisfactory. Arbitrage-free models do not attempt to explain the observed yield curve. Instead, these models take the yield curve as given. For this reason, they are sometimes labeled as **partial equilibrium models**.

The basic arbitrage-free concept can be used to solve much broader problems. These models can be extended to value many bond types, allowing for a term structure of volatilities, uncertain changes in the shape of the yield curve, adjustments for the credit risk of a bond, and much more. Yet, these many extensions are still based on the concept of arbitrage-free interest rate movements. For this reason, the principles of these models form a foundation for much of the modern progress made in financial modeling.

Example 10 addresses several basic points about modern term structure models.

EXAMPLE 10

Modern Term Structure Models

1 Which of the following would be expected to provide the *most* accurate modeling with respect to the observed term structure?

   A CIR model

   B Ho–Lee model

   C Vasicek model

2 Which of the following statements about the Vasicek model is *most* accurate? It has:

   A a single factor, the long rate.

   B a single factor, the short rate.

C two factors, the short rate and the long rate.

3 The CIR model:

A assumes interest rates are not mean reverting.

B has a drift term that differs from that of the Vasicek model.

C assumes interest rate volatility increases with increases in the level of interest rates.

Solution to 1:

B is correct. The CIR model and the Vasicek model are examples of equilibrium term structure models, whereas the Ho–Lee model is an example of an arbitrage-free term structure model. A benefit of arbitrage-free term structure models is that they are calibrated to the current term structure. In other words,the starting prices ascribed to securities are those currently found in the market. In contrast, equilibrium term structure models frequently generate term structures that are inconsistent with current market data.

Solution to 2:

B is correct. Use of the Vasicek model requires assumptions for the short-term interest rate, which are usually derived from more general assumptions about the state variables that describe the overall economy. Using the assumed process for the short-term rate, one can determine the yield on longer-term bonds by looking at the expected path of interest rates over time.

Solution to 3:

C is correct. The drift term of the CIR model is identical to that of the Vasicek model, and both models assume that interest rates are mean reverting. The big difference between the two models is that the CIR model assumes that interest rate volatility increases with increases in the level of interest rates. The Vasicek model assumes that interest rate volatility is a constant.

YIELD CURVE FACTOR MODELS

6

The effect of yield volatilities on price is an important consideration in fixed-income investment, particularly for risk management and portfolio evaluation. In this section, we will describe measuring and managing the interest rate risk of bonds.

6.1 A Bond's Exposure to Yield Curve Movement

Shaping risk is defined as the sensitivity of a bond's price to the changing shape of the yield curve. The shape of the yield curve changes continually, and yield curve shifts are rarely parallel. For active bond management, a bond investor may want to base trades on a forecasted yield curve shape or may want to hedge the yield curve risk on a bond portfolio. Shaping risk also affects the value of many options, which is very important because many fixed-income instruments have embedded options.

Exhibits 9 through 11 show historical yield curve movements for US, Japanese, and South Korean government bonds from August 2005 to July 2013. The exhibits show that the shape of the yield curve changes considerably over time. In the United States and South Korea, central bank policies in response to the Great Recession led to a significant decline in short-term yields during the 2007–2009 time period.

Long-term yields eventually followed suit, resulting in a flattening of the yield curve. Short-term and long-term Japanese yields have been low for quite some time. Note that the vertical axis values of the three exhibits differ.

Exhibit 9 Historical US Yield Curve Movements

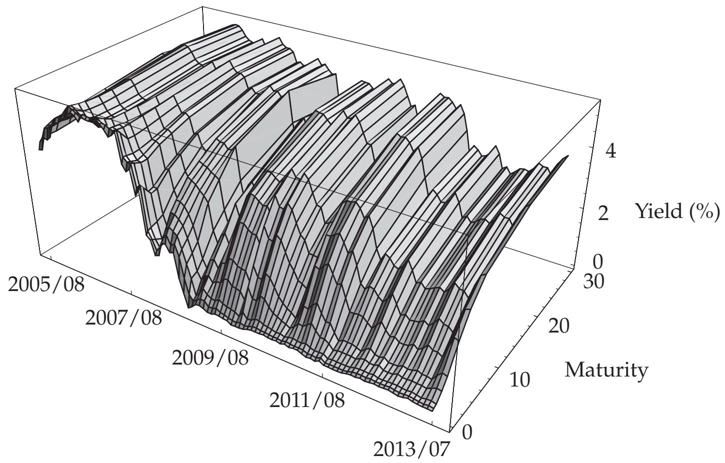


Exhibit 10 Historical Japanese Yield Curve Movements

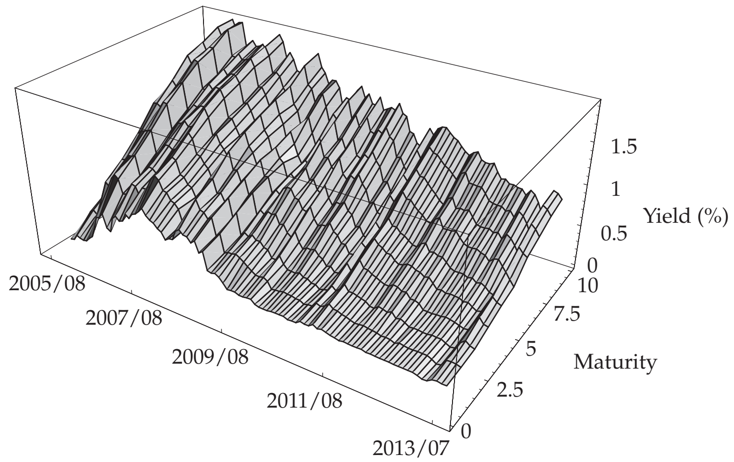
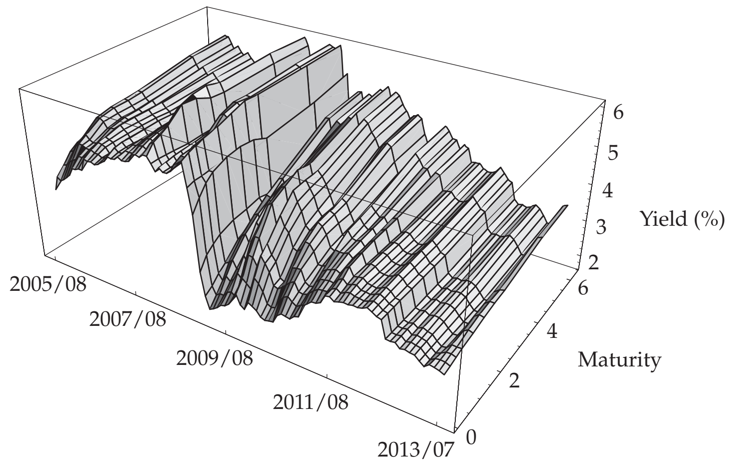


Exhibit 11 Historical Korean Yield Curve Movements



6.2 Factors Affecting the Shape of the Yield Curve

The previous section showed that the yield curve can take nearly any shape. The challenge for a fixed-income manager is to implement a process to manage the yield curve shape risk in his or her portfolio. One approach is to find a model that reduces most of the possible yield curve movements to a probabilistic combination of a few standardized yield curve movements. This section presents one of the best-known yield curve factor models.

A **yield curve factor model** is defined as a model or a description of yield curve movements that can be considered realistic when compared with historical data. Research shows that there are models that can describe these movements with some accuracy. One specific yield curve factor model is the three-factor model of Litterman and Scheinkman (1991), who found that yield curve movements are historically well described by a combination of three independent movements, which they interpreted as **level**, **steepness**, and **curvature**. The level movement refers to an upward or downward shift in the yield curve. The steepness movement refers to a non-parallel shift in the yield curve when either short-term rates change more than long-term rates or long-term rates change more than short-term rates. The curvature movement is a reference to movement in three segments of the yield curve: the short-term and long-term segments rise while the middle-term segment falls or vice versa.

The method to determine the number of factors—and their economic interpretation—begins with a measurement of the change of key rates on the yield curve, in this case 10 different points along the yield curve, as shown in Exhibits 12 and 13. The historical variance/covariance matrix of these interest rate movements is then obtained. The next step is to try to discover a number of independent factors (not to exceed the number of variables—in this case, selected points along the yield curve) that can explain the observed variance/covariance matrix. The approach that focuses on identifying the factors that best explain historical variances is known as **principal components analysis** (PCA). PCA creates a number of synthetic factors defined as (and calculated to be) statistically independent of each other; how these factors may be interpreted economically is a challenge to the researcher that can be addressed by relating movements in the factors (as we will call the principal components in this discussion) to movements in observable and easily understood variables.

