LEARNING MODULE

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Rates and Returns

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LEARNING OUTCOMES

Mastery	The candidate should be able to:
☐	interpret interest rates as required rates of return, discount rates, or opportunity costs and explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk
☐	calculate and interpret different approaches to return measurement over time and describe their appropriate uses
☐	compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures
☐	calculate and interpret annualized return measures and continuously compounded returns, and describe their appropriate uses
☐	calculate and interpret major return measures and describe their appropriate uses

INTRODUCTION

Interest rates are a critical concept in finance. In some cases, we assume a particular interest rate and in others, the interest rate remains the unknown quantity to determine. Although the pre-reads have covered the mechanics of time value of money problems, here we first illustrate the underlying economic concepts by explaining the meaning and interpretation of interest rates and then calculate, interpret, and compare different return measures.

LEARNING MODULE OVERVIEW



- An interest rate, *r*, can have three interpretations: (1) a required rate of return, (2) a discount rate, or (3) an opportunity cost. An interest rate reflects the relationship between differently dated cash flows.
- An interest rate can be viewed as the sum of the real risk-free interest rate and a set of premiums that compensate lenders for bearing distinct types of risk: an inflation premium, a default risk premium, a liquidity premium, and a maturity premium.
- The nominal risk-free interest rate is approximated as the sum of the real risk-free interest rate and the inflation premium.
- A financial asset's total return consists of two components: an income yield consisting of cash dividends or interest payments, and a return reflecting the capital gain or loss resulting from changes in the price of the financial asset.
- A holding period return, *R*, is the return that an investor earns for a single, specified period of time (e.g., one day, one month, five years).
- Multiperiod returns may be calculated across several holding periods using different return measures (e.g., arithmetic mean, geometric mean, harmonic mean, trimmed mean, winsorized mean). Each return computation has special applications for evaluating investments.
- The choice of which of the various alternative measurements of mean to use for a given dataset depends on considerations such as the presence of extreme outliers, outliers that we want to include, whether there is a symmetric distribution, and compounding.
- A money-weighted return reflects the actual return earned on an investment after accounting for the value and timing of cash flows relating to the investment.
- A time-weighted return measures the compound rate of growth of one unit of currency invested in a portfolio during a stated measurement period. Unlike a money-weighted return, a time-weighted return is not sensitive to the timing and amount of cashflows and is the preferred performance measure for evaluating portfolio managers because cash withdrawals or additions to the portfolio are generally outside of the control of the portfolio manager.
- Interest may be paid or received more frequently than annually. The periodic interest rate and the corresponding number of compounding periods (e.g., quarterly, monthly, daily) should be adjusted to compute present and future values.
- Annualizing periodic returns allows investors to compare different investments across different holding periods to better evaluate and compare their relative performance. With the number of compounding periods per year approaching infinity, the interest is compound continuously.
- Gross return, return prior to deduction of managerial and administrative expenses (those expenses not directly related to return generation), is an appropriate measure to evaluate the comparative performance of an asset manager.

- Net return, which is equal to the gross return less managerial and administrative expenses, is a better return measure of what an investor actually earned.
- The after-tax nominal return is computed as the total return minus any allowance for taxes on dividends, interest, and realized gains.
- Real returns are particularly useful in comparing returns across time periods because inflation rates may vary over time and are particularly useful for comparing investments across time periods and performance between different asset classes with different taxation.
- Leveraging a portfolio, via borrowing or futures, can amplify the portfolio's gains or losses.

INTEREST RATES AND TIME VALUE OF MONEY 2

☐ interpret interest rates as required rates of return, discount rates, or opportunity costs and explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

The time value of money establishes the equivalence between cash flows occurring on different dates. As cash received today is preferred to cash promised in the future, we must establish a consistent basis for this trade-off to compare financial instruments in cases in which cash is paid or received at different times. An **interest rate** (or yield), denoted *r*, is a rate of return that reflects the relationship between differently dated – timed – cash flows. If USD 9,500 today and USD 10,000 in one year are equivalent in value, then USD 10,000 – USD 9,500 = USD 500 is the required compensation for receiving USD 10,000 in one year rather than now. The interest rate (i.e., the required compensation stated as a rate of return) is USD 500/USD 9,500 = 0.0526 or 5.26 percent.

Interest rates can be thought of in three ways:

- First, they can be considered *required rates of return*—that is, the minimum rate of return an investor must receive to accept an investment.
- Second, interest rates can be considered *discount rates*. In the previous example, 5.26 percent is the discount rate at which USD 10,000 in one year is equivalent to USD 9,500 today. Thus, we use the terms "interest rate" and "discount rate" almost interchangeably.
- Third, interest rates can be considered *opportunity costs*. An **opportunity cost** is the value that investors forgo by choosing a course of action. In the example, if the party who supplied USD 9,500 had instead decided to spend it today, he would have forgone earning 5.26 percent by consuming rather than saving. So, we can view 5.26 percent as the opportunity cost of current consumption.

Determinants of Interest Rates

Economics tells us that interest rates are set by the forces of supply and demand, where investors supply funds and borrowers demand their use. Taking the perspective of investors in analyzing market-determined interest rates, we can view an interest rate r as being composed of a real risk-free interest rate plus a set of premiums that are required returns or compensation for bearing distinct types of risk:

r = Real risk-free interest rate + Inflation premium + Default risk premium + Liquidity premium + Maturity premium. (1)

- The **real risk-free interest rate** is the single-period interest rate for a completely risk-free security if no inflation were expected. In economic theory, the real risk-free rate reflects the time preferences of individuals for current versus future real consumption.
- The **inflation premium** compensates investors for expected inflation and reflects the average inflation rate expected over the maturity of the debt. Inflation reduces the purchasing power of a unit of currency—the amount of goods and services one can buy with it.
- The **default risk premium** compensates investors for the possibility that the borrower will fail to make a promised payment at the contracted time and in the contracted amount.
- The **liquidity premium** compensates investors for the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly. US Treasury bills (T-bills), for example, do not bear a liquidity premium because large amounts of them can be bought and sold without affecting their market price. Many bonds of small issuers, by contrast, trade infrequently after they are issued; the interest rate on such bonds includes a liquidity premium reflecting the relatively high costs (including the impact on price) of selling a position.
- The **maturity premium** compensates investors for the increased sensitivity of the market value of debt to a change in market interest rates as maturity is extended, in general (holding all else equal). The difference between the interest rate on longer-maturity, liquid Treasury debt and that on short-term Treasury debt typically reflects a positive maturity premium for the longer-term debt (and possibly different inflation premiums as well).

The sum of the real risk-free interest rate and the inflation premium is the nominal risk-free interest rate:

The nominal risk-free interest rate reflects the combination of a real risk-free rate plus an inflation premium:

(1 + nominal risk-free rate) = (1 + real risk-free rate)(1 + inflation premium).

In practice, however, the nominal rate is often approximated as the sum of the real risk-free rate plus an inflation premium:

Nominal risk-free rate = Real risk-free rate + inflation premium.

Many countries have short-term government debt whose interest rate can be considered to represent the nominal risk-free interest rate over that time horizon in that country. The French government issues BTFs, or negotiable fixed-rate discount Treasury bills (Bons du Trésor à taux fixe et à intérêts précomptés), with maturities of up to one year. The Japanese government issues a short-term Treasury bill with maturities of 6 and 12 months. The interest rate on a 90-day US T-bill, for example, represents the nominal risk-free interest rate for the United States over the next three

months. Typically, interest rates are quoted in annual terms, so the interest rate on a 90-day government debt security quoted at 3 percent is the annualized rate and not the actual interest rate earned over the 90-day period.

Whether the interest rate we use is a required rate of return, or a discount rate, or an opportunity cost, the rate encompasses the real risk-free rate and a set of risk premia that depend on the characteristics of the cash flows. The foundational set of premia consist of inflation, default risk, liquidity risk, and maturity risk. All these premia vary over time and continuously change, as does the real risk-free rate. Consequently, all interest rates fluctuate, but how much they change depends on various economic fundamentals—and on the expectation of how these various economic fundamentals can change in the future.

EXAMPLE 1

Determining Interest Rates

Exhibit 1 presents selected information for five debt securities. All five investments promise only a single payment at maturity. Assume that premiums relating to inflation, liquidity, and default risk are constant across all time horizons.

Exhibit 1: Investments Alternatives and Their Characteristics

Investment	Maturity (in years)	Liquidity	Default Risk	Interest Rate (%)
1	2	High	Low	2.0
2	2	Low	Low	2.5
3	7	Low	Low	r_3
4	8	High	Low	4.0
5	8	Low	High	6.5

Based on the information in Exhibit 1, address the following:

1. Explain the difference between the interest rates offered by Investment 1 and Investment 2.

   Solution:

   Investment 2 is identical to Investment 1 except that Investment 2 has low liquidity. The difference between the interest rate on Investment 2 and Investment 1 is 0.5 percent. This difference in the two interest rates represents a liquidity premium, which represents compensation for the lower liquidity of Investment 2 (the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly).

2. Estimate the default risk premium affecting all securities.

   Solution:

   To estimate the default risk premium, identify two investments that have the same maturity but different levels of default risk. Investments 4 and 5 both have a maturity of eight years but different levels of default risk. Investment 5, however, has low liquidity and thus bears a liquidity premium relative to Investment 4. From Part A, we know the liquidity premium is 0.5 percent. The difference between the interest rates offered by Investments 5 and 4 is 2.5 percent (6.5% – 4.0%), of which 0.5 percent is a liquidity premium. This

implies that 2.0 percent (2.5% – 0.5%) must represent a default risk premium reflecting Investment 5's relatively higher default risk.

3. Calculate upper and lower limits for the unknown interest rate for Investment 3, r_3.

   Solution:

   Investment 3 has liquidity risk and default risk comparable to Investment 2, but with its longer time to maturity, Investment 3 should have a higher maturity premium and offer a higher interest rate than Investment 2. Therefore, the interest rate on Investment 3, r_3, should thus be above 2.5 percent (the interest rate on Investment 2).

   If the liquidity of Investment 3 was high, Investment 3 would match Investment 4 except for Investment 3's shorter maturity. We would then conclude that Investment 3's interest rate should be less than the interest rate offered by Investment 4, which is 4 percent. In contrast to Investment 4, however, Investment 3 has low liquidity. It is possible that the interest rate on Investment 3 exceeds that of Investment 4 despite Investment 3's shorter maturity, depending on the relative size of the liquidity and maturity premiums. However, we would expect r_3 to be less than 4.5 percent, the expected interest rate on Investment 4 if it had low liquidity (4% + 0.5%, the liquidity premium). Thus, we should expect in the interest rate offered by Investment 3 to be between 2.5 percent and 4.5 percent.

3 RATES OF RETURN

☐ calculate and interpret different approaches to return measurement over time and describe their appropriate uses

Financial assets are frequently defined in terms of their return and risk characteristics. Comparison along these two dimensions simplifies the process of building a portfolio from among all available assets. In this lesson, we will compute, evaluate, and compare various measures of return.