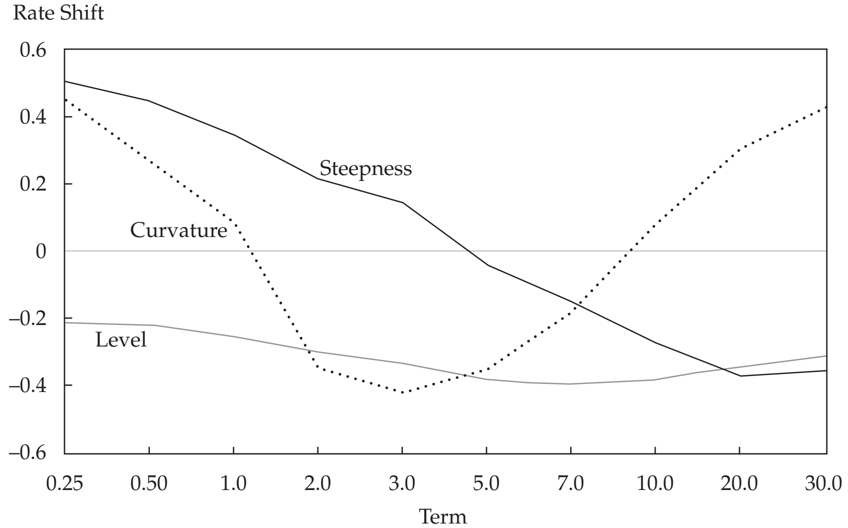
In applying this analysis to historical data for the period of August 2005–July 2013, very typical results were found, as expressed in Exhibit 12 and graphed in Exhibit 13. The first principal component explained about 77% of the total variance/covariance, and the second and third principal components (or factors) explained 17% and 3%, respectively. These percentages are more commonly recognized as R^2s, which, by the underlying assumptions of principal components analysis, can be simply summed to discover that a linear combination of the first three factors explains almost 97% of the total yield curve changes in the sample studied.

Exhibit 12 The First Three Yield Curve Factors, US Treasury Securities, August 2005–July 2013 (Entries are percents)

Time to Maturity (Years)	0.25	0.5	1	2	3	5	7	10	20	30
Factor 1 "Level"	−0.2089	−0.2199	−0.2497	−0.2977	−0.3311	−0.3756	−0.3894	−0.3779	−0.3402	−0.3102
Factor 2 "Steepness"	0.5071	0.4480	0.3485	0.2189	0.1473	−0.0371	−0.1471	−0.2680	−0.3645	−0.3514
Factor 3 "Curvature"	0.4520	0.2623	0.0878	−0.3401	−0.4144	−0.349	−0.1790	0.0801	0.3058	0.4219

Note that in Exhibit 13, the x-axis represents time to maturity in years.

Exhibit 13 The First Three Yield Curve Factors for US Treasury Securities, August 2005–July 2013



How should Exhibit 12 be interpreted? Exhibit 12 shows that for a one standard deviation positive change in the first factor (normalized to have unit standard deviation), the yield for a 0.25-year bond would decline by 0.2089%, a 0.50-year bond by 0.2199%, and so on across maturities, so that a 30-year bond would decline by 0.3102%.

Because the responses are in the same direction and by similar magnitudes, a reasonable interpretation of the first factor is that it describes (approximately) parallel shifts up and down the entire length of the yield curve.

Examining the second factor, we notice that a unitary positive standard deviation change appears to raise rates at shorter maturities (e.g., +0.5071% for 0.25-year bonds) but lowers rates at longer maturities (e.g., −0.3645% and −0.3514% for 20- and 30-year bonds, respectively). We can reasonably interpret this factor as one that causes changes in the steepness or slope of the yield curve. We note that the R^2 associated with this factor of 17% is much less important than the 77% R^2 associated with the first factor, which we associated with parallel shifts in the yield curve.

The third factor contributes a much smaller R^2 of 3%, and we associate this factor with changes in the curvature or "twist" in the curve because a unitary positive standard deviation change in this factor leads to positive yield changes at both short and long maturities but produces declines at intermediate maturities.

PCA shows similar results when applied to other government bond markets during the August 2005–July 2013 time period. Exhibits 14 and 15 reflect the results graphically for the Japanese and South Korean markets. In these instances, results can also be well explained by factors that appear to be associated, in declining order of importance, with parallel shifts, changes in steepness, and changes in curvature. Note that in Exhibits 14 and 15, as in Exhibit 13, the x-axis represents time to maturity in years.

Exhibit 14 The First Three Yield Curve Factors for Japanese Government Securities, August 2005–July 2013

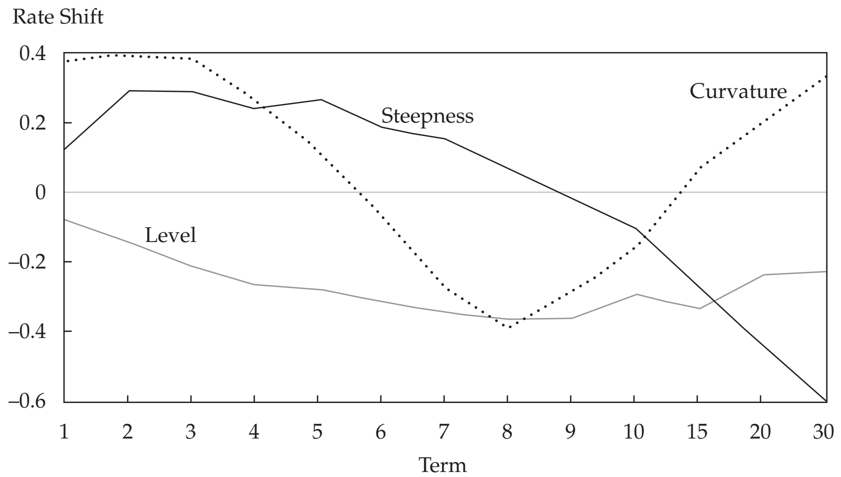
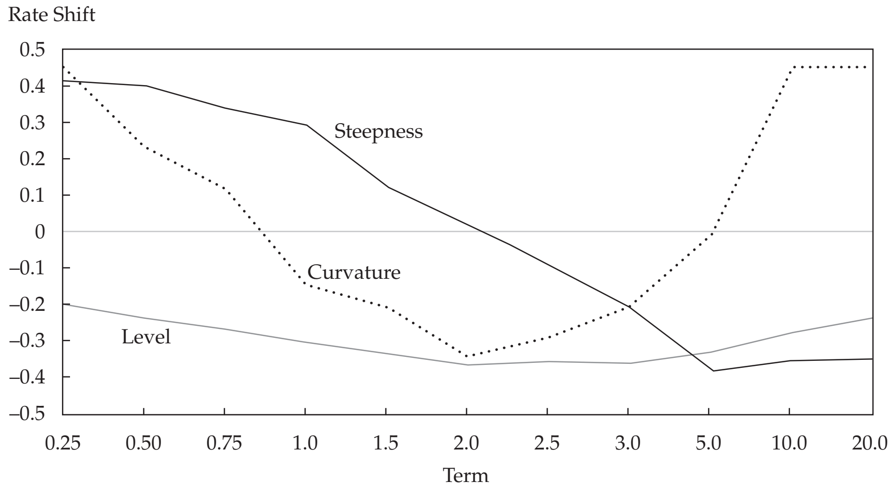


Exhibit 15 The First Three Yield Curve Factors for South Korean Government Securities, August 2005–July 2013



As in any other time series or regression model, the impact of the factors may change depending on the time period selected for study. However, if the reader selects any date within the sample period used to estimate these factors, a linear combination of the factors should explain movements of the yield curve on that date well.

6.3 The Maturity Structure of Yield Curve Volatilities

In modern fixed-income management, quantifying interest rate volatilities is important for at least two reasons. First, most fixed-income instruments and derivatives have embedded options. Option values, and hence the values of the fixed-income instrument, crucially depend on the level of interest rate volatilities. Second, fixed-income interest rate risk management is clearly an important part of any management process, and such risk management includes controlling the impact of interest rate volatilities on the instrument's price volatility.

The term structure of interest rate volatilities is a representation of the yield volatility of a zero-coupon bond for every maturity of security. This volatility curve (or "vol") or volatility term structure measures yield curve risk.