Financial assets normally generate two types of return for investors. First, they may provide periodic income through cash dividends or interest payments. Second, the price of a financial asset can increase or decrease, leading to a capital gain or loss.

Some financial assets provide return through only one of these mechanisms. For example, investors in non-dividend-paying stocks obtain their return from price movement only. Other assets only generate periodic income. For example, defined benefit pension plans and retirement annuities make income payments over the life of a beneficiary.

Holding Period Return

Returns can be measured over a single period or over multiple periods. Single-period returns are straightforward because there is only one way to calculate them. Multiple-period returns, however, can be calculated in various ways and it is important to be aware of these differences to avoid confusion.

A **holding period return**, R, is the return earned from holding an asset for a single specified period of time. The period may be one day, one week, one month, five years, or any specified period. If the asset (e.g., bond, stock) is purchased today, time ($t = 0$), at a price of 100 and sold later, say at time ($t = 1$), at a price of 105 with no dividends or other income, then the holding period return is 5 percent [(105 – 100)/100)]. If the asset also pays income of two units at time ($t = 1$), then the total return is 7 percent. This return can be generalized and shown as a mathematical expression in which P is the price and I is the income, as follows:

$$R = \frac{(P_1 - P_0) + I_1}{P_0}, \tag{1}$$

where the subscript indicates the time of the price or income; ($t = 0$) is the beginning of the period; and ($t = 1$) is the end of the period. The following two observations are important.

- We computed a capital gain of 5 percent and an income yield of 2 percent in this example. For ease of illustration, we assumed that the income is paid at time $t = 1$. If the income was received before $t = 1$, our holding period return may have been higher if we had reinvested the income for the remainder of the period.
- Return can be expressed in decimals (0.07), fractions (7/100), or as a percent (7 percent). They are all equivalent.

A holding period return can be computed for a period longer than one year. For example, an analyst may need to compute a three-year holding period return from three annual returns. In that case, the three-year holding period return is computed by compounding the three annual returns:

$$R = [(1 + R_1) \times (1 + R_2) \times (1 + R_3)] - 1,$$

where R_1, R_2, and R_3 are the three annual returns.

Arithmetic or Mean Return

Most holding period returns are reported as daily, monthly, or annual returns. When assets have returns for multiple holding periods, it is necessary to normalize returns to a common period for ease of comparison and understanding. There are different methods for aggregating returns across several holding periods. The remainder of this section presents various ways of computing average returns and discusses their applicability.

The simplest way to compute a summary measure for returns across multiple periods is to take a simple arithmetic average of the holding period returns. Thus, three annual returns of –50 percent, 35 percent, and 27 percent will give us an average of 4 percent per year = $\left(\frac{-50\% + 35\% + 27\%}{3}\right)$. The arithmetic average return is easy to compute and has known statistical properties.

In general, the arithmetic or mean return is denoted by $\overline{R}_i$ and given by the following equation for asset i, where R_{it} is the return in period t and T is the total number of periods:

$$\overline{R}_i = \frac{R_{i1} + R_{i2} + \ldots + R_{i,T-1} + R_{iT}}{T} = \frac{1}{T}\sum_{t=1}^{T} R_{it}. \tag{2}$$

Geometric Mean Return

The arithmetic mean return assumes that the amount invested at the beginning of each period is the same. In an investment portfolio, however, even if there are no cash flows into or out of the portfolio the base amount changes each year. The previous year's earnings must be added to the beginning value of the subsequent year's investment—these earnings will be "compounded" by the returns earned in that subsequent year. We can use the geometric mean return to account for the compounding of returns.

A geometric mean return provides a more accurate representation of the growth in portfolio value over a given time period than the arithmetic mean return. In general, the geometric mean return is denoted by $\overline{R}_{Gi}$ and given by the following equation for asset i:

$$\overline{R}_{Gi} = \sqrt[T]{(1+R_{i1})\times(1+R_{i2})\times\ldots\times\left(1+R_{i,T-1}\right)\times\left(1+R_{iT}\right)} - 1 \qquad (3)$$

$$= \sqrt[T]{\prod_{t=1}^{T}(1+R_t)} - 1\ ,$$

where R_{it} is the return in period t and T is the total number of periods.

In the example in the previous section, we calculated the arithmetic mean to be 4.00 percent. Using Equation 3, we can calculate the geometric mean return from the same three annual returns:

$$\overline{R}_{Gi} = \sqrt[3]{(1-0.50)\times(1+0.35)\times(1+0.27)} - 1 = -0.0500.$$

Exhibit 2 shows the actual return for each year and the balance at the end of each year using actual returns.

Exhibit 2: Portfolio Value and Performance

	Actual Return for the Year (%)	Year-End Amount	Year-End Amount Using Arithmetic Return of 4%	Year-End Amount Using Geometric Return of –5%
Year 0		EUR1.0000	EUR1.0000	EUR1.0000
Year 1	–50	0.5000	1.0400	0.9500
Year 2	35	0.6750	1.0816	0.9025
Year 3	27	0.8573	1.1249	0.8574

Beginning with an initial investment of EUR1.0000, we will have a balance of EUR0.8574 at the end of the three-year period as shown in the fourth column of Exhibit 2. Note that we compounded the returns because, unless otherwise stated, we earn a return on the balance as of the end of the prior year. That is, we will receive a return of 35 percent in the second year on the balance at the end of the first year, which is only EUR0.5000, not the initial balance of EUR1.0000. Let us compare the balance at the end of the three-year period computed using geometric returns with the balance we would calculate using the 4 percent annual arithmetic mean return from our earlier example. The ending value using the arithmetic mean return is EUR1.1249 ($=1.0000\times1.04^3$). This is much larger than the actual balance at the end of Year 3 of EUR0.8573.

In general, the arithmetic return is biased upward unless each of the underlying holding period returns are equal. The bias in arithmetic mean returns is particularly severe if holding period returns are a mix of both positive and negative returns, as in this example.

We will now look at three examples that calculate holding period returns over different time horizons.

EXAMPLE 2

Holding Period Return

1. An investor purchased 100 shares of a stock for USD34.50 per share at the beginning of the quarter. If the investor sold all of the shares for USD30.50 per share after receiving a USD51.55 dividend payment at the end of the quarter, the investor's holding period return is *closest* to:

   A. –13.0 percent.

   B. –11.6 percent.

   C. –10.1 percent.

   Solution:

   C is correct. Applying Equation 2, the holding period return is –10.1 percent, calculated as follows:
   $R = (3{,}050 - 3{,}450 + 51.55)/3{,}450 = -10.1\%$.
   The holding period return comprised of a dividend yield of 1.49 percent (= 51.55/3,450) and a capital loss of –11.59 percent (= –400/3,450).

EXAMPLE 3

Holding Period Return

1. An analyst obtains the following annual rates of return for a mutual fund, which are presented in Exhibit 3.

Exhibit 3: Mutual Fund Performance, 20X8–20X0

Year	Return (%)
20X8	14
20X9	–10
20X0	–2

The fund's holding period return over the three-year period is *closest* to:

A. 0.18 percent.

B. 0.55 percent.

C. 0.67 percent.

Solution:

B is correct. The fund's three-year holding period return is 0.55 percent, calculated as follows:

$$R = [(1 + R_1) \times (1 + R_2) \times (1 + R_3)] - 1,$$

$$R = [(1 + 0.14)(1 - 0.10)(1 - 0.02)] - 1 = 0.0055 = 0.55\%.$$

EXAMPLE 4

Geometric Mean Return

1. An analyst observes the following annual rates of return for a hedge fund, which are presented in Exhibit 4.

Exhibit 4: Hedge Fund Performance, 20X8–20X0

Year	Return (%)
20X8	22
20X9	–25
20X0	11

The fund's geometric mean return over the three-year period is *closest* to:

A. 0.52 percent.

B. 1.02 percent.

C. 2.67 percent.

Solution:

A is correct. Applying Equation 4, the fund's geometric mean return over the three-year period is 0.52 percent, calculated as follows:

$$\overline{R}_G = [(1 + 0.22)(1 - 0.25)(1 + 0.11)]^{(1/3)} - 1 = 1.0157^{(1/3)} - 1 = 0.0052 = 0.52\%.$$

EXAMPLE 5

Geometric and Arithmetic Mean Returns

1. Consider the annual return data for the group of countries in Exhibit 5.

Exhibit 5: Annual Returns for Years 1 to 3 for Selected Countries' Stock Indexes

	52-Week Return (%)			Average 3-Year Return	
Index	Year 1	Year 2	Year 3	Arithmetic	Geometric
Country A	–15.6	–5.4	6.1	–4.97	–5.38
Country B	7.8	6.3	–1.5	4.20	4.12
Country C	5.3	1.2	3.5	3.33	3.32
Country D	–2.4	–3.1	6.2	0.23	0.15
Country E	–4.0	–3.0	3.0	–1.33	–1.38
Country F	5.4	5.2	–1.0	3.20	3.16
Country G	12.7	6.7	–1.2	6.07	5.91
Country H	3.5	4.3	3.4	3.73	3.73
Country I	6.2	7.8	3.2	5.73	5.72

Index	52-Week Return (%)			Average 3-Year Return	
	Year 1	Year 2	Year 3	Arithmetic	Geometric
Country J	8.1	4.1	−0.9	3.77	3.70
Country K	11.5	3.4	1.2	5.37	5.28

Calculate the arithmetic and geometric mean returns over the three years for the following three stock indexes: Country D, Country E, and Country F.

Solution:

The arithmetic mean returns are as follows:

	Annual Return (%)			Sum $\sum_{i=1}^{3} R_i$	Arithmetic Mean Return (%)
	Year 1	Year 2	Year 3		
Country D	−2.4	−3.1	6.2	0.7	0.233
Country E	−4.0	−3.0	3.0	−4.0	−1.333
Country F	5.4	5.2	−1.0	9.6	3.200

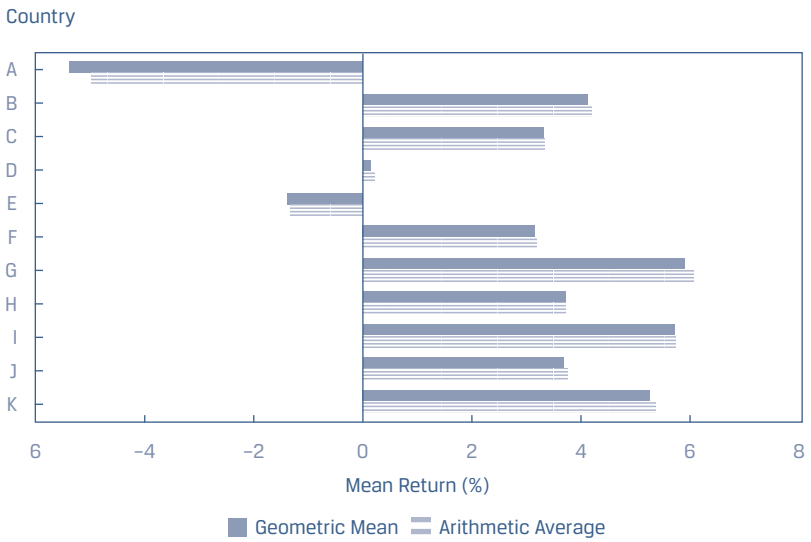
The geometric mean returns are as follows:

	1 + Return in Decimal Form $(1 + R_t)$			Product $\prod_{t}^{T}(1 + R_t)$	3rd Root $\left[\prod_{t}^{T}(1 + R_t)\right]^{1/3}$	Geometric mean return (%)
	Year 1	Year 2	Year 3			
Country D	0.976	0.969	1.062	1.00438	1.00146	0.146
Country E	0.960	0.970	1.030	0.95914	0.98619	−1.381
Country F	1.054	1.052	0.990	1.09772	1.03157	3.157

In Example 5, the geometric mean return is less than the arithmetic mean return for each country's index returns. In fact, the geometric mean is always less than or equal to the arithmetic mean with one exception: the two means will be equal is when there is no variability in the observations—that is, when all the observations in the series are the same.