Interest rate volatility is not the same for all interest rates along the yield curve. On the basis of the typical assumption of a lognormal model, the uncertainty of an interest rate is measured by the annualized standard deviation of the proportional change in a bond yield over a specified time interval. For example, if the time interval is a one-month period, then the specified time interval equals 1/12 years. This measure is called interest rate volatility, and it is denoted $\sigma(t,T)$, which is the volatility of the rate for a security with maturity T at time t. The term structure of volatilities is given by Equation 19:

$$\sigma(t,T) = \frac{\sigma\left[\Delta r(t,T)/r(t,T)\right]}{\sqrt{\Delta t}} \tag{19}$$

In Exhibit 16, to illustrate a term structure of volatility, the data series is deliberately chosen to end before the 2008 financial crisis, which was associated with some unusual volatity magnitudes.

Exhibit 16 Historical Volatility Term Structure: US Treasuries, August 2005–December 2007

Maturity (years)	0.25	0.50	1	2	3	5	7	10	20	30
$\sigma(t,T)$	0.3515	0.3173	0.2964	0.2713	0.2577	0.2154	0.1885	0.1621	0.1332	0.1169

For example, the 35.15% standard deviation for the three-month T-bill in Exhibit 16 is based on a monthly standard deviation of 0.1015 = 10.15%, which annualizes as

$$0.1015 \div \sqrt{\frac{1}{12}} = 0.3515 = 35.15\%$$

The volatility term structure typically shows that short-term rates are more volatile than long-term rates. Research indicates that short-term volatility is most strongly linked to uncertainty regarding monetary policy whereas long-term volatility is most strongly linked to uncertainty regarding the real economy and inflation. Furthermore, most of the co-movement between short-term and long-term volatilities appears to depend on the ever-changing correlations between these three determinants (monetary policy, the real economy, and inflation). During the period of August 2005–December 2007, long-term volatility was lower than short-term volatility, falling from 35.15% for the 0.25-year rate to 11.69% for the 30-year rate.

6.4 Managing Yield Curve Risks

Yield curve risk—risk to portfolio value arising from unanticipated changes in the yield curve—can be managed on the basis of several measures of sensitivity to yield curve movements. Management of yield curve risk involves changing the identified exposures to desired values by trades in security or derivative markets (the details fall under the rubric of fixed-income portfolio management and thus are outside the scope of this reading).

One available measure of yield curve sensitivity is effective duration, which measures the sensitivity of a bond's price to a small parallel shift in a benchmark yield curve. Another is based on key rate duration, which measures a bond's sensitivity to a small change in a benchmark yield curve at a specific maturity segment. A further measure can be developed on the basis of the factor model developed in Section 6.3. Using one of these last two measures allows identification and management of "shaping risk"—that is, sensitivity to changes in the shape of the benchmark yield curve—in addition to the risk associated with parallel yield curve changes, which is addressed adequately by effective duration.

To make the discussion more concrete, consider a portfolio of 1-year, 5-year, and 10-year zero-coupon bonds with $100 value in each position; total portfolio value is therefore $300. Also consider the hypothetical set of factor movements shown in the following table:

Year	1	5	10
Parallel	1	1	1
Steepness	–1	0	1
Curvature	1	0	1

In the table, a parallel movement or shift means that all the rates shift by an equal amount—in this case, by a unit of 1. A steepness movement means that the yield curve steepens with the long rate shifting up by one unit and the short rate shifting down by one unit. A curvature movement means that both the short rate and the long rate shift up by one unit whereas the medium-term rate remains unchanged. These movements need to be defined, as they are here, such that none of the movements can be a linear combination of the other two movements. Next, we address the calculation of the various yield curve sensitivity measures.

Because the bonds are zero-coupon bonds, the effective duration of each bond is the same as the maturity of the bonds.[16] The portfolio's effective duration is the weighted sum of the effective duration of each bond position; for this equally weighted portfolio, effective duration is 0.333(1 + 5 + 10) = 5.333.

To calculate **key rate durations**, consider various yield curve movements. First, suppose that the one-year rate changes by 100 bps while the other rates remain the same; the sensitivity of the portfolio to that shift is 1/[(300)(0.01)] = 0.3333. We conclude that the key rate duration of the portfolio to the one-year rate, denoted D_1, is 0.3333. Likewise, the key rate durations of the portfolio to the 5-year rate, D_5, and the 10-year rate, D_{10}, are 1.6667 and 3.3333, respectively. Note that the sum of the key rate durations is 5.333, which is the same as the effective duration of the portfolio. This fact can be explained intuitively. Key rate duration measures the portfolio risk exposure to each key rate. If all the key rates move by the same amount, then the yield curve has made a parallel shift, and as a result, the proportional change in value has to be consistent with effective duration. The related model for yield curve risk based on key rate durations would be

$$\left(\frac{\Delta P}{P}\right) \approx -D_1\Delta r_1 - D_5\Delta r_5 - D_{10}\Delta r_{10} \qquad (20)$$
$$= -0.3333\Delta r_1 - 1.6667\Delta r_5 - 3.3333\Delta r_{10}$$

Next, we can calculate a measure based on the decomposition of yield curve movements into parallel, steepness, and curvature movements made in Section 6.3. Define D_L, D_S, and D_C as the sensitivities of portfolio value to small changes in the level, steepness, and curvature factors, respectively. Based on this factor model, Equation 21 shows the proportional change in portfolio value that would result from a small change in the level factor (Δx_L), the steepness factor (Δx_S), and the curvature factor (Δx_C).

$$\left(\frac{\Delta P}{P}\right) \approx -D_L\Delta x_L - D_S\Delta x_S - D_C\Delta x_C \qquad (21)$$

Because D_L is by definition sensitivity to a parallel shift, the proportional change in the portfolio value per unit shift (the line for a parallel movement in the table) is 5.3333 = 16/[(300)(0.01)]. The sensitivity for steepness movement can be calculated as follows (see the line for steepness movement in the table). When the steepness makes an upward shift of 100 bps, it would result in a downward shift of 100 bps for the 1-year rate, resulting in a gain of \$1, and an upward shift for the 10-year rate, resulting in a loss of \$10. The change in value is therefore (1 – 10). D_S is the negative of the proportional change in price per unit change in this movement and in this case is 3.0 = −(1 – 10)/[(300)(0.01)]. Considering the line for curvature movement in the table, D_C = 3.6667 = (1 + 10)/[(300)(0.01)]. Thus, for our hypothetical bond portfolio, we can analyze the portfolio's yield curve risk using

$$\left(\frac{\Delta P}{p}\right) \approx -5.3333\Delta x_L - 3.0\Delta x_S - 3.6667\Delta x_C \qquad (22)$$

16 Exactly so under continuous compounding.

For example, if $\Delta x_L = -0.0050$, $\Delta x_S = 0.002$, and $\Delta x_C = 0.001$, the predicted change in portfolio value would be +1.7%. It can be shown that key rate durations are directly related to level, steepness, and curvature in this example and that one set of sensitivities can be derived from the other. One can use the numerical example to verify that[17]

$$D_L = D_1 + D_5 + D_{10}$$

$$D_S = -D_1 + D_{10}$$

$$D_C = D_1 + D_{10}$$

Example 11 reviews concepts from this section and the preceding sections.

EXAMPLE 11

Term Structure Dynamics

1 The most important factor in explaining changes in the yield curve has been found to be:

A level.

B curvature.

C steepnesss.

2 A movement of the yield curve in which the short rate decreases by 150 bps and the long rate decreases by 50 bps would *best* be described as a:

A flattening of the yield curve resulting from changes in level and steepness.

B steepening of the yield curve resulting from changes in level and steepness.

C steepening of the yield curve resulting from changes in steepness and curvature.

3 A movement of the yield curve in which the short- and long-maturity sectors increase by 100 bps and 75 bps, respectively, but the intermediate-maturity sector increases by 10 bps, is *best* described as involving a change in:

A level only.

B curvature only.

C level and curvature.

4 Typically, short-term interest rates:

A are less volatile than long-term interest rates.

B are more volatile than long-term interest rates.

C have about the same volatility as long-term rates.

---

17 To see this, decompose Δr_1, Δr_5, and Δr_{10} into three factors—parallel, steepness, and curvature—based on the hypothetical movements in the table.

$$\Delta r_1 = \Delta x_L - \Delta x_s + \Delta x_C$$

$$\Delta r_5 = \Delta x_L$$

$$\Delta r_{10} = \Delta x_L + \Delta x_S + \Delta x_C$$

When we plug these equations into the expression for portfolio change based on key rate duration and simplify, we get

$$\frac{\Delta P}{P} = -D_1(\Delta x_L - \Delta x_S + \Delta x_C) - D_5(\Delta x_L) - D_{10}(\Delta x_L + \Delta x_S + \Delta x_C)$$

$$= -(D_1 + D_5 + D_{10})\Delta x_L - (-D_1 + D_{10})\Delta x_S - (D_1 + D_{10})\Delta x_C$$

5 Suppose for a given portfolio that key rate changes are considered to be changes in the yield on 1-year, 5-year, and 10-year securities. Estimated key rate durations are D_1 = 0.50, D_2 = 0.70, and D_3 = 0.90. What is the percentage change in the value of the portfolio if a parallel shift in the yield curve results in all yields declining by 50 bps?

A −1.05%.

B +1.05%.

C +2.10%.

Solution to 1:

A is correct. Research shows that upward and downward shifts in the yield curve explain more than 75% of the total change in the yield curve.

Solution to 2:

B is correct. Both the short-term and long-term rates have declined, indicating a change in the level of the yield curve. Short-term rates have declined more than long-term rates, indicating a change in the steepness of the yield curve.

Solution to 3:

C is correct. Both the short-term and long-term rates have increased, indicating a change in the level of the yield curve. However, intermediate rates have increased less than both short-term and long-term rates, indicating a change in curvature.

Solution to 4:

B is correct. A possible explanation is that expectations for long-term inflation and real economic activity affecting longer-term interest rates are slower to change than those related to shorter-term interest rates.

Solution to 5:

B is correct. A decline in interest rates would lead to an increase in bond portfolio value: −0.50(−0.005) − 0.70(−0.005) − 0.90(−0.005) = 0.0105 = 1.05%.

SUMMARY

- The spot rate for a given maturity can be expressed as a geometric average of the short-term rate and a series of forward rates.
- Forward rates are above (below) spot rates when the spot curve is upward (downward) sloping, whereas forward rates are equal to spot rates when the spot curve is flat.
- If forward rates are realized, then all bonds, regardless of maturity, will have the same one-period realized return, which is the first-period spot rate.
- If the spot rate curve is upward sloping and is unchanged, then each bond "rolls down" the curve and earns the forward rate that rolls out of its pricing (i.e., a T^*-period zero-coupon bond earns the T^*-period forward rate as it rolls down to be a $T^* - 1$ period security). This implies an expected return in excess of short-maturity bonds (i.e., a term premium) for longer-maturity bonds if the yield curve is upward sloping.

- Active bond portfolio management is consistent with the expectation that today's forward curve does not accurately reflect future spot rates.
- The swap curve provides another measure of the time value of money.
- The swap markets are significant internationally because swaps are frequently used to hedge interest rate risk exposure.
- The swap spread, the I-spread, and the Z-spread are bond quoting conventions that can be used to determine a bond's price.
- Swap curves and Treasury curves can differ because of differences in their credit exposures, liquidity, and other supply/demand factors.
- The local expectations theory, liquidity preference theory, segmented markets theory, and preferred habitat theory provide traditional explanations for the shape of the yield curve.
- Modern finance seeks to provide models for the shape of the yield curve and the use of the yield curve to value bonds (including those with embedded options) and bond-related derivatives. General equilibrium and arbitrage-free models are the two major types of such models.
- Arbitrage-free models are frequently used to value bonds with embedded options. Unlike equilibrium models, arbitrage-free models begin with the observed market prices of a reference set of financial instruments, and the underlying assumption is that the reference set is correctly priced.
- Historical yield curve movements suggest that they can be explained by a linear combination of three principal movements: level, steepness, and curvature.
- The volatility term structure can be measured using historical data and depicts yield curve risk.
- The sensitivity of a bond value to yield curve changes may make use of effective duration, key rate durations, or sensitivities to parallel, steepness, and curvature movements. Using key rate durations or sensitivities to parallel, steepness, and curvature movements allows one to measure and manage shaping risk.

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PRACTICE PROBLEMS

1 Given spot rates for one-, two-, and three-year zero coupon bonds, how many forward rates can be calculated?

2 Give two interpretations for the following forward rate: The two-year forward rate one year from now is 2%.

3 Describe the relationship between forward rates and spot rates if the yield curve is flat.

4 A Define the yield to maturity for a coupon bond.

B Is it possible for a coupon bond to earn less than the yield to maturity if held to maturity?

5 If a bond trader believes that current forward rates overstate future spot rates, how might he or she profit from that conclusion?

6 Explain the strategy of riding the yield curve.

7 What are the advantages of using the swap curve as a benchmark of interest rates relative to a government bond yield curve?

8 Describe how the Z-spread can be used to price a bond.

9 What is the TED spread and what type of risk does it measure?

10 According to the local expectations theory, what would be the difference in the one-month total return if an investor purchased a five-year zero-coupon bond versus a two-year zero-coupon bond?

11 Compare the segmented market and the preferred habitat term structure theories.

12 A List the three factors that have empirically been observed to affect Treasury security returns and explain how each of these factors affects returns on Treasury securities.

B What has been observed to be the most important factor in affecting Treasury returns?

C Which measures of yield curve risk can measure shaping risk?

13 Which forward rate cannot be computed from the one-, two-, three-, and four-year spot rates? The rate for a:

A one-year loan beginning in two years.

B two-year loan beginning in two years.

C three-year loan beginning in two years.

14 Consider spot rates for three zero-coupon bonds: $r(1)$ = 3%, $r(2)$ = 4%, and $r(3)$ = 5%. Which statement is correct? The forward rate for a one-year loan beginning in one year will be:

A less than the forward rate for a one-year loan beginning in two-years.

B greater than the forward rate for a two-year loan beginning in one-year.

C greater than the forward rate for a one-year loan beginning in two-years.

15 If one-period forward rates are decreasing with maturity, the yield curve is *most likely*:

A flat.

B upward-sloping.

C downward sloping.

The following information relates to Questions 16–29

A one-year zero-coupon bond yields 4.0%. The two- and three-year zero-coupon bonds yield 5.0% and 6.0% respectively.

16 The rate for a one-year loan beginning in one year is *closest* to:

A 4.5%.

B 5.0%.

C 6.0%.

17 The forward rate for a two-year loan beginning in one year is *closest* to:

A 5.0%.

B 6.0%.

C 7.0%.

18 The forward rate for a one-year loan beginning in two years is *closest* to:

A 6.0%.

B 7.0%.

C 8.0%.

19 The five-year spot rate is not given above; however, the forward price for a two-year zero-coupon bond beginning in three years is known to be 0.8479. The price today of a five-year zero-coupon bond is *closest* to:

A 0.7119.

B 0.7835.

C 0.9524.

20 The one-year spot rate $r(1)$ = 4%, the forward rate for a one-year loan beginning in one year is 6%, and the forward rate for a one-year loan beginning in two years is 8%. Which of the following rates is *closest* to the three-year spot rate?

A 4.0%

B 6.0%

C 8.0%

21 The one-year spot rate $r(1)$ = 5% and the forward price for a one-year zero-coupon bond beginning in one year is 0.9346. The spot price of a two-year zero-coupon bond is *closest* to:

A 0.87.

B 0.89.

C 0.93.

22 In a typical interest rate swap contract, the swap rate is *best* described as the interest rate for the:

A fixed-rate leg of the swap.

B floating-rate leg of the swap.

C difference between the fixed and floating legs of the swap.

23 A two-year fixed-for-floating Libor swap is 1.00% and the two-year US Treasury bond is yielding 0.63%. The swap spread is *closest* to:

A 37 bps.

B 100 bps.

C 163 bps.

24 The swap spread is quoted as 50 bps. If the five-year US Treasury bond is yielding 2%, the rate paid by the fixed payer in a five-year interest rate swap is *closest* to:

A 0.50%.

B 1.50%.

C 2.50%.

25 If the three-month T-bill rate drops and the Libor rate remains the same, the relevant TED spread:

A increases.

B decreases.

C does not change.

26 Given the yield curve for US Treasury zero-coupon bonds, which spread is *most* helpful pricing a corporate bond? The:

A Z-Spread.

B TED spread.

C Libor–OIS spread.

27 A four-year corporate bond with a 7% coupon has a Z-spread of 200 bps. Assume a flat yield curve with an interest rate for all maturities of 5% and annual compounding. The bond will *most likely* sell:

A close to par.

B at a premium to par.

C at a discount to par.

28 The Z-spread of Bond A is 1.05% and the Z-spread of Bond B is 1.53%. All else equal, which statement *best* describes the relationship between the two bonds?

A Bond B is safer and will sell at a lower price.

B Bond B is riskier and will sell at a lower price.

C Bond A is riskier and will sell at a higher price.