In general, the difference between the arithmetic and geometric means increases with the variability within the sample; the more disperse the observations, the greater the difference between the arithmetic and geometric means. Casual inspection of the returns in Exhibit 5 and the associated graph of means in Exhibit 6 suggests a greater variability for Country A's index relative to the other indexes, and this is confirmed with the greater deviation of the geometric mean return (−5.38 percent) from the arithmetic mean return (−4.97 percent). How should the analyst interpret these results?

Exhibit 6: Arithmetic and Geometric Mean Returns for Country Stock Indexes, Years 1 to 3



The geometric mean return represents the growth rate or compound rate of return on an investment. One unit of currency invested in a fund tracking the Country B index at the beginning of Year 1 would have grown to (1.078)(1.063)(0.985) = 1.128725 units of currency, which is equal to 1 plus Country B's geometric mean return of 4.1189 percent compounded over three periods: $[1 + 0.041189]^3 = 1.128725$. This math confirms that the geometric mean is the compound rate of return. With its focus on the actual return of an investment over a multiple-period horizon, the geometric mean is of key interest to investors. The arithmetic mean return, focusing on average single-period performance, is also of interest. Both arithmetic and geometric means have a role to play in investment management, and both are often reported for return series.

For reporting historical returns, the geometric mean has considerable appeal because it is the rate of growth or return we would have to earn each year to match the actual, cumulative investment performance. Suppose we purchased a stock for EUR100 and two years later it was worth EUR100, with an intervening year at EUR200. The geometric mean of 0 percent is clearly the compound rate of growth during the two years, which we can confirm by compounding the returns: $[(1 + 1.00)(1 - 0.50)]^{1/2} - 1 = 0\%$. Specifically, the ending amount is the beginning amount times $(1 + R_G)^2$.

However, the arithmetic mean, which is [100% + −50%]/2 = 25% in the previous example, can distort our assessment of historical performance. As we noted, the arithmetic mean is always greater than or equal to the geometric mean. If we want to estimate the average return over a one-period horizon, we should use the arithmetic mean because the arithmetic mean is the average of one-period returns. If we want to estimate the average returns over more than one period, however, we should use the geometric mean of returns because the geometric mean captures how the total returns are linked over time.

The Harmonic Mean

The **harmonic mean**, X_H, is another measure of central tendency. The harmonic mean is appropriate in cases in which the variable is a rate or a ratio. The terminology "harmonic" arises from its use of a type of series involving reciprocals known as a harmonic series.

Harmonic Mean Formula. The harmonic mean of a set of observations X_1, X_2, …, X_n is:

$$\bar{X}_H = \frac{n}{\sum_{i=1}^{n}(1/X_i)}, \tag{4}$$

with $X_i > 0$ for $i = 1, 2, \ldots, n$.

The harmonic mean is the value obtained by summing the reciprocals of the observations,

$$\sum_{i=1}^{n}(1/X_i),$$

the terms of the form $1/X_i$, and then averaging their sum by dividing it by the number of observations, n, and, then finally, taking the reciprocal of that average,

$$\frac{n}{\sum_{i=1}^{n}(1/X_i)}.$$

The harmonic mean may be viewed as a special type of weighted mean in which an observation's weight is inversely proportional to its magnitude. For example, if there is a sample of observations of 1, 2, 3, 4, 5, 6, and 1,000, the harmonic mean is 2.8560. Compared to the arithmetic mean of 145.8571, we see the influence of the outlier (the 1,000) to be much less than in the case of the arithmetic mean. So, the harmonic mean is quite useful as a measure of central tendency in the presence of outliers.

The harmonic mean is used most often when the data consist of rates and ratios, such as P/Es. Suppose three peer companies have P/Es of 45, 15, and 15. The arithmetic mean is 25, but the harmonic mean, which gives less weight to the P/E of 45, is 19.3.

The harmonic mean is a relatively specialized concept of the mean that is appropriate for averaging ratios ("amount per unit") when the ratios are repeatedly applied to a fixed quantity to yield a variable number of units. The concept is best explained through an illustration. A well-known application arises in the investment strategy known as **cost averaging**, which involves the periodic investment of a fixed amount of money. In this application, the ratios we are averaging are prices per share at different purchase dates, and we are applying those prices to a constant amount of money to yield a variable number of shares. An illustration of the harmonic mean to cost averaging is provided in Example 6.

EXAMPLE 6

Cost Averaging and the Harmonic Mean

1. Suppose an investor invests EUR1,000 each month in a particular stock for $n = 2$ months. The share prices are EUR10 and EUR15 at the two purchase dates. What was the average price paid for the security?

   Solution:

   Purchase in the first month = EUR1,000/EUR10 = 100 shares.

   Purchase in the second month = EUR1,000/EUR15 = 66.67 shares.

   The investor purchased a total of 166.67 shares for EUR2,000, so the average price paid per share is EUR2,000/166.67 = EUR12.
   The average price paid is in fact the harmonic mean of the asset's prices at the purchase dates. Using Equation 5, the harmonic mean price is 2/[(1/10) + (1/15)] = EUR12. The value EUR12 is less than the arithmetic mean purchase price (EUR10 + EUR15)/2 = EUR12.5.

Because they use the same data but involve different progressions in their respective calculations, the arithmetic, geometric, and harmonic means are mathematically related to one another. We will not go into the proof of this relationship, but the basic result follows:

Arithmetic mean × Harmonic mean = (Geometric mean)^2.

Unless all the observations in a dataset are the same value, the harmonic mean is always less than the geometric mean, which, in turn, is always less than the arithmetic mean.

The harmonic mean only works for non-negative numbers, so when working with returns that are expressed as positive or negative percentages, we first convert the returns into a compounding format, assuming a reinvestment, as (1 + R), as was done in the geometric mean return calculation, and then calculate (1 + harmonic mean), and subtract 1 to arrive at the harmonic mean return.

EXAMPLE 7

Calculating the Arithmetic, Geometric, and Harmonic Means for P/Es

Each year in December, a securities analyst selects her 10 favorite stocks for the next year. Exhibit 7 presents the P/Es, the ratio of share price to projected earnings per share (EPS), for her top 10 stock picks for the next year.

Exhibit 7: Analyst's 10 Favorite Stocks for Next Year

Stock	P/E
Stock 1	22.29
Stock 2	15.54
Stock 3	9.38
Stock 4	15.12
Stock 5	10.72
Stock 6	14.57
Stock 7	7.20
Stock 8	7.97
Stock 9	10.34
Stock 10	8.35

1. Calculate the arithmetic mean P/E for these 10 stocks.

Solution:

The arithmetic mean is calculated as:

121.48/10 = 12.1480.

2. Calculate the harmonic mean P/E for the 10 stocks.

Solution:

The harmonic mean is calculated as:

$$\overline{X}_H = \frac{n}{\sum_{i=1}^{n}(1/X_i)},$$

$$\overline{X}_H = \frac{10}{\left(\frac{1}{22.29}\right) + \left(\frac{1}{15.54}\right) + \ldots + \left(\frac{1}{10.34}\right) + \left(\frac{1}{8.35}\right)},$$

$X_H = 10/0.9247 = 10.8142.$

In finance, the weighted harmonic mean is used when averaging rates and other multiples, such as the P/E ratio, because the harmonic mean gives equal weight to each data point, and reduces the influence of outliers. These calculations can be performed using Excel:

- To calculate the arithmetic mean or average return, the =AVERAGE(return1, return2, …) function can be used.
- To calculate the geometric mean return, the =GEOMEAN(return1, return2, …) function can be used.
- To calculate the harmonic mean return, the =HARMEAN(return1, return2, …) function can be used.

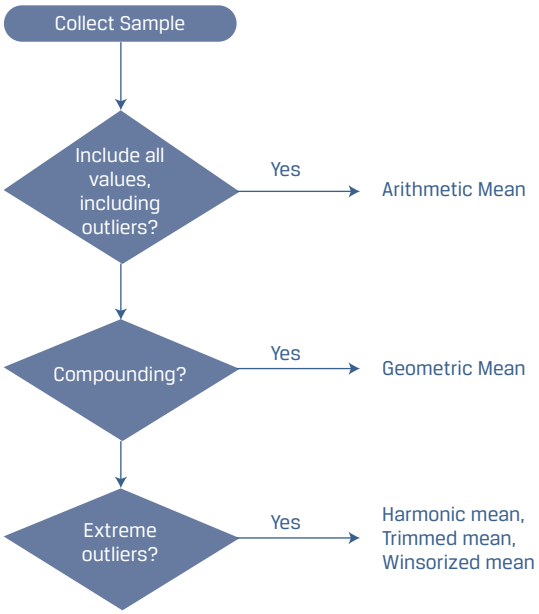
In addition to arithmetic, geometric, and harmonic means, two other types of means can be used. Both the trimmed and the winsorized means seek to minimize the impact of outliers in a dataset. Specifically, the **trimmed mean** removes a small defined percentage of the largest and smallest values from a dataset containing our observation before calculating the mean by averaging the remaining observations.

A winsorized mean replaces the extreme observations in a dataset to limit the effect of the outliers on the calculations. The **winsorized mean** is calculated after replacing extreme values at both ends with the values of their nearest observations, and then calculating the mean by averaging the remaining observations.

However, the key question is: Which mean to use in what circumstances? The choice of which mean to use depends on many factors, as we describe in Exhibit 8:

- Are there outliers that we want to include?
- Is the distribution symmetric?
- Is there compounding?
- Are there extreme outliers?

Exhibit 8: Deciding Which Measure to Use



QUESTION SET

A fund had the following returns over the past 10 years:

Exhibit 9: 10-Year Returns

Year	Return
1	4.5%
2	6.0%
3	1.5%
4	–2.0%
5	0.0%
6	4.5%
7	3.5%
8	2.5%
9	5.5%
10	4.0%

1. The arithmetic mean return over the 10 years is *closest* to:

   A. 2.97 percent.

   B. 3.00 percent.

   C. 3.33 percent.

   Solution:

   B is correct. The arithmetic mean return is calculated as follows:

$\overline{R}$ = 30.0%/10 = 3.0%.