29 Which term structure model can be calibrated to closely fit an observed yield curve?

A The Ho–Lee Model

B The Vasicek Model

C The Cox–Ingersoll–Ross Model

The following information relates to Questions 30–36

Jane Nguyen is a senior bond trader and Christine Alexander is a junior bond trader for an investment bank. Nguyen is responsible for her own trading activities and also for providing assignments to Alexander that will develop her skills and create profitable trade ideas. Exhibit 1 presents the current par and spot rates.

Exhibit 1 Current Par and Spot Rates

Maturity	Par Rate	Spot Rate
One year	2.50%	2.50%
Two years	2.99%	3.00%
Three years	3.48%	3.50%
Four years	3.95%	4.00%
Five years	4.37%	

Note: Par and spot rates are based on annual-coupon sovereign bonds.

Nguyen gives Alexander two assignments that involve researching various questions:

Assignment 1 What is the yield to maturity of the option-free, default risk–free bond presented in Exhibit 2? Assume that the bond is held to maturity, and use the rates shown in Exhibit 1.

Exhibit 2 Selected Data for $1,000 Par Bond

Bond Name	Maturity (*T*)	Coupon
Bond Z	Three years	6.00%

Note: Terms are today for a *T*-year loan.

Assignment 2 Assuming that the projected spot curve two years from today will be below the current forward curve, is Bond Z fairly valued, undervalued, or overvalued?

After completing her assignments, Alexander asks about Nguyen's current trading activities. Nguyen states that she has a two-year investment horizon and will purchase Bond Z as part of a strategy to ride the yield curve. Exhibit 1 shows Nguyen's yield curve assumptions implied by the spot rates.

30 Based on Exhibit 1, the five-year spot rate is *closest to*:

A 4.40%.

B 4.45%.

C 4.50%.

31 Based on Exhibit 1, the market is *most likely* expecting:

A deflation.

B inflation.

C no risk premiums.

32 Based on Exhibit 1, the forward rate of a one-year loan beginning in three years is *closest to*:
- A 4.17%.
- B 4.50%.
- C 5.51%.

33 Based on Exhibit 1, which of the following forward rates can be computed?
- A A one-year loan beginning in five years
- B A three-year loan beginning in three years
- C A four-year loan beginning in one year

34 For Assignment 1, the yield to maturity for Bond Z is *closest* to the:
- A one-year spot rate.
- B two-year spot rate.
- C three-year spot rate.

35 For Assignment 2, Alexander should conclude that Bond Z is currently:
- A undervalued.
- B fairly valued.
- C overvalued.

36 By choosing to buy Bond Z, Nguyen is *most likely* making which of the following assumptions?
- A Bond Z will be held to maturity.
- B The three-year forward curve is above the spot curve.
- C Future spot rates do not accurately reflect future inflation.

The following information relates to Questions 37–41

Laura Mathews recently hired Robert Smith, an investment adviser at Shire Gate Advisers, to assist her in investing. Mathews states that her investment time horizon is short, approximately two years or less. Smith gathers information on spot rates for on-the-run annual-coupon government securities and swap spreads, as presented in Exhibit 1. Shire Gate Advisers recently published a report for its clients stating its belief that, based on the weakness in the financial markets, interest rates will remain stable, the yield curve will not change its level or shape for the next two years, and swap spreads will also remain unchanged.

Exhibit 1 Government Spot Rates and Swap Spreads

	Maturity (years)			
	1	2	3	4
Government spot rate	2.25%	2.70%	3.30%	4.05%
Swap spread	0.25%	0.30%	0.45%	0.70%

Smith decides to examine the following three investment options for Mathews:

Investment 1:	Buy a government security that would have an annualized return that is nearly risk free. Smith is considering two possible implementations: a two-year investment or a combination of two one-year investments.
Investment 2:	Buy a four-year, zero-coupon corporate bond and then sell it after two years. Smith illustrates the returns from this strategy using the swap rate as a proxy for corporate yields.
Investment 3:	Buy a lower-quality, two-year corporate bond with a coupon rate of 4.15% and a Z-spread of 65 bps.

When Smith meets with Mathews to present these choices, Mathews tells him that she is somewhat confused by the various spread measures. She is curious to know whether there is one spread measure that could be used as a good indicator of the risk and liquidity of money market securities during the recent past.

37 In his presentation of Investment 1, Smith could show that under the no-arbitrage principle, the forward price of a one-year government bond to be issued in one year is *closest* to:

A 0.9662.

B 0.9694.

C 0.9780.

38 In presenting Investment 1, using Shire Gate Advisers' interest rate outlook, Smith could show that riding the yield curve provides a total return that is *most likely*:

A lower than the return on a maturity-matching strategy.

B equal to the return on a maturity-matching strategy.

C higher than the return on a maturity-matching strategy.

39 In presenting Investment 2, Smith should show a total return *closest* to:

A 4.31%.

B 5.42%.

C 6.53%.

40 The bond in Investment 3 is *most likely* trading at a price of:

A 100.97.

B 101.54.

C 104.09.

41 The *most* appropriate response to Mathews question regarding a spread measure is the:

A Z-spread.

B Treasury–Eurodollar (TED) spread.

C Libor–OIS (overnight indexed swap) spread.

The following information relates to Questions 42–48

Rowan Madison is a junior analyst at Cardinal Capital. Sage Winter, a senior portfolio manager and Madison's supervisor, meets with Madison to discuss interest rates and review two bond positions in the firm's fixed-income portfolio.

Winter begins the meeting by asking Madison to state her views on the term structure of interest rates. Madison responds:

> "Yields are a reflection of expected spot rates and risk premiums. Investors demand risk premiums for holding long-term bonds, and these risk premiums increase with maturity."

Winter next asks Madison to describe features of equilibrium and arbitrage-free term structure models. Madison responds by making the following statements:

Statement 1	"Equilibrium term structure models are factor models that use the observed market prices of a reference set of financial instruments, assumed to be correctly priced, to model the market yield curve."
Statement 2	"In contrast, arbitrage-free term structure models seek to describe the dynamics of the term structure by using fundamental economic variables that are assumed to affect interest rates."

Winter asks Madison about her preferences concerning term structure models. Madison states:

> "I prefer arbitrage-free models. Even though equilibrium models require fewer parameters to be estimated relative to arbitrage-free models, arbitrage-free models allow for time-varying parameters. In general, this allowance leads to arbitrage-free models being able to model the market yield curve more precisely than equilibrium models."

Winter tells Madison that, based on recent changes in spreads, she is concerned about a perceived increase in counterparty risk in the economy and its effect on the portfolio. Madison asks Winter:

> "Which spread measure should we use to assess changes in counterparty risk in the economy?"

Winter is also worried about the effect of yield volatility on the portfolio. She asks Madison to identify the economic factors that affect short-term and long-term rate volatility. Madison responds:

> "Short-term rate volatility is mostly linked to uncertainty regarding monetary policy, whereas long-term rate volatility is mostly linked to uncertainty regarding the real economy and inflation."

Finally, Winter asks Madison to analyze the interest rate risk portfolio positions in a 5-year and a 20-year bond. Winter requests that the analysis be based on level, slope, and curvature as term structure factors. Madison presents her analysis in Exhibit 1.

Exhibit 1 Three-Factor Model of Term Structure

	Time to Maturity (years)	
Factor	**5**	**20**
Level	–0.4352%	–0.5128%
Steepness	–0.0515%	–0.3015%
Curvature	0.3963%	0.5227%

Note: Entries indicate how yields would change for a one standard deviation increase in a factor.

Winter asks Madison to perform two analyses:

Analysis 1: Calculate the expected change in yield on the 20-year bond resulting from a two standard deviation increase in the steepness factor.

Analysis 2: Calculate the expected change in yield on the five-year bond resulting from a one standard deviation decrease in the level factor and a one standard deviation decrease in the curvature factor.

42 Madison's views on the term structure of interest rates are *most* consistent with the:

A local expectations theory.

B segmented markets theory.

C liquidity preference theory.

43 Which of Madison's statement(s) regarding equilibrium and arbitrage-free term structure models is *incorrect*?

A Statement 1 only

B Statement 2 only

C Both Statement 1 and Statement 2

44 Is Madison correct in describing key differences in equilibrium and arbitrage-free models as they relate to the number of parameters and model accuracy?

A Yes.

B No, she is incorrect about which type of model requires fewer parameter estimates.

C No, she is incorrect about which type of model is more precise at modeling market yield curves.

45 The *most appropriate* response to Madison's question regarding the spread measure is the:

A Z-spread.

B Treasury–Eurodollar (TED) spread.

C Libor–OIS (overnight indexed swap) spread.

46 Is Madison's response regarding the factors that affect short-term and long-term rate volatility correct?

A Yes.

B No, she is incorrect regarding factors linked to long-term rate volatility.