2. The geometric mean return over the 10 years is *closest* to:

A. 2.94 percent.

B. 2.97 percent.

C. 3.00 percent.

Solution:

B is correct. The geometric mean return is calculated as follows:

$$\overline{R}_G = \sqrt[10]{(1 + 0.045) \times (1 + 0.06) \times \ldots \times (1 + 0.055) \times (1 + 0.04)} - 1$$

$$\overline{R}_G = \sqrt[10]{1.3402338} - 1 = 2.9717\%.$$

MONEY-WEIGHTED AND TIME-WEIGHTED RETURN

4

☐ compare the money-weighted and time-weighted rates of return and evaluate the performance of portfolios based on these measures

The arithmetic and geometric return computations do not account for the timing of cash flows into and out of a portfolio. For example, suppose an investor experiences the returns shown in Exhibit 2. Instead of only investing EUR1.0 at the start (Year 0) as was the case in Exhibit 2, suppose the investor had invested EUR10,000 at the start, EUR1,000 in Year 1, and EUR1,000 in Year 2. In that case, the return of –50 percent in Year 1 significantly hurts her given the relatively large investment at the start. Conversely, if she had invested only EUR100 at the start, the absolute effect of the –50 percent return on the total return is drastically reduced.

Calculating the Money Weighted Return

The **money-weighted return** accounts for the money invested and provides the investor with information on the actual return she earns on her investment. The money-weighted return and its calculation are similar to the internal rate of return and a bond's yield to maturity. Amounts invested are cash outflows from the investor's perspective and amounts returned or withdrawn by the investor, or the money that remains at the end of an investment cycle, is a cash inflow for the investor.

For example, assume that an investor invests EUR100 in a mutual fund at the beginning of the first year, adds another EUR950 at the beginning of the second year, and withdraws EUR350 at the end of the second year. The cash flows are presented in Exhibit 10.

Exhibit 10: Portfolio Balances across Three Years

Year	1	2	3
Balance from previous year	EUR0	EUR50	EUR1,000
New investment by the investor (cash inflow for the mutual fund) at the start of the year	100	950	0
Net balance at the beginning of year	100	1,000	1,000
Investment return for the year	–50%	35%	27%
Investment gain (loss)	–50	350	270
Withdrawal by the investor (cash outflow for the mutual fund) at the end of the year	0	–350	0
Balance at the end of year	EUR50	EUR1,000	EUR1,270

The **internal rate of return** is the discount rate at which the sum of present values of cash flows will equal zero. In general, the equation may be expressed as follows:

$$\sum_{t=0}^{T} \frac{CF_t}{(1+\text{IRR})^t} = 0, \tag{5}$$

where T is the number of periods, CF_t is the cash flow at time t, and IRR is the internal rate of return or the money-weighted rate of return.

A cash flow can be positive or negative; a positive cash flow is an inflow where money flows to the investor, whereas a negative cash flow is an outflow where money flows away from the investor. The cash flows are expressed as follows, where each cash inflow or outflow occurs at the end of each year. Thus, CF_0 refers to the cash flow at the end of Year 0 or beginning of Year 1, and CF_3 refers to the cash flow at end of Year 3 or beginning of Year 4. Because cash flows are being discounted to the present—that is, end of Year 0 or beginning of Year 1—the period of discounting CF_0 is zero.

$$\begin{aligned}
&CF_0 = -100\\
&CF_1 = -950\\
&CF_2 = +350\\
&CF_3 = +1,270\\
&\frac{CF_0}{(1+\text{IRR})^0} + \frac{CF_1}{(1+\text{IRR})^1} + \frac{CF_2}{(1+\text{IRR})^2} + \frac{CF_3}{(1+\text{IRR})^3}.\\
&= \frac{-100}{1} + \frac{-950}{(1+\text{IRR})^1} + \frac{+350}{(1+\text{IRR})^2} + \frac{+1270}{(1+\text{IRR})^3} = 0\\
&\text{IRR} = 26.11\%
\end{aligned}$$

The investor's internal rate of return, or the money-weighted rate of return, is 26.11 percent, which tells the investor what she earned on the actual euros invested for the entire period on an annualized basis. This return is much greater than the arithmetic and geometric mean returns because only a small amount was invested when the mutual fund's return was –50 percent.

All the above calculations can be performed using Excel using the =IRR(values) function, which results in an IRR of 26.11 percent.

Money-Weighted Return for a Dividend-Paying Stock

Next, we'll illustrate calculating the money-weighted return for a dividend paying stock. Consider an investment that covers a two-year horizon. At time $t = 0$, an investor buys one share at a price of USD200. At time $t = 1$, he purchases an additional share at a price of USD225. At the end of Year 2, $t = 2$, he sells both shares at a price of USD235. During both years, the stock pays a dividend of USD5 per share. The $t = 1$ dividend is not reinvested. Exhibit 11 outlines the total cash inflows and outflows for the investment.

Exhibit 11: Cash Flows for a Dividend-Paying Stock

Time	Outflows
0	USD200 to purchase the first share
1	USD225 to purchase the second share

Time	Inflows
1	USD5 dividend received from first share (and not reinvested)
2	USD10 dividend (USD5 per share × 2 shares) received
2	USD470 received from selling two shares at USD235 per share

To solve for the money-weighted return, the first step is to group net cash flows by time. For this example, we have –USD200 for the $t = 0$ net cash flow, –USD220 = –USD225 + USD5 for the $t = 1$ net cash flow, and USD480 for the $t = 2$ net cash flow. After entering these cash flows, we use the spreadsheet's (such as Excel) or calculator's IRR function to find that the money-weighted rate of return is 9.39 percent.

$$CF_0 = -200$$
$$CF_1 = -220$$
$$CF_2 = +480$$
$$\frac{CF_0}{(1+\text{IRR})^0} + \frac{CF_1}{(1+\text{IRR})^1} + \frac{CF_2}{(1+\text{IRR})^2} .$$
$$= \frac{-200}{1} + \frac{-220}{(1+\text{IRR})^1} + \frac{480}{(1+\text{IRR})^2} = 0$$
$$\text{IRR} = 9.39\%$$

All these calculations can be performed using Excel using the =IRR(values) function, which results in an IRR of 9.39 percent.

Now we take a closer look at what has happened to the portfolio during each of the two years.

In the first year, the portfolio generated a one-period holding period return of (USD5 + USD225 – USD200)/USD200 = 15%. At the beginning of the second year, the amount invested is USD450, calculated as USD225 (per share price of stock) × 2 shares, because the USD5 dividend was spent rather than reinvested.

At the end of the second year, the proceeds from the liquidation of the portfolio are USD470 plus USD10 in dividends (as outlined in Exhibit 11). So, in the second year the portfolio produced a holding period return of (USD10 + USD470 – USD450)/USD450 = 6.67%. The mean holding period return was (15% + 6.67%)/2 = 10.84%.

The money-weighted rate of return, which we calculated as 9.39 percent, puts a greater weight on the second year's relatively poor performance (6.67 percent) than the first year's relatively good performance (15 percent), as more money was invested in the second year than in the first. That is the sense in which returns in this method of calculating performance are "money weighted."

Although the money-weighted return is an accurate measure of what the investor earned on the money invested, it is limited in its applicability to other situations. For example, it does not allow for a return comparison between different individuals or different investment opportunities. Importantly, two investors in the *same* mutual fund or with the same portfolio of underlying investments may have different money-weighted returns because they invested different amounts in different years.

EXAMPLE 8

Computation of Returns

Ulli Lohrmann and his wife, Suzanne Lohrmann, are planning for retirement and want to compare the past performance of a few mutual funds they are considering for investment. They believe that a comparison over a five-year period would be appropriate. They gather information on a fund they are considering, the Rhein Valley Superior Fund, which is presented in Exhibit 12.

Exhibit 12: Rhein Valley Superior Fund Performance

Year	Assets under Management at the Beginning of Year (euros)	Annual Return (%)
1	30 million	15
2	45 million	–5
3	20 million	10
4	25 million	15
5	35 million	3

The Lohrmanns are interested in aggregating this information for ease of comparison with other funds.

Exhibit 13: Rhein Valley Superior Fund Annual Returns and Investments (euro millions)

Year	1	2	3	4	5
Balance from previous year	0	34.50	42.75	22.00	28.75
New investment by the investor (cash inflow for the Rhein fund)	30.00	10.50	0	3.00	6.25
Withdrawal by the investor (cash outflow for the Rhein fund)	0	0	–22.75	0	0
Net balance at the beginning of year	30.00	45.00	20.00	25.00	35.00
Investment return for the year	15%	–5%	10%	15%	3%
Investment gain (loss)	4.50	–2.25	2.00	3.75	1.05
Balance at the end of year	34.50	42.75	22.00	28.75	36.05

1. Compute the fund's holding period return for the five-year period.

Solution:

The five-year holding period return is calculated as:

$$R = (1 + R_1)(1 + R_2)(1 + R_3)(1 + R_4)(1 + R_5) - 1$$

$$R = (1.15)(0.95)(1.10)(1.15)(1.03) - 1 =$$

$$R = 0.4235 = 42.35\%.$$

2. Compute the fund's arithmetic mean annual return.

Solution:

The arithmetic mean annual return is calculated as:

$$\overline{R}_i = \frac{15\% - 5\% + 10\% + 15\% + 3\%}{5} = 7.60\%.$$

3. Compute the fund's geometric mean annual return. How does it compare with the arithmetic mean annual return?

Solution:

The geometric mean annual return can be computed as:

$$\overline{R}_{Gi} = \sqrt[5]{1.15 \times 0.95 \times 1.10 \times 1.15 \times 1.03} - 1,$$

$$\overline{R}_{Gi} = \sqrt[5]{1.4235} - 1 = 0.0732 = 7.32\%.$$

Thus, the geometric mean annual return is 7.32 percent, which is slightly less than the arithmetic mean return of 7.60 percent.

4. The Lohrmanns want to earn a minimum annual return of 5 percent. The annual returns and investment amounts are presented in Exhibit 13. Is the money-weighted annual return greater than 5 percent?

Solution:

To calculate the money-weighted rate of return, tabulate the annual returns and investment amounts to determine the cash flows, as shown in Exhibit 13:

$CF_0 = –30.00$, $CF_1 = –10.50$, $CF_2 = +22.75$, $CF_3 = –3.00$, $CF_4 = –6.25$, $CF_5 = +36.05$.

We can use the given 5 percent return to see whether or not the present value of the net cash flows is positive. If it is positive, then the money-weighted rate of return is greater than 5 percent, because a 5 percent discount rate could not reduce the present value to zero.

$$\frac{-30.00}{(1.05)^0} + \frac{-10.50}{(1.05)^1} + \frac{22.75}{(1.05)^2} + \frac{-3.00}{(1.05)^3} + \frac{-6.25}{(1.05)^4} + \frac{36.05}{(1.05)^5} = 1.1476.$$

Because the value is positive, the money-weighted rate of return is greater than 5 percent. The exact money-weighted rate of return (found by setting the above equation equal to zero) is 5.86 percent.
These calculations can be performed using Excel using the =IRR(cash flows) function, which results in an IRR of 5.86 percent.

Time-Weighted Returns

An investment measure that is not sensitive to the additions and withdrawals of funds is the time-weighted rate of return. The **time-weighted rate of return** measures the compound rate of growth of USD1 initially invested in the portfolio over a stated measurement period. For the evaluation of portfolios of publicly traded securities, the time-weighted rate of return is the preferred performance measure as it neutralizes the effect of cash withdrawals or additions to the portfolio, which are generally outside of the control of the portfolio manager.

Computing Time-Weighted Returns

To compute an exact time-weighted rate of return on a portfolio, take the following three steps:

1. Price the portfolio immediately prior to any significant addition or withdrawal of funds. Break the overall evaluation period into subperiods based on the dates of cash inflows and outflows.
2. Calculate the holding period return on the portfolio for each subperiod.
3. Link or compound holding period returns to obtain an annual rate of return for the year (the time-weighted rate of return for the year). If the investment is for more than one year, take the geometric mean of the annual returns to obtain the time-weighted rate of return over that measurement period.

Let us return to our dividend stock money-weighted example in the section, "Money-Weighted Return for a Dividend-Paying Stock" and calculate the time-weighted rate of return for that investor's portfolio based on the information included in Exhibit 11. In that example, we computed the holding period returns on the portfolio, Step 2 in the procedure for finding the time-weighted rate of return. Given that the portfolio earned returns of 15 percent during the first year and 6.67 percent during the second year, what is the portfolio's time-weighted rate of return over an evaluation period of two years?

We find this time-weighted return by taking the geometric mean of the two holding period returns, Step 3 in the previous procedure. The calculation of the geometric mean exactly mirrors the calculation of a compound growth rate. Here, we take the product of 1 plus the holding period return for each period to find the terminal value at $t = 2$ of USD1 invested at $t = 0$. We then take the square root of this product and subtract 1 to get the geometric mean return. We interpret the result as the annual compound growth rate of USD1 invested in the portfolio at $t = 0$. Thus, we have:

$$(1 + \text{Time-weighted return})^2 = (1.15)(1.0667)$$

$$\text{Time-weighted return} = \sqrt{(1.15)(1.0667)} - 1 = 10.76\%.$$

The time-weighted return on the portfolio was 10.76 percent, compared with the money-weighted return of 9.39 percent, which gave larger weight to the second year's return. We can see why investment managers find time-weighted returns more meaningful. If a client gives an investment manager more funds to invest at an unfavorable time, the manager's money-weighted rate of return will tend to be depressed. If a client adds funds at a favorable time, the money-weighted return will tend to be elevated. The time-weighted rate of return removes these effects.

In defining the steps to calculate an exact time-weighted rate of return, we said that the portfolio should be valued immediately prior to any significant addition or withdrawal of funds. With the amount of cash flow activity in many portfolios, this task can be costly. We can often obtain a reasonable approximation of the time-weighted rate of return by valuing the portfolio at frequent, regular intervals, particularly if additions and withdrawals are unrelated to market movements.

The more frequent the valuation, the more accurate the approximation. Daily valuation is commonplace. Suppose that a portfolio is valued daily over the course of a year. To compute the time-weighted return for the year, we first compute each day's holding period return. We compute 365 such daily returns, denoted R_1, R_2, …, R_{365}. We obtain the annual return for the year by linking the daily holding period returns in the following way: $(1 + R_1) \times (1 + R_2) \times \ldots \times (1 + R_{365}) - 1$. If withdrawals and additions to the portfolio happen only at day's end, this annual return is a precise time-weighted rate of return for the year. Otherwise, it is an approximate time-weighted return for the year.

If we have several years of data, we can calculate a time-weighted return for each year individually, as above. If R_i is the time-weighted return for year i, we calculate an annualized time-weighted return as the geometric mean of N annual returns, as follows:

$$R_{TW} = [(1 + R_1) \times (1 + R_2) \times \ldots \times (1 + R_N)]^{1/N} - 1. \quad (6)$$

Example 9 illustrates the calculation of the time-weighted rate of return.

EXAMPLE 9

Time-Weighted Rate of Return

Strubeck Corporation sponsors a pension plan for its employees. It manages part of the equity portfolio in-house and delegates management of the balance to Super Trust Company. As chief investment officer of Strubeck, you want to review the performance of the in-house and Super Trust portfolios over the last four quarters. You have arranged for outflows and inflows to the portfolios to be made at the very beginning of the quarter. Exhibit 14 summarizes the inflows and outflows as well as the two portfolios' valuations. In Exhibit 11, the ending value is the portfolio's value just prior to the cash inflow or outflow at the beginning of the quarter. The amount invested is the amount each portfolio manager is responsible for investing.

Exhibit 14: Cash Flows for the In-House Strubeck Account and the Super Trust Account (US dollars)

	Quarter			
	1	2	3	4
Panel A: In-House Account				
Beginning value	4,000,000	6,000,000	5,775,000	6,720,000
Beginning of period inflow (outflow)	1,000,000	(500,000)	225,000	(600,000)
Amount invested	5,000,000	5,500,000	6,000,000	6,120,000
Ending value	6,000,000	5,775,000	6,720,000	5,508,000
Panel B: Super Trust Account				
Beginning value	10,000,000	13,200,000	12,240,000	5,659,200
Beginning of period inflow (outflow)	2,000,000	(1,200,000)	(7,000,000)	(400,000)
Amount invested	12,000,000	12,000,000	5,240,000	5,259,200
Ending value	13,200,000	12,240,000	5,659,200	5,469,568

1. Calculate the time-weighted rate of return for the in-house account.

Solution:

To calculate the time-weighted rate of return for the in-house account, we compute the quarterly holding period returns for the account and link them into an annual return. The in-house account's time-weighted rate of return is 27.01 percent, calculated as follows:

1Q HPR: $r_1 = (\text{USD}6{,}000{,}000 - \text{USD}5{,}000{,}000) / \text{USD}5{,}000{,}000 = 0.20$
2Q HPR: $r_2 = (\text{USD}5{,}775{,}000 - \text{USD}5{,}500{,}000) / \text{USD}5{,}500{,}000 = 0.05$
3Q HPR: $r_3 = (\text{USD}6{,}720{,}000 - \text{USD}6{,}000{,}000) / \text{USD}6{,}000{,}000 = 0.12$
4Q HPR: $r_4 = (\text{USD}5{,}508{,}000 - \text{USD}6{,}120{,}000) / \text{USD}6{,}120{,}000 = -0.10$

$$R_{\text{TW}} = (1 + r_1)(1 + r_2)(1 + r_3)(1 + r_4) - 1,$$

$$R_{\text{TW}} = (1.20)(1.05)(1.12)(0.90) - 1 = 0.2701 \text{ or } 27.01\%.$$

2. Calculate the time-weighted rate of return for the Super Trust account.

Solution:

The account managed by Super Trust has a time-weighted rate of return of 26.02 percent, calculated as follows:

1Q HPR: $r_1 = (USD13{,}200{,}000 - USD12{,}000{,}000) / USD12{,}000{,}000 = 0.10$
2Q HPR: $r_2 = (USD12{,}240{,}000 - USD12{,}000{,}000) / USD12{,}000{,}000 = 0.02$
3Q HPR: $r_3 = (USD5{,}659{,}200 - USD5{,}240{,}000) / USD5{,}240{,}000 = 0.08$
4Q HPR: $r_4 = (USD5{,}469{,}568 - USD5{,}259{,}200) / USD5{,}259{,}200 = 0.04$

$$R_{\text{TW}} = (1 + r_1)(1 + r_2)(1 + r_3)(1 + r_4) - 1,$$

$$R_{\text{TW}} = (1.10)(1.02)(1.08)(1.04) - 1 = 0.2602 \text{ or } 26.02\%.$$

The in-house portfolio's time-weighted rate of return is higher than the Super Trust portfolio's by 99 basis points. Note that 27.01 percent and 26.02 percent might be rounded to 27 percent and 26 percent, respectively. The impact of the rounding the performance difference (100 bp vs. 99 bp) may seem as trivial, yet it's impact on a large portfolio may be substantive.

Having worked through this exercise, we are ready to look at a more detailed case.

EXAMPLE 10

Time-Weighted and Money-Weighted Rates of Return Side by Side

Your task is to compute the investment performance of the Walbright Fund for the most recent year. The facts are as follows:

- On 1 January, the Walbright Fund had a market value of USD100 million.
- During the period 1 January to 30 April, the stocks in the fund generated a capital gain of USD10 million.
- On 1 May, the stocks in the fund paid a total dividend of USD2 million. All dividends were reinvested in additional shares.
- Because the fund's performance had been exceptional, institutions invested an additional USD20 million in Walbright on 1 May, raising assets under management to USD132 million (USD100 + USD10 + USD2 + USD20).
- On 31 December, Walbright received total dividends of USD2.64 million. The fund's market value on 31 December, not including the USD2.64 million in dividends, was USD140 million.
- The fund made no other interim cash payments during the year.

1. Compute the Walbright Fund's time-weighted rate of return.

Solution:

Because interim cash flows were made on 1 May, we must compute two interim total returns and then link them to obtain an annual return. Exhibit 15 lists the relevant market values on 1 January, 1 May, and 31 December, as well as the associated interim four-month (1 January to 1 May) and eight-month (1 May to 31 December) holding period returns.

Exhibit 15: Cash Flows for the Walbright Fund

1 January	Beginning portfolio value = USD100 million
1 May	Dividends received before additional investment = USD2 million
	Ending portfolio value = USD110 million
	Four-month holding period return: $R = \frac{USD2 + USD10}{USD100} = 12\%$
	New investment = USD20 million
	Beginning market value for last two-thirds of the year = USD132 million
31 December	Dividends received = USD2.64 million
	Ending portfolio value = USD140 million
	Eight-month holding period return: $R = \frac{USD2.64 + USD140 - USD132}{USD132} = 8.06\%$

Now we must geometrically link the four- and eight-month holding period returns to compute an annual return. We compute the time-weighted return as follows:

$$R_{TW} = 1.12 \times 1.0806 - 1 = 0.2103.$$

In this instance, we compute a time-weighted rate of return of 21.03 percent for one year. The four-month and eight-month intervals combine to equal one year. (Note: Taking the square root of the product 1.12 × 1.0806 would be appropriate only if 1.12 and 1.0806 each applied to one full year.)