C No, she is incorrect regarding factors linked to short-term rate volatility.

47 Based on Exhibit 1, the results of Analysis 1 should show the yield on the 20-year bond decreasing by:

A 0.3015%.

B 0.6030%.

C 0.8946%.

48 Based on Exhibit 1, the results of Analysis 2 should show the yield on the five-year bond:

A decreasing by 0.8315%.

B decreasing by 0.0389%.

C increasing by 0.0389%.

The following information relates to Questions 49–57

Liz Tyo is a fund manager for an actively managed global fixed-income fund that buys bonds issued in Countries A, B, and C. She and her assistant are preparing the quarterly markets update. Tyo begins the meeting by distributing the daily rates sheet, which includes the current government spot rates for Countries A, B, and C as shown in Exhibit 1.

Exhibit 1 Today's Government Spot Rates

Maturity	Country A	Country B	Country C
One year	0.40%	–0.22%	14.00%
Two years	0.70	–0.20	12.40
Three years	1.00	–0.12	11.80
Four years	1.30	–0.02	11.00
Five years	1.50	0.13	10.70

Tyo asks her assistant how these spot rates were obtained. The assistant replies, "Spot rates are determined through the process of bootstrapping. It entails backward substitution using par yields to solve for zero-coupon rates one by one, in order from latest to earliest maturities."

Tyo then provides a review of the fund's performance during the last year and comments, "The choice of an appropriate benchmark depends on the country's characteristics. For example, although Countries A and B have both an active government bond market and a swap market, Country C's private sector is much bigger than its public sector, and its government bond market lacks liquidity."

Tyo further points out, "The fund's results were mixed; returns did not benefit from taking on additional risk. We are especially monitoring the riskiness of the corporate bond holdings. For example, our largest holdings consist of three four-year corporate bonds (Bonds 1, 2, and 3) with identical maturities, coupon rates, and other contract terms. These bonds have Z-spreads of 0.55%, 1.52%, and 1.76%, respectively."

Tyo continues, "We also look at risk in terms of the swap spread. We considered historical three-year swap spreads for Country B, which reflect that market's credit and liquidity risks, at three different points in time." Tyo provides the information in Exhibit 2.

Exhibit 2 Selected Historical Three-Year Rates for Country B

Period	Government Bond Yield (%)	Fixed-for-Floating Libor Swap (%)
1 Month ago	–0.10	0.16
6 Months ago	–0.08	0.01
12 Months ago	–0.07	0.71

Tyo then suggests that the firm was able to add return by riding the yield curve. The fund plans to continue to use this strategy but only in markets with an attractive yield curve for this strategy.

She moves on to present her market views on the respective yield curves for a five-year investment horizon.

Country A: "The government yield curve has changed little in terms of its level and shape during the last few years, and I expect this trend to continue. We assume that future spot rates reflect the current forward curve for all maturities."

Country B: "Because of recent economic trends, I expect a reversal in the slope of the current yield curve. We assume that future spot rates will be higher than current forward rates for all maturities."

Country C: "To improve liquidity, Country C's central bank is expected to intervene, leading to a reversal in the slope of the existing yield curve. We assume that future spot rates will be lower than today's forward rates for all maturities."

Tyo's assistant asks, "Assuming investors require liquidity premiums, how can a yield curve slope downward? What does this imply about forward rates?"

Tyo answers, "Even if investors require compensation for holding longer-term bonds, the yield curve can slope downward—for example, if there is an expectation of severe deflation. Regarding forward rates, it can be helpful to understand yield curve dynamics by calculating implied forward rates. To see what I mean, we can use Exhibit 1 to calculate the forward rate for a two-year Country C loan beginning in three years."

49 Did Tyo's assistant accurately describe the process of bootstrapping?

A Yes.

B No, with respect to par yields.

C No, with respect to backward substitution.

50 The swap curve is a better benchmark than the government spot curve for:

A Country A.

B Country B.

C Country C.

51 Based on the given *Z*-spreads for Bonds 1, 2, and 3, which bond has the greatest credit and liquidity risk?

A Bond 1

B Bond 2

C Bond 3

52 Based on Exhibit 2, the implied credit and liquidity risks as indicated by the historical three-year swap spreads for Country B were the lowest:

A 1 month ago.

B 6 months ago.

C 12 months ago.

53 Based on Exhibit 1 and Tyo's expectations, which country's term structure is currently best for traders seeking to ride the yield curve?

A Country A

B Country B

C Country C

54 Based on Exhibit 1 and assuming Tyo's market views on yield curve changes are realized, the forward curve of which country will lie below its spot curve?

A Country A

B Country B

C Country C

55 Based on Exhibit 1 and Tyo's expectations for the yield curves, Tyo *most likely* perceives the bonds of which country to be fairly valued?

A Country A

B Country B

C Country C

56 With respect to their discussion of yield curves, Tyo and her assistant are *most likely* discussing which term structure theory?

A Pure expectations theory

B Local expectations theory

C Liquidity preference theory

57 Tyo's assistant should calculate a forward rate *closest* to:

A 9.07%.

B 9.58%.

C 9.97%.

SOLUTIONS

1 Three forward rates can be calculated from the one-, two- and three-year spot rates. The rate on a one-year loan that begins at the end of Year 1 can be calculated using the one- and two-year spot rates; in the following equation one would solve for $f(1,1)$:

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$

The rate on a one-year loan that starts at the end of Year 2 can be calculated from the two- and three-year spot rates; in the following equation one would solve for $f(2,1)$:

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$

Additionally, the rate on a two-year loan that begins at the end of Year 1 can be computed from the one- and three-year spot rates; in the following equation one would solve for $f(1,2)$:

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

2 For the two-year forward rate one year from now of 2%, the two interpretations are as follows:

- 2% is the rate that will make an investor indifferent between buying a three-year zero-coupon bond or investing in a one-year zero-coupon bond and when it matures reinvesting in a zero-coupon bond that matures in two years.
- 2% is the rate that can be locked in today by buying a three-year zero-coupon bond rather than investing in a one-year zero-coupon bond and when it matures reinvesting in a zero-coupon bond that matures in two years.

3 A flat yield curve implies that all spot interest rates are the same. When the spot rate is the same for every maturity, successive applications of the forward rate model will show all the forward rates will also be the same and equal to the spot rate.

4 **A** The yield to maturity of a coupon bond is the expected rate of return on a bond if the bond is held to maturity, there is no default, and the bond and all coupons are reinvested at the original yield to maturity.

B Yes, it is possible. For example, if reinvestment rates for the future coupons are lower than the initial yield to maturity, a bond holder may experience lower realized returns.

5 If forward rates are higher than expected future spot rates the market price of the bond will be lower than the intrinsic value. This is because, everything else held constant, the market is currently discounting the bonds cash flows at a higher rate than the investor's expected future spot rates. The investor can capitalize on this by purchasing the undervalued bond. If expected future spot rates are realized, then bond prices should rise, thus generating gains for the investor.

6 The strategy of riding the yield curve is one in which a bond trader attempts to generate a total return over a given investment horizon that exceeds the return to bond with maturity matched to the horizon. The strategy involves buying a bond with maturity more distant than the investment horizon. Assuming an upward sloping yield curve, if the yield curve does not change level or shape, as the bond approaches maturity (or rolls down the yield curve) it will be priced at successively lower yields. So as long as the bond is held for a period less than maturity, it should generate higher returns because of price gains.

7 Some countries do not have active government bond markets with trading at all maturities. For those countries without a liquid government bond market but with an active swap market, there are typically more points available to construct a swap curve than a government bond yield curve. For those markets, the swap curve may be a superior benchmark.

8 The Z-spread is the constant basis point spread added to the default-free spot curve to correctly price a risky bond. A Z-spread of 100bps for a particular bond would imply that adding a fixed spread of 100bps to the points along the spot yield curve will correctly price the bond. A higher Z-spread would imply a riskier bond.

9 The TED spread is the difference between a Libor rate and the US T-Bill rate of matching maturity. It is an indicator of perceived credit risk in the general economy. I particular, because sovereign debt instruments are typically the benchmark for the lowest default risk instruments in a given market, and loans between banks (often at Libor) have some counterparty risk, the TED spread is considered to at least in part reflect default (or counterparty) risk in the banking sector.

10 The local expectations theory asserts that the total return over a one-month horizon for a five-year zero-coupon bond would be the same as for a two-year zero-coupon bond.

11 Both theories attempt to explain the shape of any yield curve in terms of supply and demand for bonds. In segmented market theory, bond market participants are limited to purchase of maturities that match the timing of their liabilities. In the preferred habitat theory, participants have a preferred maturity for asset purchases, but may deviate from it if they feel returns in other maturities offer sufficient compensation for leaving their preferred maturity segment.