2. Compute the Walbright Fund's money-weighted rate of return.

Solution:

To calculate the money-weighted return, we need to find the discount rate that sets the sum of the present value of cash inflows and outflows equal to zero. The initial market value of the fund and all additions to it are treated as cash outflows. (Think of them as expenditures.) Withdrawals, receipts, and the ending market value of the fund are counted as inflows. (The ending market value is the amount investors receive on liquidating the fund.) Because interim cash flows have occurred at four-month intervals, we must solve for the four-month internal rate of return. Exhibit 15 details the cash flows and their timing.

$$CF_0 = -100$$

$$CF_1 = -20$$

$$CF_2 = 0$$

$CF_3 = 142.64$

CF_0 refers to the initial investment of USD100 million made at the beginning of the first four-month interval on 1 January. CF_1 refers to the cash flows made at end of the first four-month interval or the beginning of the second four-month interval on 1 May. Those cash flows include a cash inflow of USD2 million for the dividend received and cash outflows of USD22 million for the dividend reinvested and additional investment, respectively. The second four-month interval had no cash flow so CF_2 is equal to zero. CF_3 refers to the cash inflows at the end of the third four-month interval. Those cash inflows include a USD2.64 million dividend received and the fund's terminal market value of USD140 million.
Using a spreadsheet or IRR-enabled calculator, we use −100, −20, 0, and USD142.64 for the $t = 0$, $t = 1$, $t = 2$, and $t = 3$ net cash flows, respectively. Using either tool, we get a four-month IRR of 6.28 percent.

$$\frac{CF_0}{(1+IRR)^0}+\frac{CF_1}{(1+IRR)^1}+\frac{CF_2}{(1+IRR)^2}+\frac{CF_3}{(1+IRR)^3}=0$$
$$\frac{-100}{1}+\frac{-20}{(1+IRR)^1}+\frac{0}{(1+IRR)^2}+\frac{142.64}{(1+IRR)^3}=0 \quad .$$
$$IRR = 6.28\%$$

The quick way to annualize this four-month return is to multiply it by 3. A more accurate way is to compute it on a compounded basis as: $(1.0628)^3 - 1 = 0.2005$ or 20.05 percent.
These calculations can also be performed using Excel using the =IRR(cash flows) function, which results in an IRR of 6.28 percent.

3. Interpret the differences between the Fund's time-weighted and money-weighted rates of return.

Solution:

In this example, the time-weighted return (21.03 percent) is greater than the money-weighted return (20.05 percent). The Walbright Fund's performance was relatively poorer during the eight-month period, when the fund had more money invested, than the overall performance. This fact is reflected in a lower money-weighted rate of return compared with the time-weighted rate of return, as the money-weighted return is sensitive to the timing and amount of withdrawals and additions to the portfolio.

The accurate measurement of portfolio returns is important to the process of evaluating portfolio managers. In addition to considering returns, however, analysts must also weigh risk. When we worked through Example 9, we stopped short of suggesting that in-house management was superior to Super Trust because it earned a higher time-weighted rate of return. A judgment as to whether performance was "better" or "worse" must include the risk dimension, which will be covered later in your study materials.

5 ANNUALIZED RETURN

☐ calculate and interpret annualized return measures and continuously compounded returns, and describe their appropriate uses

The period during which a return is earned or computed can vary and often we have to annualize a return that was calculated for a period that is shorter (or longer) than one year. You might buy a short-term treasury bill with a maturity of three months, or you might take a position in a futures contract that expires at the end of the next quarter. How can we compare these returns?

In many cases, it is most convenient to annualize all available returns to facilitate comparison. Thus, daily, weekly, monthly, and quarterly returns are converted to annualized returns. Many formulas used for calculating certain values or prices also require all returns and periods to be expressed as annualized rates of return. For example, the most common version of the Black–Scholes option-pricing model requires annualized returns and periods to be in years.

Non-annual Compounding

Recall that interest may be paid semiannually, quarterly, monthly, or even daily. To handle interest payments made more than once a year, we can modify the present value formula as follows. Here, R_s is the quoted interest rate and equals the periodic interest rate multiplied by the number of compounding periods in each year. In general, with more than one compounding period in a year, we can express the formula for present value as follows:

$$\text{PV} = \text{FV}_N\left(1 + \frac{R_s}{m}\right)^{-mN}, \tag{7}$$

where

m = number of compounding periods per year,

R_S = quoted annual interest rate, and

N = number of years.

The formula in Equation 7 is quite similar to the simple present value formula. As we have already noted, present value and future value factors are reciprocals. Changing the frequency of compounding does not alter this result. The only difference is the use of the periodic interest rate and the corresponding number of compounding periods.

The following example presents an application of monthly compounding.

EXAMPLE 11

The Present Value of a Lump Sum with Monthly Compounding

The manager of a Canadian pension fund knows that the fund must make a lump-sum payment of CAD5 million 10 years from today. She wants to invest an amount today in a guaranteed investment contract (GIC) so that it will grow to the required amount. The current interest rate on GICs is 6 percent a year, compounded monthly.

1. How much should she invest today in the GIC?

   Solution:

   By applying Equation 7, the required present value is calculated as follow:

$$\begin{aligned}
&\text{FV}_N = \text{CAD } 5{,}000{,}000\\
&R_s = 6\% = 0.06\\
&m = 12\\
&R_s/m = 0.06/12 = 0.005\\
&N = 10\\
&mN = 12(10) = 120\\
&\text{PV} = \text{FV}_N\left(1+\frac{R_s}{m}\right)^{-mN}\\
&= CAD\ 5{,}000{,}000\,(1.005)^{-120}\\
&= CAD\ 5{,}000{,}000(0.549633)\\
&= CAD\ 2{,}748{,}163.67
\end{aligned}$$

In applying Equation 7, we use the periodic rate (in this case, the monthly rate) and the appropriate number of periods with monthly compounding (in this case, 10 years of monthly compounding, or 120 months).

Annualizing Returns

To annualize any return for a period shorter than one year, the return for the period must be compounded by the number of periods in a year. A monthly return is compounded 12 times, a weekly return is compounded 52 times, and a quarterly return is compounded 4 times. Daily returns are normally compounded 365 times. For an uncommon number of days, we compound by the ratio of 365 to the number of days.

If the weekly return is 0.2 percent, then the compound annual return is 10.95 percent (there are 52 weeks in a year):

$$\begin{aligned}
&R_{annual} = \left(1+R_{weekly}\right)^{52} - 1 = (1+0.2\%)^{52} - 1\\
&= (1.002)^{52} - 1 = 0.1095 = 10.95\%
\end{aligned}$$

If the return for 15 days is 0.4 percent, then the annualized return is 10.20 percent, assuming 365 days in a year:

$$\begin{aligned}
&R_{annual} = \left(1+R_{15}\right)^{365/15} - 1 = (1+0.4\%)^{365/15} - 1\\
&= (1.004)^{365/15} - 1 = 0.1020 = 10.20\%
\end{aligned}$$

A general equation to annualize returns is given, where c is the number of periods in a year. For a quarter, $c = 4$ and for a month, $c = 12$:

$$R_{annual} = \left(1+R_{period}\right)^{c} - 1. \tag{8}$$

How can we annualize a return when the holding period return is more than one year? For example, how do we annualize an 18-month holding period return? Because one year contains two-thirds of 18-month periods, $c = 2/3$ in the above equation. For example, an 18-month return of 20 percent can be annualized as follows:

$$R_{annual} = \left(1+R_{18month}\right)^{2/3} - 1 = (1+0.20)^{2/3} - 1 = 0.1292 = 12.92\%.$$

Similar expressions can be constructed when quarterly or weekly returns are needed for comparison instead of annual returns. In such cases, c is equal to the number of holding periods in a quarter or in a week. For example, assume that you want to convert daily returns to weekly returns or annual returns to weekly returns for comparison between weekly returns. To convert daily returns to weekly returns, $c = 5$, assume that there are five trading days in a week. However, daily return calculations can be annualized differently. For example, five can be used for trading-day-based calculations, giving approximately 250 trading days a year; seven can be used on calendar-day-based calculations. Specific methods used conform to specific business practices, market

conventions, and standards. To convert annual returns to weekly returns, $c = 1/52$. The expressions for annual returns can then be rewritten as expressions for weekly returns as follows:

$$R_{weekly} = \left(1 + R_{daily}\right)^{5} - 1;\ R_{weekly} = \left(1 + R_{annual}\right)^{1/52} - 1. \quad (9)$$

One major limitation of annualizing returns is the implicit assumption that returns can be repeated precisely, that is, money can be reinvested repeatedly while earning a similar return. This type of return is not always possible. An investor may earn a return of 5 percent during a week because the market rose sharply that week, but it is highly unlikely that he will earn a return of 5 percent every week for the next 51 weeks, resulting in an annualized return of 1,164.3 percent ($= 1.05^{52} - 1$). Therefore, it is important to annualize short-term returns with this limitation in mind.

EXAMPLE 12

Annualized Returns

An analyst seeks to evaluate three securities she has held in her portfolio for different periods of time.

- Over the past 100 days, Security A has earned a return of 6.2 percent.
- Security B has earned 2 percent over the past four weeks.
- Security C has earned a return of 5 percent over the past three months.

1. Compare the relative performance of the three securities.

   Solution:

   To facilitate comparison, the three securities' returns need to be annualized:

   - Annualized return for Security A: $R_{SA} = (1 + 0.062)^{365/100} - 1 = 0.2455 = 24.55\%$
   - Annualized return for Security B: $R_{SB} = (1 + 0.02)^{52/4} - 1 = 0.2936 = 29.36\%$
   - Annualized return for Security C: $R_{SC} = (1 + 0.05)^{4} - 1 = 0.2155 = 21.55\%$

   Security B generated the highest annualized return.

EXAMPLE 13

Exchange-Traded Fund Performance

An investor is evaluating the returns of three recently formed exchange-traded funds. Selected return information on the exchange-traded funds (ETFs) is presented in Exhibit 16.

Exhibit 16: ETF Performance Information

ETF	Time Since Inception	Return Since Inception (%)
1	146 days	4.61
2	5 weeks	1.10
3	15 months	14.35

1. Which ETF has the highest annualized rate of return?

   A. ETF 1

   B. ETF 2

   C. ETF 3

   Solution:

   B is correct. The annualized rate of return for the three ETFs are as follows:

   ETF 1 annualized return = $(1.0461^{365/146}) - 1 = 11.93\%$

   ETF 2 annualized return = $(1.0110^{52/5}) - 1 = 12.05\%$

   ETF 3 annualized return = $(1.1435^{12/15}) - 1 = 11.32\%$

   Despite having the lowest value for the periodic rate, ETF 2 has the highest annualized rate of return because of the reinvestment rate assumption and the compounding of the periodic rate.

Continuously Compounded Returns

An important concept is the continuously compounded return associated with a holding period return, such as R_1. The **continuously compounded return** associated with a holding period return is the natural logarithm of one plus that holding period return, or equivalently, the natural logarithm of the ending price over the beginning price (the price relative). Note that here we are using r to refer specifically to continuously compounded returns, but other textbooks and sources may use a different notation.

If we observe a one-week holding period return of 0.04, the equivalent continuously compounded return, called the one-week continuously compounded return, is ln(1.04) = 0.039221; EUR1.00 invested for one week at 0.039221 continuously compounded gives EUR1.04, equivalent to a 4 percent one-week holding period return.

The continuously compounded return from t to $t + 1$ is

$$r_{t,t+1} = \ln(P_{t+1}/P_t) = \ln(1 + R_{t,t+1}). \quad (10)$$

For our example, an asset purchased at time t for a P_0 of USD30 and the same asset one period later, $t + 1$, has a value of P_1 of USD34.50 has a continuously compounded return given by $r_{0,1} = \ln(P_1/P_0) = \ln(1 + R_{0,1}) = \ln(\text{USD34.50/USD30}) = \ln(1.15) = 0.139762$.

Thus, 13.98 percent is the continuously compounded return from $t = 0$ to $t = 1$. The continuously compounded return is smaller than the associated holding period return. If our investment horizon extends from $t = 0$ to $t = T$, then the continuously compounded return to T is

$$r_{0,T} = \ln(P_T/P_0). \quad (11)$$

Applying the exponential function to both sides of the equation, we have $\exp(r_{0,T}) = \exp[\ln(P_T/P_0)] = P_T/P_0$, so

$$P_T = P_0\exp(r_{0,T}).$$

We can also express P_T/P_0 as the product of price relatives:

$$P_T/P_0 = (P_T/P_{T-1})(P_{T-1}/P_{T-2}) \ldots (P_1/P_0). \quad (12)$$

Taking logs of both sides of this equation, we find that the continuously compounded return to time T is the sum of the one-period continuously compounded returns:

$$r_{0,T} = r_{T-1,T} + r_{T-2,T-1} + \ldots + r_{0,1}. \quad (13)$$

Using holding period returns to find the ending value of a USD1 investment involves the multiplication of quantities (1 + holding period return). Using continuously compounded returns involves addition (as shown in Equation 13), which is a desirable property of continuously compounded returns and which we will use throughout the curriculum.

OTHER MAJOR RETURN MEASURES AND THEIR APPLICATIONS

6

☐ calculate and interpret major return measures and describe their appropriate uses

The statistical measures of return discussed in the previous section are generally applicable across a wide range of assets and time periods. Special assets, however, such as mutual funds, and other considerations, such as taxes or inflation, may require more specific return measures.

Although it is not possible to consider all types of special measures, we will discuss the effect of fees (gross versus net returns), taxes (pre-tax and after-tax returns), inflation (nominal and real returns), and the effect of **leverage**. Many investors use mutual funds or other external entities (i.e., investment vehicles) for investment. In those cases, funds charge management fees and expenses to the investors. Consequently, gross and net-of-fund-expense returns should also be considered. Of course, an investor may be interested in the net-of-expenses after-tax real return, which is in fact what an investor truly receives. We consider these additional return measures in the following sections.

Gross and Net Return

A gross return is the return on assets managed less any trading expenses and commissions. Gross return is intended to reflect the investment skill of the manager. Expenses including management fees, custody fees, and taxes are not included in the gross return because they may be different for different investors. For example, most asset managers provide lower management fee rates to larger accounts. Excluding these expenses in gross returns provides a basis for evaluation and comparison of investment management skill.

Net return is a measure of what the investment vehicle (e.g., mutual fund) has earned for the investor. Net return accounts for (i.e., deducts) all managerial and administrative expenses that reduce an investor's return. Because individual investors are most concerned about the net return (i.e., what they actually receive), small mutual funds with a limited amount of assets under management are at a disadvantage compared with the larger funds that can spread their largely fixed administrative expenses over a larger asset base. As a result, many small mutual funds waive part of the expenses to keep the funds competitive.

Pre-Tax and After-Tax Nominal Return

All return measures discussed up to this point are pre-tax nominal returns—that is, no adjustment has been made for taxes or inflation. In general, all returns are pre-tax nominal returns unless they are otherwise designated.

Many investors are concerned about the possible tax liability associated with their returns because taxes reduce the net return that they receive. Capital gains and income may be taxed differently, depending on the jurisdiction. Capital gains come in two forms: short-term capital gains and long-term capital gains. Long-term capital gains receive preferential tax treatment in a number of countries. Interest income is taxed as ordinary income in most countries. Dividend income may be taxed as ordinary income, may have a lower tax rate, or may be exempt from taxes depending on the country and the type of investor. The after-tax nominal return is computed as the total return minus any allowance for taxes on dividends, interest, and realized gains. Bonds issued at a discount to the par value may be taxed based on accrued gains instead of realized gains.

Because taxes are paid on realized capital gains and income, the investment manager can minimize the tax liability by selecting appropriate securities (e.g., those subject to more favorable taxation, all other investment considerations equal) and reducing trading turnover. Therefore, taxable investors evaluate investment managers based on the after-tax nominal return.

Real Returns

Previously this learning module approximated the relationship between the nominal rate and the real rate by the following relationship:

$$(1 + \text{nominal risk–free rate}) = (1 + \text{real risk–free rate})(1 + \text{inflation premium}).$$

This relationship can be extended to link the relationship between nominal and real returns. Specifically, the nominal return consists of a real risk-free rate of return to compensate for postponed consumption; inflation as loss of purchasing power; and a risk premium for assuming risk. Frequently, the real risk-free return and the risk premium are combined to arrive at the real "risky" rate and is simply referred to as the real return, or:

$$(1 + \text{real return}) = (1 + \text{real risk–free rate})(1 + \text{risk premium}). \quad (14)$$

Real returns are particularly useful in comparing returns across time periods because inflation rates may vary over time. Real returns are also useful in comparing returns among countries when returns are expressed in local currencies instead of a constant investor currency and when inflation rates vary between countries (which are usually the case).

Finally, the after-tax real return is what the investor receives as compensation for postponing consumption and assuming risk after paying taxes on investment returns. As a result, the after-tax real return becomes a reliable benchmark for making investment decisions. Although it is a measure of an investor's benchmark return, it is not commonly calculated by asset managers because it is difficult to estimate a general tax component applicable to all investors. For example, the tax component depends on an investor's specific taxation rate (marginal tax rate), how long the investor holds an investment (long-term versus short-term), and the type of account the asset is held in (tax-exempt, tax-deferred, or normal).

EXAMPLE 14

Computation of Special Returns

Let's return to Example 8. After reading this section, Mr. Lohrmann decided that he was not being fair to the fund manager by including the asset management fee and other expenses because the small size of the fund would put it at a competitive disadvantage. He learns that the fund spends a fixed amount of EUR500,000 every year on expenses that are unrelated to the manager's performance.

Mr. Lohrmann has become concerned that both taxes and inflation may reduce his return. Based on the current tax code, he expects to pay 20 percent tax on the return he earns from his investment. Historically, inflation has been around 2 percent and he expects the same rate of inflation to be maintained.

1. Estimate the annual gross return for the first year by adding back the fixed expenses.

   Solution:

   The gross return for the first year is higher by 1.67 percent (= EUR500,000/EUR30,000,000) than the 15 percent% investor return reported by the fund. Thus, the gross return for the first year is 16.67 percent (= 15% + 1.67%).

2. What is the net return that investors in the Rhein Valley Superior Fund earned during the five-year period?

   Solution:

   The investor return reported by the mutual fund is the net return of the fund after accounting for all direct and indirect expenses. The net return is also the pre-tax nominal return because it has not been adjusted for taxes or inflation. From Example. 8, the net return for the five-year holding period was calculated as 42.35 percent.

3. What is the after-tax net return for the first year that investors earned from the Rhein Valley Superior Fund? Assume that all gains are realized at the end of the year and the taxes are paid immediately at that time.

   Solution:

   The net return earned by investors during the first year was 15 percent. Applying a 20 percent tax rate, the after-tax return that accrues to investors is 12 percent [= 15% – (0.20 × 15%)].

4. What is the after-tax real return that investors would have earned in the fifth year?

   Solution:

   The after-tax return earned by investors in the fifth year is 2.4 percent [= 3% – (0.20 × 3%)]. Inflation reduces the return by 2 percent so the after-tax real return earned by investors in the fifth year is 0.39 percent, as shown:

   $$\frac{(1 + 2.40\%)}{(1 + 2.00\%)} - 1 = \frac{(1 + 0.0240)}{(1 + 0.0200)} - 1 = 1.0039 - 1 = 0.0039 = 0.39\%.$$

   Note that taxes are paid before adjusting for inflation.

Leveraged Return

In the previous calculations, we have assumed that the investor's position in an asset is equal to the total investment made by an investor using his or her own money. This section differs in that the investor creates a leveraged position.

There are two ways of creating a claim on asset returns that are greater than the investment of one's own money. First, an investor may trade futures contracts in which the money required to take a position may be as little as 10 percent of the notional value of the asset. In this case, the leveraged return, the return on the investor's own money, is 10 times the actual return of the underlying security. Both the gains and losses are amplified by a factor of 10.

Investors can also invest more than their own money by borrowing money to purchase the asset. This approach is easily done in stocks and bonds, and very common when investing in real estate. If half (50 percent) of the money invested is borrowed, then the gross return to the investor is doubled, but the interest to be paid on borrowed money must be deducted to calculate the net return.

Using borrowed capital, debt, the size of the leveraged position increases by the additional, borrowed capital. If the total investment return earned on the leveraged portfolio, R_P, exceeds the borrowing cost on debt, r_D, taking on leverage increases the return on the portfolio. Denoting the return on a leveraged portfolio as R_L, then the return can be calculated as follows:

$$R_L = \frac{\text{Portfolio return}}{\text{Portfolio equity}} = \frac{[R_P \times (V_E + V_B) - (V_B \times r_D)]}{V_E} = R_p + \frac{V_B}{V_E}(R_p - r_D), \qquad (15)$$

where V_E is the equity of the portfolio and V_B is the debt or borrowed funds. If $R_P < r_D$ then leverage decreases R_L.

For example, for a EUR10 million equity portfolio that generates an 8 percent total investment return, R_P, over one year and is financed 30 percent with debt at 5 percent, then the leveraged return, R_L, is:

$$R_L = R_p + \frac{V_B}{V_E}(R_p - r_D) = 8\% + \frac{EUR3 \text{ million}}{EUR7 \text{ million}}(8\% - 5\%) = 8\% + 0.43 \times 3\% = 9.29\%.$$

EXAMPLE 15

Return Calculations

An analyst observes the following historic asset class geometric returns:

Exhibit 17: Asset Class Geometric Return

Asset Class	Geometric Return (%)
Equities	8.0
Corporate Bonds	6.5
Treasury bills	2.5
Inflation	2.1

1. The real rate of return for equities is *closest* to:

   A. 5.4 percent.

   B. 5.8 percent.

C. 5.9 percent.

Solution:

B is correct. The real rate of return for equities is calculated as follows:

$(1 + 0.080)/(1 + 0.0210) - 1 = 5.8\%$.

2. The real rate of return for corporate bonds is *closest* to:

A. 4.3 percent.

B. 4.4 percent.

C. 4.5 percent.

Solution:

A is correct. The real rate of return for corporate bonds is calculated as follows:

$(1 + 0.065)/(1 + 0.0210) - 1 = 4.3\%$.

3. The risk premium for equities is closest to:

A. 5.4 percent.

B. 5.5 percent.

C. 5.6 percent.

Solution:

A is correct. The risk premium for equities is calculated as follows:

$(1 + 0.080)/(1 + 0.0250) - 1 = 5.4\%$.