12 A Studies have shown that there have been three factors that affect Treasury returns: (1) changes in the level of the yield curve, (2) changes in the slope of the yield curve, and (3) changes in the curvature of the yield curve. Changes in the level refer to upward or downward shifts in the yield curve. For example, an upward shift in the yield curve is likely to result in lower returns across all maturities. Changes in the slope of the yield curve relate to the steepness of the yield curve. Thus, if the yield curve steepens it is likely to result in higher returns for short maturity bonds and lower returns for long maturity bonds. An example of a change in the curvature of the yield curve is a situation where rates fall at the short and long end of the yield curve while rising for intermediate maturities. In this situation returns on short and long maturities are likely to rise to rise while declining for intermediate maturity bonds.

B Empirically, the most important factor is the change in the level of interest rates.

C Key rate durations and a measure based on sensitivities to level, slope, and curvature movements can address shaping risk, but effective duration cannot.

13 C is correct. There is no spot rate information to provide rates for a loan that terminates in five years. That is $f(2,3)$ is calculated as follows:

$$f(2,3) = \sqrt[3]{\frac{[1+r(5)]^5}{[1+r(2)]^2}} - 1$$

The equation above indicates that in order to calculate the rate for a three-year loan beginning at the end of two years you need the five year spot rate $r(5)$ and the two-year spot rate $r(2)$. However $r(5)$ is not provided.

14 A is correct. The forward rate for a one-year loan beginning in one-year $f(1,1)$ is $1.04^2/1.03 - 1 = 5\%$. The rate for a one-year loan beginning in two-years $f(2,1)$ is $1.05^3/1.04^2 - 1 = 7\%$. This confirms that an upward sloping yield curve is consistent with an upward sloping forward curve.

15 C is correct. If one-period forward rates are decreasing with maturity then the forward curve is downward sloping. This turn implies a downward sloping yield curve where longer term spot rates $r(T + T^*)$ are less than shorter term spot rates $r(T)$.

16 C is correct. From the forward rate model, we have

$$[1 + r(2)]^2 = [1 + r(1)]^1[1 + f(1,1)]^1$$

Using the one- and two-year spot rates, we have

$$(1 + .05)^2 = (1 + .04)^1[1 + f(1,1)]^1, \text{ so } \frac{(1+.05)^2}{(1+.04)^1} - 1 = f(1,1) = 6.010\%$$

17 C is correct. From the forward rate model,

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,2)]^2$$

Using the one and three-year spot rates, we find

$$(1 + 0.06)^3 = (1 + 0.04)^1[1 + f(1,2)]^2, \text{ so } \sqrt{\frac{(1+0.06)^3}{(1+0.04)^1}} - 1 = f(1,2) = 7.014\%$$

18 C is correct. From the forward rate model,

$$[1 + r(3)]^3 = [1 + r(2)]^2[1 + f(2,1)]^1$$

Using the two and three-year spot rates, we find

$$(1 + 0.06)^3 = (1 + 0.05)^2[1 + f(2,1)]^1, \text{ so } \frac{(1+0.06)^3}{(1+0.05)^2} - 1 = f(2,1) = 8.029\%$$

19 A is correct. We can convert spot rates to spot prices to find $P(3) = \frac{1}{(1.06)^3} =$ 0.8396. The forward pricing model can be used to find the price of the five-year zero as $P(T^* + T) = P(T^*)F(T^*,T)$, so $P(5) = P(3)F(3,2) = 0.8396 \times 0.8479 =$ 0.7119.

20 B is correct. Applying the forward rate model, we find

$$[1 + r(3)]^3 = [1 + r(1)]^1[1 + f(1,1)]^1[1 + f(2,1)]^1$$

So $[1 + r(3)]^3 = (1 + 0.04)^1(1 + 0.06)^1(1 + 0.08)^1$, $\sqrt[3]{1.1906} - 1 = r(3) = 5.987\%$.

21 B is correct. We can convert spot rates to spot prices and use the forward pricing model, so have $P(1) = \dfrac{1}{(1.05)^1} = 0.9524$. The forward pricing model is $P(T^* + T) = P(T^*)F(T^*,T)$ so $P(2) = P(1)F(1,1) = 0.9524 \times 0.9346 = 0.8901$.

22 A is correct. The swap rate is the interest rate for the fixed-rate leg of an interest rate swap.

23 A is correct. The swap spread = 1.00% – 0.63% = 0.37% or 37 bps.

24 C is correct. The fixed leg of the five-year fixed-for-floating swap will be equal to the five-year Treasury rate plus the swap spread: 2% + 0.5% = 2.5%.

25 A is correct. The TED spread is the difference between the three-month Libor rate and the three-month Treasury bill rate. If the T-bill rate falls and Libor does not change, the TED spread will increase.

26 A is correct. The Z-spread is the single rate which, when added to the rates of the spot yield curve, will provide the correct discount rates to price a particular risky bond.

27 A is correct. The 200bps Z-spread can be added to the 5% rates from the yield curve to price the bond. The resulting 7% discount rate will be the same for all of the bond's cash-flows, since the yield curve is flat. A 7% coupon bond yielding 7% will be priced at par.

28 B is correct. The higher Z-spread for Bond B implies it is riskier than Bond A. The higher discount rate will make the price of Bond B lower than Bond A.

29 A is correct. The Ho–Lee model is arbitrage-free and can be calibrated to closely match the observed term structure.

30 B is correct. The five-year spot rate is determined by using forward substitution and using the known values of the one-year, two-year, three-year, and four-year spot rates as follows:

$$1 = \frac{0.0437}{(1.025)} + \frac{0.0437}{(1.030)^2} + \frac{0.0437}{(1.035)^3} + \frac{0.0437}{(1.040)^4} + \frac{1 + 0.0437}{\left[1 + r(5)\right]^5}$$

$$r(5) = \sqrt[5]{\frac{1.0437}{0.8394}} - 1 = 4.453\%$$

31 B is correct. The spot rates imply an upward-sloping yield curve, $r(3) > r(2) > r(1)$. Because nominal yields incorporate a premium for expected inflation, an upward-sloping yield curve is generally interpreted as reflecting a market expectation of increasing, or at least level, future inflation (associated with relatively strong economic growth).

32 C is correct. A one-year loan beginning in three years, or $f(3,1)$, is calculated as follows:

$$\left[1 + r(3 + 1)\right]^{(3+1)} = \left[1 + r(3)\right]^3 \left[1 + f(3,1)\right]^1$$

$$\left[1.040\right]^4 = \left[1.035\right]^3 \left[1 + f(3,1)\right]^1$$

$$f(3,1) = \frac{(1.04)^4}{(1.035)^3} - 1 = 5.514\%$$

33 C is correct. Exhibit 1 provides five years of par rates, from which the spot rates for $r(1)$, $r(2)$, $r(3)$, $r(4)$, and $r(5)$ can be derived. Thus the forward rate $f(1,4)$ can be calculated as follows:

$$f(1,4) = \sqrt[4]{\frac{[1 + r(5)]^5}{[1 + r(1)]}} - 1$$

34 C is correct. The yield to maturity, $y(3)$, of Bond Z should be a weighted average of the spot rates used in the valuation of the bond. Because the bond's largest cash flow occurs in Year 3, $r(3)$ will have a greater weight than $r(1)$ and $r(2)$ in determining $y(3)$.

Using the spot rates:

$$\text{Price} = \frac{\$60}{(1.025)^1} + \frac{\$60}{(1.030)^2} + \frac{\$1{,}060}{(1.035)^3} = \$1{,}071.16$$

Using the yield to maturity:

$$\text{Price} = \frac{\$60}{[1 + y(3)]^1} + \frac{\$60}{[1 + y(3)]^2} + \frac{\$1{,}060}{[1 + y(3)]^3} = \$1{,}071.16$$

Using a calculator, the compute result is $y(3) = 3.46\%$, which is closest to the three-year spot rate of 3.50%.

35 A is correct. Alexander projects that the spot curve two years from today will be below the current forward curve, which implies that her expected future spot rates beyond two years will be lower than the quoted forward rates. Alexander would perceive Bond Z to be undervalued in the sense that the market is effectively discounting the bond's payments at a higher rate than she would and the bond's market price is below her estimate of intrinsic value.

36 B is correct. Nguyen's strategy is to ride the yield curve, which is appropriate when the yield curve is upward sloping. The yield curve implied by Exhibit 1 is upward sloping, which implies that the three-year forward curve is above the current spot curve. When the yield curve slopes upward, as a bond approaches maturity or "rolls down the yield curve," the bond is valued at successively lower yields and higher prices.