4. The risk premium for corporate bonds is *closest* to:

A. 3.5 percent.

B. 3.9 percent.

C. 4.0 percent.

Solution:

B is correct. The risk premium for corporate bonds is calculated as follows:

$(1 + 0.0650)/(1 + 0.0250) - 1 = 3.9\%$.

PRACTICE PROBLEMS

1. The nominal risk-free rate is *best* described as the sum of the real risk-free rate and a premium for:

   A. maturity.

   B. liquidity.

   C. expected inflation.

2. Which of the following risk premiums is most relevant in explaining the difference in yields between 30-year bonds issued by the US Treasury and 30-year bonds issued by a small, private US corporate issuer?

   A. Inflation

   B. Maturity

   C. Liquidity

3. Consider the following annual return for Fund Y over the past five years:

Exhibit 1: Five-Year Annual Returns

Year	Return (%)
Year 1	19.5
Year 2	–1.9
Year 3	19.7
Year 4	35.0
Year 5	5.7

   The geometric mean return for Fund Y is *closest* to:

   A. 14.9 percent.

   B. 15.6 percent.

   C. 19.5 percent.

4. A portfolio manager invests EUR5,000 annually in a security for four years at the following prices:

Exhibit 1: Five-Year Purchase Prices

Year	Purchase Price (euros per unit)
Year 1	62.00
Year 2	76.00

Year	Purchase Price (euros per unit)
Year 3	84.00
Year 4	90.00

The average price is *best* represented as the:

A. harmonic mean of EUR76.48.

B. geometric mean of EUR77.26.

C. arithmetic average of EUR78.00.

5. Which of the following statements regarding arithmetic and geometric means is correct?

A. The geometric mean will exceed the arithmetic mean for a series with non-zero variance.

B. The geometric mean measures an investment's compound rate of growth over multiple periods.

C. The arithmetic mean measures an investment's terminal value over multiple periods.

6. A fund receives investments at the beginning of each year and generates returns for three years as follows:

Exhibit 1: Investments and Returns for Three Years

Year of Investment	Assets under Management at the Beginning of each year	Return during Year of Investment
1	USD1,000	15%
2	USD4,000	14%
3	USD45,000	–4%

Which return measure over the three-year period is negative?

A. Geometric mean return

B. Time-weighted rate of return

C. Money-weighted rate of return

7. At the beginning of Year 1, a fund has USD10 million under management; it earns a return of 14 percent for the year. The fund attracts another net USD100 million at the start of Year 2 and earns a return of 8 percent for that year. The money-weighted rate of return of the fund is *most likely* to be:

A. less than the time-weighted rate of return.

B. the same as the time-weighted rate of return.

C. greater than the time-weighted rate of return.

8. An investor is evaluating the returns of three recently formed ETFs. Selected

return information on the ETFs is presented in Exhibit 1:

Exhibit 1: Returns on ETFs

ETF	Time Since Inception	Return Since Inception (%)
1	125 days	4.25
2	8 weeks	1.95
3	16 months	17.18

Which ETF has the highest annualized rate of return?

A. ETF 1

B. ETF 2

C. ETF 3

9. The price of a stock at $t = 0$ is USD208.25 and at $t = 1$ is USD186.75. The continuously compounded rate of return, $r_{1,T}$ for the stock from $t = 0$ to $t = 1$ is *closest* to:

A. –10.90 percent.

B. –10.32 percent.

C. 11.51 percent.

10. A USD25 million equity portfolio is financed 20 percent with debt at a cost of 6 percent annual cost. If that equity portfolio generates a 10 percent annual total investment return, then the leveraged return is:

A. 11.0 percent.

B. 11.2 percent.

C. 13.2 percent

11. An investment manager's gross return is:

A. an after-tax nominal, risk-adjusted return.

B. the return earned by the manager prior to deduction of trading expenses.

C. an often used measure of an investment manager's skill because it does not include expenses related to management or administration.

12. The strategy of using leverage to enhance investment returns:

A. amplifies gains but not losses.

B. doubles the net return if half of the invested capital is borrowed.

C. increases total investment return only if the return earned exceeds the borrowing cost.

13. At the beginning of the year, an investor holds EUR10,000 in a hedge fund. The investor borrowed 25 percent of the purchase price, EUR2,500, at an annual interest rate of 6 percent and expects to pay a 30 percent tax on the return she

earns from his investment. At the end of the year, the hedge fund reported the information in Exhibit 1:

Exhibit 1: Hedge Fund Investment

Gross return	8.46%
Trading expenses	1.10%
Managerial and administrative expenses	1.60%

The investor's after-tax return on the hedge fund investment is closest to:

A. 3.60 percent.

B. 3.98 percent.

C. 5.00 percent.

SOLUTIONS

1. C is correct. The nominal risk-free rate is approximated as the sum of the real risk-free interest rate and an inflation premium.

2. C is correct. US Treasury bonds are highly liquid, whereas the bonds of small issuers trade infrequently and the interest rate includes a liquidity premium. This liquidity premium reflects the relatively high costs (including the impact on price) of selling a position. As the two bond issues have the same 30-year maturity, the observed difference in yields would not be solely explained by maturity. Further, the inflation premium embedded in the yield of both bonds is likely to be similar given they are both US-based bonds with the same maturity.

3. A is correct. The geometric mean return for Fund Y is calculated as follows:

   $\overline{R}_G = [(1 + 0.195) \times (1 - 0.019) \times (1 + 0.197) \times (1 + 0.350) \times (1 + 0.057)]^{(1/5)} - 1$
   $= 14.9\%.$

4. A is correct. The harmonic mean is appropriate for determining the average price per unit as it gives equal weight to each data point and reduces the potential influence of outliers. It is calculated as follows:

   $\overline{X}_H = 4/[(1/62.00) + (1/76.00) + (1/84.00) + (1/90.00)] = \text{EUR}76.48.$

5. B is correct. The geometric mean compounds the periodic returns of every period, giving the investor a more accurate measure of the terminal value of an investment.

6. C is correct. The money-weighted rate of return considers both the timing and amounts of investments into the fund. To calculate the money-weighted rate of return, tabulate the annual returns and investment amounts to determine the cash flows.

Year	1	2	3
Balance from previous year	0	USD1,150	USD4,560
New investment	USD1,000	USD2,850	USD40,440
Net balance at the beginning of year	USD1,000	USD4,000	USD45,000
Investment return for the year	15%	14%	–4%
Investment gain (loss)	USD150	USD560	–USD1,800
Balance at the end of year	USD1,150	USD4,560	USD43,200

$CF_0 = -\text{USD}1{,}000$, $CF_1 = -\text{USD}2{,}850$, $CF_2 = -\text{USD}40{,}440$, $CF_3 = +\text{USD}43{,}200$.

$$CF_0 = -1{,}000$$
$$CF_1 = -2{,}850$$
$$CF_2 = -40{,}440$$
$$CF_3 = +43{,}200$$

$$\frac{CF_0}{(1+\text{IRR})^0} + \frac{CF_1}{(1+\text{IRR})^1} + \frac{CF_2}{(1+\text{IRR})^2} + \frac{CF_3}{(1+IRR)^3}$$
$$= \frac{-1{,}000}{1} + \frac{-2{,}850}{(1+\text{IRR})^1} + \frac{-40{,}440}{(1+\text{IRR})^2} + \frac{43{,}200}{(1+\text{IRR})^3} = 0$$

Solving for *IRR* results in a value of *IRR* = –2.22 percent.

Note that A and B are incorrect because the time-weighted rate of return (TWR) of the fund is the same as the geometric mean return of the fund and is positive:

$R_{TW} = \sqrt[3]{(1.15)(1.14)(0.96)} - 1 = 7.97\%.$

7. A is correct. Computation of the money-weighted return, *r*, requires finding the discount rate that sums the present value of cash flows to zero. Because most of the investment came during Year 2, the money-weighted return will be biased toward the performance of Year 2 when the return was lower. The cash flows are as follows:

$CF_0 = -10$

$CF_1 = -100$

$CF_2 = +120.31$

The terminal value is determined by summing the investment returns for each period [(10 × 1.14 × 1.08) + (100 × 1.08)].

$$\frac{CF_0}{(1+IRR)^0} + \frac{CF_1}{(1+IRR)^1} + \frac{CF_2}{(1+IRR)^2}$$
$$\frac{-10}{1} + \frac{-100}{(1+IRR)^1} + \frac{120.31}{(1+IRR)^2} = 0.$$

This results in a value of *IRR* = 8.53 percent.

The time-weighted return of the fund is calculated as follows:

$R_{TW} = \sqrt[2]{(1.14)(1.08)} - 1 = 10.96\%.$

8. B is correct. The annualized rate of return for

ETF 1 annualized return = $(1.0425^{365/125}) - 1 = 12.92\%$

ETF 2 annualized return = $(1.0195^{52/8}) - 1 = 13.37\%$

ETF 3 annualized return = $(1.1718^{12/16}) - 1 = 12.63\%$

Despite having the lowest value for the periodic rate, ETF 2 has the highest annualized rate of return because of the reinvestment rate assumption and the compounding of the periodic rate.

9. A is correct. The continuously compounded return from $t = 0$ to $t = 1$ is $r_{0,1} = \ln(S_1/S_0) = \ln(186.75/208.25) = -0.10897 = -10.90\%$.

10. A is correct.

$$R_L = R_p + \frac{V_B}{V_E}(R_p - r_D)$$

$$= 10\% + \frac{USD5\ million}{USD20\ million}(10\% - 6\%)$$

$$= 10\% + 0.25 \times 4\% = 11.0\%.$$

11. C is correct. Gross returns are calculated on a pre-tax basis; trading expenses *are* accounted for in the computation of gross returns as they contribute directly to the returns earned by the manager. A is incorrect because investment managers' gross returns are pre-tax and not adjusted for risk. B is incorrect because managers' gross returns do reflect the deduction of trading expenses since they contribute directly to the return earned by the manager.

12. C is correct. The use of leverage can increase an investor's return if the total investment return earned on the leveraged investment exceeds the borrowing cost on debt. A is incorrect because leverage amplifies both gains and losses. B is incorrect because, if half of the invested capital is borrowed, then the investor's gross (not net) return would double.

13. C is correct. The first step is to compute the investor's net return from the hedge fund investment. The net return is the fund's gross return less managerial and administrative expenses of 1.60 percent, or 8.46% – 1.60% = 6.86%. Note that trading expenses are already reflected in the gross return, so they are not subtracted.
The second step is to compute the investor's leveraged return (the investor borrowed EUR2,500 (25 percent) of the purchase), calculated as: follows

$$R_L = R_p + \frac{V_B}{V_E}(R_p - r_D)$$

$$R_L = 6.86\% + \frac{EUR2,500}{EUR7,500}(6.86\% - 6\%)$$

$R_L = 6.86\% + 0.33 \times 0.86\% = 7.15\%$.

The final step is to compute the after-tax return:
After-tax return = 7.15% (1 – 0.30) = 5.00%.