37 B is correct. The forward pricing model is based on the no-arbitrage principle and is used to calculate a bond's forward price based on the spot yield curve. The spot curve is constructed by using annualized rates from option-free and default risk–free zero-coupon bonds.

Equation 2: $P(T^* + T) = P(T^*)F(T^*,T)$; we need to solve for $F(1,1)$.

$P(1) = 1/(1 + 0.0225)^1$ and $P(2) = 1/(1 + 0.0270)^2$,

$F(1,1) = P(2)/P(1) = 0.9481/0.9780 = 0.9694$.

38 C is correct. When the spot curve is upward sloping and its level and shape are expected to remain constant over an investment horizon (Shire Gate Advisers' view), buying bonds with a maturity longer than the investment horizon (i.e., riding the yield curve) will provide a total return greater than the return on a maturity-matching strategy.

39 C is correct. The swap spread is a common way to indicate credit spreads in a market. The four-year swap rate (fixed leg of an interest rate swap) can be used as an indication of the four-year corporate yield. Riding the yield curve by purchasing a four-year zero-coupon bond with a yield of 4.75% {i.e., 4.05% + 0.70%, [$P_4 = 100/(1 + 0.0475)^4 = 83.058$]} and then selling it when it becomes a two-year zero-coupon bond with a yield of 3.00% {i.e., 2.70% + 0.30%, [$P_2 = 100/(1 + 0.0300)^2 = 94.260$]} produces an annual return of 6.53%: $(94.260/83.058)^{0.5} - 1.0 = 0.0653$.

40 B is correct. The Z-spread is the constant basis point spread that is added to the default-free spot curve to price a risky bond. A Z-spread of 65 bps for a particular bond would imply adding a fixed spread of 65 bps to maturities along the spot curve to correctly price the bond. Therefore, for the two-year bond, $r(1)$ = 2.90% (i.e., 2.25% + 0.65%), $r(2)$ = 3.35% (i.e., 2.70% + 0.65%), and the price of the bond with an annual coupon of 4.15% is as follows:

$$P = 4.15/(1 +0.029)^1 + 4.15/(1 + 0.0335)^2 + 100/(1 + 0.0335)^2,$$

$$P = 101.54.$$

41 C is correct. The Libor–OIS spread is considered an indicator of the risk and liquidity of money market securities. This spread measures the difference between Libor and the OIS rate.

42 C is correct. Liquidity preference theory asserts that investors demand a risk premium, in the form of a liquidity premium, to compensate them for the added interest rate risk they face when buying long-maturity bonds. The theory also states that the liquidity premium increases with maturity.

43 C is correct. Both statements are incorrect because Madison incorrectly describes both types of models. Equilibrium term structure models are factor models that seek to describe the dynamics of the term structure by using fundamental economic variables that are assumed to affect interest rates. Arbitrage-free term structure models use observed market prices of a reference set of financial instruments, assumed to be correctly priced, to model the market yield curve.

44 A is correct. Consistent with Madison's statement, equilibrium term structure models require fewer parameters to be estimated relative to arbitrage-free models, and arbitrage-free models allow for time-varying parameters. Consequently, arbitrage-free models can model the market yield curve more precisely than equilibrium models.

45 B is correct. The TED spread, calculated as the difference between Libor and the yield on a T-bill of matching maturity, is an indicator of perceived credit risk in the general economy. An increase (decrease) in the TED spread signals that lenders believe the risk of default on interbank loans is increasing (decreasing). Therefore, the TED spread can be thought of as a measure of counterparty risk.

46 A is correct. Madison's response is correct; research indicates that short-term rate volatility is mostly linked to uncertainty regarding monetary policy, whereas long-term rate volatility is mostly linked to uncertainty regarding the real economy and inflation.

47 B is correct. Because the factors in Exhibit 1 have been standardized to have unit standard deviations, a two standard deviation increase in the steepness factor will lead to the yield on the 20-year bond decreasing by 0.6030%, calculated as follows:

$$\text{Change in 20-year bond yield} = -0.3015\% \times 2 = -0.6030\%.$$

48 C is correct. Because the factors in Exhibit 1 have been standardized to have unit standard deviations, a one standard deviation decrease in both the level factor and the curvature factor will lead to the yield on the five-year bond increasing by 0.0389%, calculated as follows:

Change in five-year bond yield = 0.4352% – 0.3963% = 0.0389%.

49 C is correct. The assistant states that bootstrapping entails *backward* substitution using par yields to solve for zero-coupon rates one by one, in order from latest to earliest maturities. Bootstrapping entails *forward* substitution, however, using par yields to solve for zero-coupon rates one by one, in order from earliest to latest maturities.

50 C is correct. Country C's private sector is much bigger than the public sector, and the government bond market in Country C currently lacks liquidity. Under such circumstances, the swap curve is a more relevant benchmark for interest rates.

51 C is correct. Although swap spreads provide a convenient way to measure risk, a more accurate measure of credit and liquidity risk is called the zero-spread (Z-spread). It is the constant spread that, added to the implied spot yield curve, makes the discounted cash flows of a bond equal to its current market price. Bonds 1, 2, and 3 are otherwise similar but have Z-spreads of 0.55%, 1.52%, and 1.76%, respectively. Bond 3 has the highest Z-spread, implying that this bond has the greatest credit and liquidity risk.

52 B is correct. The historical three-year swap spread for Country B was the lowest six months ago. Swap spread is defined as the spread paid by the fixed-rate payer of an interest rate swap over the rate of the "on the run" (most recently issued) government bond security with the same maturity as the swap. The lower (higher) the swap spread, the lower (higher) the return that investors require for credit and/or liquidity risks.

The fixed rate of the three-year fixed-for-floating Libor swap was 0.01% six months ago, and the three-year government bond yield was –0.08% six months ago. Thus the swap spread six months ago was 0.01% – (–0.08%) = 0.09%.

One month ago, the fixed rate of the three-year fixed-for-floating Libor swap was 0.16%, and the three-year government bond yield was –0.10%. Thus the swap spread one month ago was 0.16% – (–0.10%) = 0.26%.

Twelve months ago, the fixed rate of the three-year fixed-for-floating Libor swap was 0.71%, and the three-year government bond yield was –0.07%. Thus, the swap spread 12 months ago was 0.71% – (–0.07%) = 0.78%.

53 A is correct. Country A's yield curve is upward sloping—a condition for the strategy—and more so than Country B's.

54 B is correct. The yield curve for Country B is currently upward sloping, but Tyo expects a reversal in the slope of the current yield curve. This means she expects the resulting yield curve for Country B to slope downward, which implies that the resulting forward curve would lie below the spot yield curve. The forward curve lies below the spot curve in scenarios in which the spot curve is downward sloping; the forward curve lies above the spot curve in scenarios in which the spot curve is upward sloping.

A is incorrect because the yield curve for Country A is currently upward sloping and Tyo expects that the yield curve will maintain its shape and level. That expectation implies that the resulting forward curve would be above the spot yield curve.

C is incorrect because the yield curve for Country C is currently downward sloping and Tyo expects a reversal in the slope of the current yield curve. This means she expects the resulting yield curve for Country C to slope upward, which implies that the resulting forward curve would be above the spot yield curve.

55 A is correct. Tyo's projected spot curve assumes that future spot rates reflect, or will be equal to, the current forward rates for all respective maturities. This assumption implies that the bonds for Country A are fairly valued because the market is effectively discounting the bond's payments at spot rates that match those projected by Tyo.

B and C are incorrect because Tyo's projected spot curves for the two countries do not match the current forward rates for all respective maturities. In the case of Country B, she expects future spot rates to be higher (than the current forward rates that the market is using to discount the bond's payments). For Country C, she expects future spot rates to be lower (than the current forward rates). Hence, she perceives the Country B bond to be currently overvalued and the Country C bond to be undervalued.

56 C is correct. Liquidity preference theory suggests that liquidity premiums exist to compensate investors for the added interest rate risk that they face when lending long term and that these premiums increase with maturity. Tyo and her assistant are assuming that liquidity premiums exist.

57 A is correct. From the forward rate model, $f(3,2)$, is found as follows:

$$[1 + r(5)]^5 = [1 + r(3)]^3[1 + f(3,2)]^2$$

Using the three-year and five-year spot rates, we find

$$(1 + 0.107)^5 = (1 + 0.118)^3[1 + f(3,2)]^2\text{, so}$$

$$\sqrt{\frac{(1 + 0.107)^5}{(1 + 0.118)^3}} - 1 = f(3,2) = 9.07\%$$